Efficient 2D and 3D calculation algorithms for incompressible flow around cylinders

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In this paper a solution method for the pressure Poisson equation which follows from the spatial and time discretization of the nonstationary Navier-Stokes equation for an incompressible fluid based on an algebraically defined multigrid method will be described. The method is used to compute and to investigate the flow field around a cylinder and will be used for **L**arge **E**ddy **S**imulations.

0. Introduction

The mathematical modelling of laminar and turbulent flows requires robust and efficient numerical methods for solving the Navier-Stokes equations. The aim of the discussed mathematical and numerical modelling is to develop and to investigate special techniques for the solution of algebraic equation systems resulting from discretization over O-type grids and rectangular grids.

We have investigated the low Reynolds number flow around single cylinders and groups of cylinders with a 2D calculation method for the unsteady incompressible Navier–Stokes equation in the primitive variable formulation.

The fulfilment of the continuity equation at every time step we get by an iterative method (pressure velocity correction of SMAC, PISO et cetera type). This method gives sufficient good results (in sense of time efficiency) for the 2D modelling with a resolution of the spatial computation region of nearly 30.000 – 50.000 grid points.

For the case of **LES** or **DNS** modelling of flow around cylinders we have to solve problems with more then 1.000.000 grid points. Instead of the above discussed pressure velocity correction method we use more direct methods to fulfil the continuity equation at every time step that means we use a Poisson equation to realize the continuity equation for an incompressible fluid. This is the entry point of the application of efficient solution methods for a special class of equation systems with sparse matrices.

The rate of convergence and computation times of the used solution methods are compared to those of other often used pressure velocity iteration methods.

1. Some remarks on the mathematical model and the practical background

The unsteady Navier-Stokes equations for an incompressible fluid are of the form

$$\frac{\partial \vec{u}}{\partial t} + \nabla \cdot \vec{u}\vec{u} = -\nabla p + \nabla \cdot \nu \nabla \vec{u} + \vec{f} \qquad (1)$$

$$\nabla \cdot \vec{u} = 0. \qquad (2)$$

in the flow region $\Omega \subset R^3$. At the time $t = t_0$ the initial value

$$\vec{u}(x, t_0) = \vec{u}_0(x) \qquad (3)$$

is given. $\vec{u}$ and p stand for the velocity field and the modified pressure (pressure over density), ν is the kinematic viscosity, $\vec{f}$ is a force vector. On the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ the solution must fulfil the inflow and no-slip boundary conditions

$$\vec{u}(x,t) = \vec{g}_1(x,t) \quad , x \in \Gamma_1 \, , \qquad (4)$$

and suitable outflow boundary conditions shortly summarized as

$$a \cdot \frac{\partial^2 \vec{u}}{\partial \vec{x}^2} + b \cdot \frac{\partial \vec{u}}{\partial \vec{x}} = \vec{g}_2(x,t) \quad , x \in \Gamma_2 \, , \qquad (5)$$

where $\vec{x}$ is the direction of the inflow velocity u_∞. Related to the continuity equation (2) it is nessecary to hold the condition

$$\int_\Gamma \vec{n} \cdot \vec{u} d\gamma = 0 \tag{6}$$

for the boundary values. The condition (6) means that the inflow mass fluxes over the boundary are equal to the outflow mass fluxes.

The pde system (1)–(6) describes the nonsteady 3D flow and we are interested in the investigation of the structure of vortices in the wake region of a circular cylinder.

In the case of the LES we have to consider instead of the equation system (1)-(2) the equation system for the overlined resolved values of the velocity and the pressure

$$\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i}\,\overline{u_j}) =$$

$$-\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \overline{u}}{\partial x_j}) + f_i \tag{7}$$

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0, \tag{8}$$

with the SGS stress of the form

$$\tau_{ij} = \overline{u_i u_j} - \overline{u_i}\,\overline{u_j}\,. \tag{9}$$

For the subgrid scale modelling of τ_{ij} the SGS-models of Smagorinski and Germano are under discussion.

