Smoothing of data using Mumford-Shah type functionals

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Abstract. As results of 3d sensors using the ”time of flight” technology we get noise information of the shape of objects together with intensities. From the mathematical point of view these information are scalar fields d and I.
For filtering and smoothing of images we minimize a Mumford-Shah functional by solving the boundary value problem of the relevant Euler-Lagrange equations.
For the numerical solution of the boundary value problem we use a finite volume discretization and we solve the resulting nonlinear equation system on the finite volume grid by Newtons method and the steepest descent method.

Keywords: image processing, shape from shading, Mumford-Shah functional, solution of nonlinear pde's
PACS: 35J40, 35J35, 65N06, 65F10

INTRODUCTION

In the shape-from-shading problem, the intensity image of a surface, S, is given as an input. We assume that the surface has an uniform albedo and the reflectance of the surface is proportional to the cosine of the angle between the surface normal and the direction of the light source (Lambertian surface). As another assumption we have an illumination of the surface by a single distant light source. Shadows and self-reflections can be neglected and the direction of the light source is known. The problem consists of determining the shape of S by calculating the distance of the points on the surface relative to some reference plane normal to the viewing direction. Let I denote the observed image intensity defined over a domain Ω contained in the reference plane. Let S be defined by the function $f : \Omega \to \mathbb{R}$. Note that the definition of f depends on the choice of the reference plane. Let N denote the outward surface (unit) normal and let l denote the unit vector in the direction of the light source. Then the reflectance map is of the form

$$R = \gamma N \cdot l \,. \tag{1}$$

If the image intensity is normalized we have $\gamma = 1$.

FUSION OF RANGE AND SHAPE

Fusion without regularization

Firstly we consider fusion without regularization. $d : \Omega \to \mathbb{R}$ denotes the range data coming from a 3d sensor. For the fusion of range and shape we have to minimize the functional

$$F(f) = \int_\Omega [(f-d)^2 + \alpha^2 (R(f) - I)^2]\, d\Omega, \tag{2}$$

and ($\alpha \in \mathbb{R}$ is weighting parameter) in the case of Lambertian surface we have with $l = (l_1, l_2, l_3)$

$$\begin{aligned} R &= N \cdot l \\ &= \frac{1}{\sqrt{1+|\nabla f|^2}}(-f_x, -f_y, 1)\cdot(l_1,l_2,l_3) = \frac{1}{\sqrt{1+|\nabla f|^2}}(-f_x l_1 - f_y l_2 + l_3) \end{aligned} \tag{3}$$

(the subscripts $x, y, \ldots$ denote the partial derivatives). The functional (2) ist the simplest way of the combination of image irradiance equation

$$|\nabla f| = \sqrt{\frac{1}{I^2} - 1}$$

with the range data. As a convention for a compact notation we introduce $\nabla_{f_x,f_y}R$ as the gradient of R with respect to the variables $f_x,\ f_y$ and this results in

$$\nabla_{f_x,f_y}R = -\frac{1}{\sqrt{1+|\nabla f|^2}}\binom{l_1}{l_2} - \frac{N\cdot l}{\sqrt{1+|\nabla f|^2}}\nabla f\ . \tag{4}$$

The necessary condition for the minimum of F

$$\frac{d}{d\varepsilon}F(f+\varepsilon v)|_{\varepsilon=0} = 0$$

implies the Euler-Lagrange equation

$$-\nabla\cdot V = d - f \quad \text{in } \Omega \tag{5}$$

with

$$V = \alpha^2(R-I)\nabla_{f_x,f_y}R$$

and the boundary condition

$$\nu\cdot V = 0 \quad \text{on } \partial\Omega\ , \tag{6}$$

where ν denotes the outward normal to the boundary $\partial\Omega$.