It's written below u and p for resolved velocity and ν for the sum of the SGS-viscosity and the molecular viscosity in the case of LES-modelling. The special application fields of the mathematical model (1)–(5) require the use of a cylinder symmetrical coordinate system. Thus we have to consider the equations (1) and (2) in the (φ, r, z)-coordinate system. We understand now the velocity vector as $\vec{u} = (u, v, w) = (u_\varphi, u_r, u_z)$.

2. Principle of spatial and time discretization of the pde system

The spatial discretization of (1) and (2) was made by a finite volume method. We use four staggered grids for the three velocity components and the pressure. Using the Gauss integral theorem we get by integration over a volume element of the φ-grid the following approximation for the φ-component of the impulse transport equation (1) for the u-velocity component

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{r_j}[u^2_{i+1jk}]_{\overline{\varphi}} + \frac{1}{r_j}[(r\,v\,u)_{i+\frac{1}{2}j+\frac{1}{2}k}]_{\overline{r}} \\ &+[(w\,u)_{i+\frac{1}{2}jk+\frac{1}{2}}]_{\overline{z}} + \frac{1}{r_j}(v\,u)_{i+\frac{1}{2}jk} = \\ &-\frac{1}{r_j}[p_{i+1jk}]_{\overline{\varphi}} + \frac{1}{r_j}[r_{j+\frac{1}{2}}\nu\, u_{i+\frac{1}{2}j+1k_{\overline{r}}}]_{\overline{r}} \\ &+\frac{1}{r_j^2}[\nu\, u_{i+\frac{3}{2}jk_{\overline{\varphi}}}]_{\overline{\varphi}} + [\nu\, u_{i+\frac{1}{2}jk+1_{\overline{z}}}]_{\overline{z}} \\ &+\frac{\nu}{r_j^2}[v_{i+1jk}]_{\overline{\varphi}} - \frac{\nu}{r_j^2}u_{i+\frac{1}{2}jk} \;+\; f_u\,. \end{aligned} \tag{10}$$

In the equation (10) we use the notation

$$[g_{i+\alpha j+\beta k+\gamma}]_{\overline{\varphi}} = \frac{g_{i+\alpha j+\beta k+\gamma} - g_{i+\alpha-1j+\beta k+\gamma}}{\varphi_{i+\alpha} - \varphi_{i+\alpha-1}}$$

and so on. For the r-velocity component follows from the integration of the second component of the impuls transfer equation over a volume element of the r-grid

$$\begin{aligned} &\frac{\partial v}{\partial t} + \frac{1}{r_{j+\frac{1}{2}}}\{[r_{j+1}\,v^2_{ij+1k}]_{\overline{r}} + [(v\,u)_{i+\frac{1}{2}j+\frac{1}{2}k}]_{\overline{\varphi}}\} \\ &+[(w\,v)_{ij+\frac{1}{2}k+\frac{1}{2}}]_{\overline{z}} + \frac{1}{r_{j+\frac{1}{2}}}v^2_{ij+\frac{1}{2}k} = \\ &-[p_{ij+1k}]_{\overline{r}} + \frac{1}{r_{j+\frac{1}{2}}}[\nu\, v_{i+1j+\frac{1}{2}k_{\overline{\varphi}}}]_{\overline{\varphi}} \\ &+\frac{1}{r_{j+\frac{1}{2}}}[r_{j+1}\nu\, v_{ij+\frac{3}{2}k_{\overline{r}}}]_{\overline{r}} + [\nu\, v_{ij+\frac{1}{2}k+1_{\overline{z}}}]_{\overline{z}} \\ &+\frac{\nu}{r^2_{j+\frac{1}{2}}}[u_{i+\frac{1}{2}j+\frac{1}{2}k}]_{\overline{\varphi}} - \frac{\nu}{r^2_{j+\frac{1}{2}}}v_{ij+\frac{1}{2}k} \;+\; f_v\,. \end{aligned} \tag{11}$$

For the z-velocity component we get by integration over the volume element of the z-grid an equation similar to (10) and (11). A more detailed formulation of the spatial discretization is given in [1].