Regularization and discontinuity preservation

If we add a regularization term of the form

$$\int\limits_{\Omega}[|\nabla f_x|^2 + |\nabla f_y|^2]\,d\Omega$$

on the functional (2) we have the effect that meaningful discontinuities of the shape will be smoothed. If the discontinuities should be preserved in [2] the functional

$$\hat{G} = \int\limits_{\Omega}[(f-d)^2 + \alpha^2(R(f)-I)^2]\,d\Omega + \lambda^4\int\limits_{\Omega\backslash\Gamma}[|\nabla f_x|^2 + |\nabla f_y|^2]\,d\Omega + \beta|\Gamma| \tag{7}$$

was proposed, where Γ stands for the locus of the surface creases and $|\Gamma|$ denotes the length of Γ. The functional $\hat{G}$ has to be minimized over f with a minimal $|\Gamma|$. λ and $\beta > 0$ are given parameters. The minimization of (7) is difficult to realize and following [3] the term $|\Gamma|$ is replaced by

$$\frac{1}{2}\int\limits_{\Omega}[\rho|\nabla s|^2 + \frac{s^2}{\rho}]\,d\Omega\ , \tag{8}$$

where the continuous scalar field s varies between zero and one and may be interpretd as the probability of the presence of Γ. The term (8) can also be interpreted as blurring of Γ with a blurring radius of ρ. The introduction of term (8) instead of $|\Gamma|$ requires the modification of the second integral of (7)

$$\int\limits_{\Omega\backslash\Gamma}[|\nabla f_x|^2 + |\nabla f_y|^2]\,d\Omega \qquad \Longleftrightarrow \qquad \int\limits_{\Omega}[|\nabla f_x|^2 + |\nabla f_y|^2](1-s)^2\,d\Omega\ .$$

Thus we have to minimze the functional

$$\begin{aligned} G(f,s) &= \int\limits_{\Omega}[(f-d)^2 + \alpha^2(R(f)-I)^2]\,d\Omega \\ &\quad + \lambda^4\int\limits_{\Omega}[|\nabla f_x|^2 + |\nabla f_y|^2](1-s)^2\,d\Omega + \frac{\beta}{2}\int\limits_{\Omega}[\rho|\nabla s|^2 + \frac{s^2}{\rho}]\,d\Omega\ . \end{aligned} \tag{9}$$

The evaluation of the necessary conditions for the minimum of G

$$\frac{d}{d\varepsilon}G(f+\varepsilon v,s)|_{\varepsilon=0}=0\,, \qquad \frac{d}{d\varepsilon}G(f,s+\varepsilon v)|_{\varepsilon=0}=0$$

implies after a longer technical computation the boundary value problem with the equations

$$-\nabla\cdot\nabla s+\frac{s}{\rho^2}=\frac{2\lambda^4}{\beta\rho}[|\nabla f_x|^2+|\nabla f_y|^2](1-s)-\nabla\cdot W=\frac{1}{\lambda^4}(d-f) \tag{10}$$

in Ω and the boundary conditions

$$\begin{aligned} \nu\cdot\nabla s &= 0 \qquad (11)\\ \nu\cdot W-[(1-s)^2 f_{\nu\tau}]_\tau &= 0 \qquad (12)\\ f_{\nu\nu} &= 0 \qquad (13)\end{aligned}$$

on $\partial\Omega$ and $f_{\nu\tau}=0$ at corners of Ω. The subscripts ν and τ stand for the derivatives in the outward normal and tangential direction. W is here the vector field

$$W=\frac{\alpha^2}{\lambda^4}(R-I)\nabla_{f_x,f_y}R-\begin{pmatrix}\nabla\cdot[(1-s)^2\nabla f_x]\\ \nabla\cdot[(1-s)^2\nabla f_y]\end{pmatrix}. \tag{14}$$

NUMERICAL MODEL

The numerical solution procedure for the systems (5) and (10), (10) consists in

1) spatial discretization by **FV/FD-method**, and
2) nonlinear system solver **steepest descent/Newton's method**,

where the relevant boundary conditions close the nonlinear equation systems. The discretization of the mixed derivatives of the elliptical main part of the equations was done following the recommendation of [1] to preserve the M-matrix property of the resulting stiffness matrix of the main part discretization.
To get appropriate approximations for an initial iterate of Newton's method we use, for example in the case with regularization, some iterations based on the equations

$$\begin{aligned}\frac{\partial f}{\partial t} &= c_f[\nabla\cdot W-\frac{1}{\lambda^4}(d-f)]\\ \frac{\partial s}{\partial t} &= c_s[\nabla\cdot\nabla s-\frac{s}{\rho^2}+\frac{2\lambda^4}{\beta\rho}[|\nabla f_x|^2+|\nabla f_y|^2](1-s)]\end{aligned}$$

c_f, c_s are constants controlling the rate of gradient descent and t is an artificial time parameter.