Now we integrate the continuity equation (2) over the volume elements of the pressure-grid and when we represent the values of the velocity by the values at the center of the boundary elements of the volume elements we will have the following discretization of the continuity equation

$$\nabla_h \cdot \vec{u} := \frac{1}{r_j} u_{i+\frac{1}{2}jk\overline{\varphi}} + \frac{1}{r_j}[(r\,v)_{ij+\frac{1}{2}k}]_{\overline{r}} \qquad (12)$$
$$+ w_{ijk+\frac{1}{2}\overline{z}} = 0\,.$$

If we denote the FV-discretization of ∇ and $\nabla\cdot$ by ∇_h and $\nabla_h\cdot$ we can summarize the the FV-approximation of (1) and (2) to

$$\frac{\partial \vec{u}}{\partial t} + \nabla_h \cdot \vec{u}\vec{u} = -\nabla_h p \qquad (13)$$
$$+\nabla_h \cdot (\nu \nabla_h \vec{u}) + \vec{f}$$
$$\nabla_h \cdot \vec{u} = 0. \qquad (14)$$

The equation system (13)-(14) closed by the discretization of the boundary conditions (4)-(5) is now to integrate by time with the initial velocity field (3).
The time discretization of this system we do by a two stage semiimplicit splitting method for the equation (13).

$$\frac{\tilde{\vec{u}} - \vec{u}}{\tau} + \sigma_1 \nabla_h \cdot \tilde{\vec{u}}\tilde{\vec{u}} + (1-\sigma_1)\nabla_h \cdot \vec{u}\vec{u} = -\nabla_h p \, (15)$$
$$+\sigma_2 \nabla_h \cdot (\nu \nabla_h \tilde{\vec{u}}) + (1-\sigma_2)\nabla_h \cdot (\nu \nabla_h \vec{u}) + \vec{f},$$
$$\frac{\vec{u}^{\,n+1} - \tilde{\vec{u}}}{\tau} + \nabla_h (p^{n+1} - p) = 0, \, (16)$$
$$\nabla_h \cdot \vec{u}^{\,n+1} = 0. \, (17)$$

τ is the time step, $\sigma_{1,2}$ are weighting factors. Quantities with the upper index $n+1$ are taken at the $(n+1)$th time level, quantities without an index are values at the nth time level, $\tilde{\vec{u}}$ stands for a predicted velocity between nth and $(n+1)$th time level.
If we suppose that the equation (15) was solved - in the case $\sigma_1 = \sigma_2 = 0$ the prediction of $\tilde{\vec{u}}$ is an explicit procedure - we have to determine $\vec{u}^{\,n+1}$ and $\delta\, p^{n+1} = p^{n+1} - p$ by solving the equation system (16)-(17).
A way to by–pass the unpleasant equation system (16)-(17) is the solution of the Poisson equation

$$-\nabla_h^2\, \delta p^{n+1} = -\frac{1}{\tau}\nabla_h \cdot \tilde{\vec{u}}, \qquad (18)$$

at every time level. As the result of putting equation (16) into (17) for boundary volume elements we will have for the Poisson equation (18) boundary conditions of Neumann–type, that means that the sum of off diagonal elements of every row is equal to the negative value of the diagonal entry, and we will have a non regular coefficient matrix A for determination of δp^{n+1}.
The Matrix A is of type $(n3d, n3d)$ where $n3d$ ist the number of volume elements of the pressure grid. The analytic property of pressure choice from a manifold by fixing the value p_0 at an arbitrary point corresponds with the property of A that the rank of A is equal to $n3d-1$.
For the solvability of (18) this means that the discrete integral over the right hand side must be vanish or

$$\int_\Omega \nabla_h \cdot \tilde{\vec{u}}\, d\Omega = 0. \qquad (19)$$

The solvability condition (19) is fulfilled if we have a conservative approximation of the continuity equation. This we will show now.
The definition of conservativity of approximation of (2) is given by

$$\int_\Omega \nabla_h \cdot \vec{u}\, d\Omega = \int_\Gamma \vec{n} \cdot \vec{u}\, d\gamma\,. \qquad (20)$$

If we declare ω_{ijk} by $r_j \delta\varphi_i \delta r_j \delta z_k$ in our cylinder coordinate system it follows from the approximation (12) and the definition (20)

$$\sum_{i,j,k\in\Omega} \omega_{ijk}\, (\nabla_h \cdot \vec{u}) =$$
$$\sum_{\Gamma+} \omega_{ijk}\, \vec{n} \cdot \Big(\frac{u_{i+\frac{1}{2}jk}}{r_j \delta\varphi_i}, \frac{r_{j+\frac{1}{2}} v_{ij+\frac{1}{2}k}}{r_j \delta r_j}, \frac{w_{ijk+\frac{1}{2}}}{\delta z_k}\Big) -$$
$$(21)$$
$$\sum_{\Gamma-} \omega_{ijk}\, \vec{n} \cdot \Big(\frac{u_{i-\frac{1}{2}jk}}{r_j \delta\varphi_i}, \frac{r_{j-\frac{1}{2}} v_{ij-\frac{1}{2}k}}{r_j \delta r_j}, \frac{w_{ijk-\frac{1}{2}}}{\delta z_k}\Big).$$

The right hand side of (21) is zero since it is the approximation of the flux condition (6).
In our cylinder coordinate system the equation (18) is of the symmetric form

$$\omega_{ijk}\Big(\frac{1}{r_j}[r_{j+\frac{1}{2}}\, [p_{ij+1k}]_{\overline{r}}]_{\overline{r}} + \frac{1}{r_j^2}[[p_{i+1jk}]_{\overline{\varphi}}]_{\overline{\varphi}} \qquad (22)$$
$$+[[p_{ijk+1}]_{\overline{z}}]_{\overline{z}}\Big) = \frac{1}{\tau}\omega_{ijk}\nabla_h \cdot \tilde{\vec{u}}$$

for inner grid points (i,j,k) of the pressure grid. For boundary points of the pressure grid the equation (22) must be cut in a natural way.

3. Some remarks to the choice of boundary conditions for the velocity field

On the inflow boundary we work with a given velocity $\vec{u}_\infty$. In the case of LES we must take $\vec{u}_\infty$ from the experiment or by filtering of suitable DNS data, for example those of Kleiser et al. [2] of channel flow simulation.
We've noted above that the boundary values of the velocity must fulfil the conditions (6) or (19) because of the balance of inflow and outflow mass fluxes resp. the solvability condition for the equation (18). Following this a prediction rule for the normal component of the velocity on the outflow boundary is given for example by

$$u_{i+\frac{1}{2}jk} = u_{i-\frac{1}{2}jk} \tag{23}$$
$$-\delta\varphi_i([(rv)_{ij+\frac{1}{2}k}]_{\overline{r}} + r_j w_{ijk+\frac{1}{2}\overline{z}}).$$

Together with a linear extrapolation of the tangential velocity component, that means for example

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} = 0 \tag{24}$$

we have boundary conditions which fulfil the required conditions (6), (19) and work very well.

4. The solution of the pressure Poisson equation

For regularization of A we set the pressure at one grid point constant by multiplying the corresponding diagonal entry of A and the right hand b side by a large number. This way of regularization from A to A' and b to b' secures the symmetry of the coefficient matrix and the equation system structure as a very important supposition for the possibility of multigrid preconditioning and we can solve the regularized equation system

$$A' \; \delta p \; = \; b' \tag{25}$$

by a cg–method with preconditioning.

In the case of cartesian or curvilinear rectangular grids the matrix A' has seven non zero diagonals (in 2D–case five non zero diagonals) and it is possible to use preconditioners which use the arising rectangular or quadrilateral grid structure. In the Institute of applied Analysis and Stochastics, there exist multigrid and ILU decompositon codes [3] which can be used for this purpose.
If we have a discretization of a flow around a circular cylinder in the (ϕ, r, z)–coordinate system we get a cyclic matrix with nine non zero diagonals because of the existence of a coupling plane