QUALITATIVE PROPERTIES OF THE PROBLEM AND THE DISCRETIZATION

It is possible to show the existence of a solution of the minimum problems supposing regular regions Ω but the uniqueness is not given. For the existence proof one argues with the the existence theory of the strong monotony and local Lipschitz continuity of the differential operator. The consistence of the FV/FD-approximation is a canonical result of the construction based on the Gauß-Ostrogradski formula.

RESULTS

In the following we show some results of the application of the fusion problem with regularization for constructed shapes. The fig. 1 show the ideal image with discontinuities(left) and the result of the solution of the boundary value problem (10), (10) (right) starting with a very noise image of the shape. The fig. 2 show the ideal image of a smooth shape (left) and the result of the solution of the boundary value problem (10), (10) starting with a very noise image of the shape. The handling of images generated by a time-of-flight sensor are now under consideration.

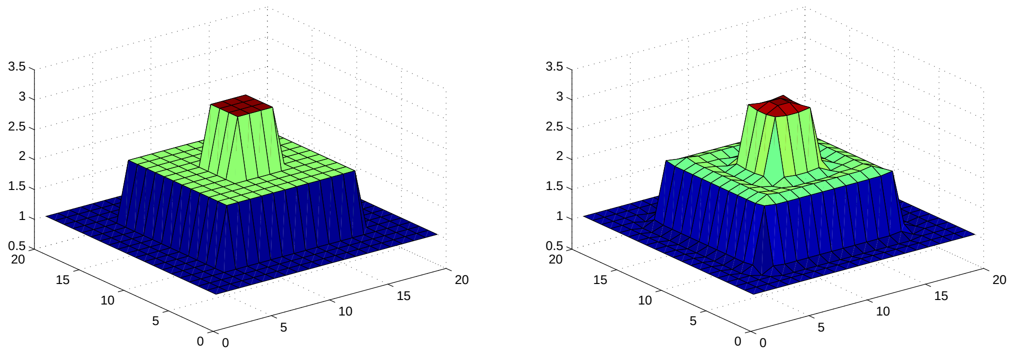

FIGURE 1. Shapes with discontinuities

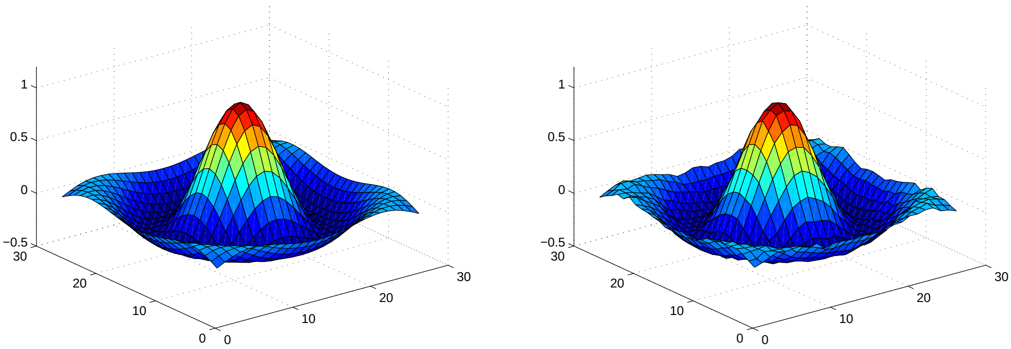

FIGURE 2. Smooth shapes

CONCLUSION

With the solution of the boundary value problem with the Euler-Lagrange equations of the Mumford-Shahtype functionals a promising method for smoothing and filtering of noisy sensor data is given. The steepest descent method was successful but the convergence rate is poor. Newton's method was successful if the initial iterate was of a ”good quality”. On fine grids both methods are computational not very efficient. These problems could be solved using multigrid techniques which require the problem solution on coarse grids only. For the implementation of these methods, for example on safety engineering units there must be further investigations for the improvement of the numerical solution methods up to real time applications. This will be the focus of our further investigations.

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