$$\{(\varphi, r, z) \;\mid\; \varphi = 0 \text{ and } \varphi = 2\pi\}$$

or periodic boundary conditions in the φ-direction

$$\delta p\,(\varphi, r, z) = \delta p\,(\varphi + 2\pi, r, z).$$

If we consider the flow around an infinitly long circular cylinder we usually assume periodicity in the z-direction with a period length l_z and we have also in z-direction periodic boundary conditions like

$$\delta p\,(\varphi, r, z) = \delta p\,(\varphi, r, z + l_z)$$

or we can say that

$$\{(\varphi, r, z) \;\mid\; z = 0 \text{ and } z = l_z\}$$

is a "coupling" plane. In such a case the matrix A' is cyclic in two directions and there exist eleven non zero diagonals.
There have been no codes at hand which are able to manage this grid structure, so for the first investigations we choosed the following approach which allows to use the rectangular grid preconditioners in this case, too. We have a matrix splitting

$$A' = L - N \tag{26}$$

where N consists of the sign reversed off-diagonal entries across the coupling planes and thus is positive. L is irreducibly diagonally dominant, has nonpositive off diagonal entries and a positive main diagonal, so it is an M-matrix. This implies that (26) is a regular splitting, and the iterative method (instead of δp we write now x)

$$x_{i+1} = x_i - L^{-1}(A'x_i - b') \tag{27}$$

converges to the solution of (25) [4]. So, for instance, one can iteratively invert L in each step of (27), a procedure we had in mind at the first stages of our code developement, which indeed did converge.

A more efficient procedure is the following. The convergence of (27) for the symmetric problem (25) and the symmetric matrix L suggests that there exists a spectral equivalence (written here as matrix inequality)

$$\lambda^- L \leq A' \leq \lambda^+ L \tag{28}$$

with the relative condition number $\kappa(L, A') = \lambda^+/\lambda^-$ in moderate regions. If now B is a symmetric positive definite preconditioner for L with

$$\mu^- B \leq L \leq \mu^+ B \tag{29}$$

then it is obviously spectrally equivalent to A via

$$\mu^- \lambda^- B \leq A' \leq \mu^+ \lambda^+ B. \tag{30}$$

From this follows the convergence of the preconditioned Richardson method

$$x_{i+1} = x_i - B^{-1}(A' x_i - b') \tag{31}$$

for solving (25). But because of the symmetry of the matrix and the preconditioners it is better to use a preconditioned conjugate gradient method. As the preconditioner B^{-1} among others it was used one step of a multigrid method to solve

$$Ly = r$$

after [5]. This is a semi-algebraic multigrid method which works on rectangular resp. quadrilateral meshes. So it is well fitted to the discretization described by (22). Up to now, this method has been implemented for two- and three-dimensional rectangular grids, so it in fact can be used in the sense described above.

The comparison of the time efficiency of the described cg/mg-method with a fill in procedure of a solver for banded matrices shows only small differences between the two solution methods if we have at least five grid levels. Because of round off errors in the case of the use of elimination methods for banded matrices it is nessecary to predict a correction by solving the equation system with the residuum as the right hand side. Thus the cg/mg-method is much more efficient then elimination method.

The methods using the pressure Poisson equation (18) are especially in the beginning of the non-stationary solution process to prefer the above mentioned pressure-velocity correction methods of SMAC or PISO type.

Essential reserves of time efficiency we expect in an genuine O-grid implementation which would have the same structure and should result in a large improvement of the covergence rates.

5. Some results of the flow calculation

The figures (2)-(6) of the next page of this paper show some velocity fields of the time development (2D) of vortex structures in the wake region of a circular cylinder for $Re = 1.000$ with a spatial grid of (129×129) nodes. For the computed 30.000 time steps we need approximatly 500' on a 16 Mflops workstation.

The 3D version of the method is just validated for low Re numbers and a (φ, r, z)-grid of (65,65,65) nodes and numerical LES expirements will be prepared.

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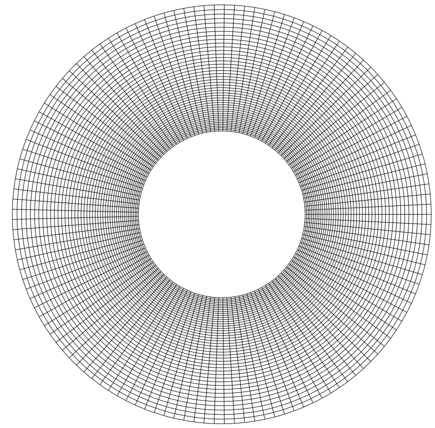

Figure 1: part of the spatial grid (129×45)

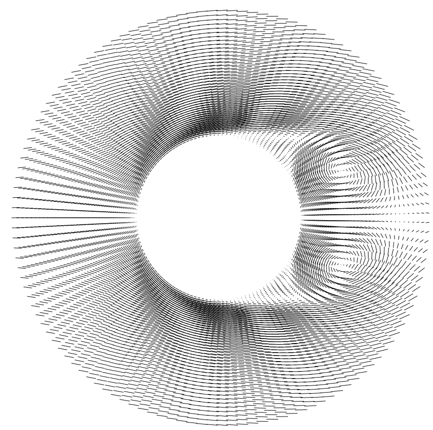

Figure 2: part of velocity field, $t = 1001 \cdot \tau$

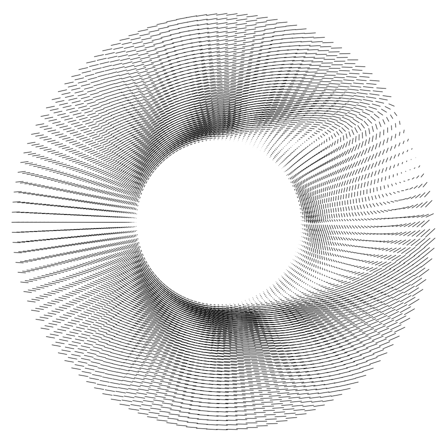

Figure 3: part of velocity field, $t = 15001 \cdot \tau$

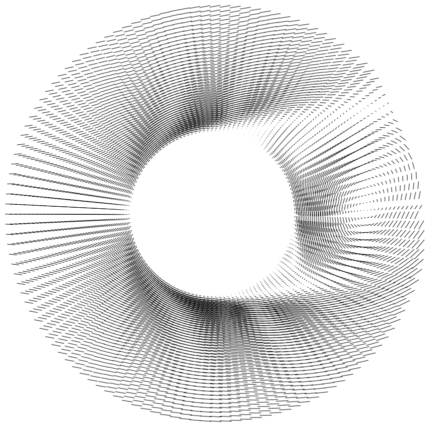

Figure 4: part of velocity field, $t = 17001 \cdot \tau$

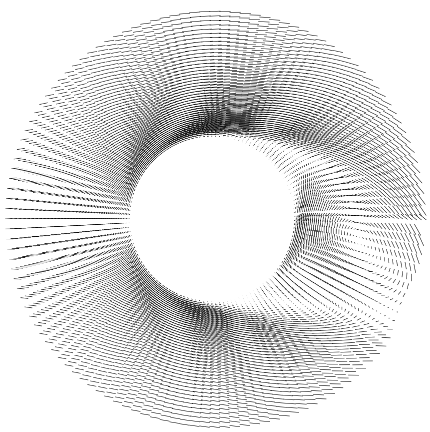

Figure 5: part of velocity field, $t = 18001 \cdot \tau$

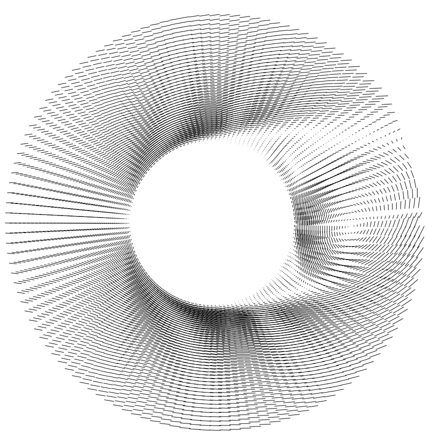

Figure 6: part of velocity field, $t = 19001 \cdot \tau$