On the convergence and stability of the pressure-velocity-iteration of Chorin type with natural boundary conditions

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Abstract

Due to the implementation of numerical solution algorithms for the nonstationary Navier-Stokes equations of an incompressible fluid on massively parallel computers iterative methods are of special interest.

A red-black pressure velocity iteration which allows an efficient parallelization based on a domain decomposition [3] will be analyzed in this paper.

We prove the equivalence of the pressure-velocity-iteration (PUI) by Chorin/Hirt/Cook [1][2] with a SOR-iteration to solve a poisson equation for the pressure. We show this on a 2D rectangle with some special outflow boundary conditions and Dirichlet data for the velocity elsewhere. This equivalence allows us to prove the convergence of that iteration scheme. We also discuss the stablity of the occuring discrete Laplacian in discrete Sobolev spaces.

1 Introduction

In the sequel the 2D consideration is used only for reasons of simplicity. The results of the paper can be generalised for the 3D case.

On the rectangle $\Omega = [a,b] \times [c,d]$ the nonstationary, incompressible Navier-Stokes equations (NSE) are given:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{1}$$

On $\Gamma_D = \{a\} \times [c,d] \bigcup \{c,d\} \times [a,b]$ we have Dirichlet-data for $\mathbf{u}$ and on the outflow $\Gamma_{out} = \{b\} \times]c,d[$ we use a natural boundary condition of [5] or [6]:

$$\mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D \tag{2}$$

$$\nu \frac{\partial u}{\partial x} = p - \bar{p} \text{ on } \Gamma_{out} \tag{3}$$

$$\frac{\partial v}{\partial x} = 0 \text{ on } \Gamma_{out} \tag{4}$$

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If a solution $p \in H^1(\Omega)$ of (1) exists, it is unique, since

$$\int_c^d p(b,y)dy = \nu \int_c^d \frac{\partial u}{\partial x} dy + (d-c)\bar{p} = \tag{5}$$

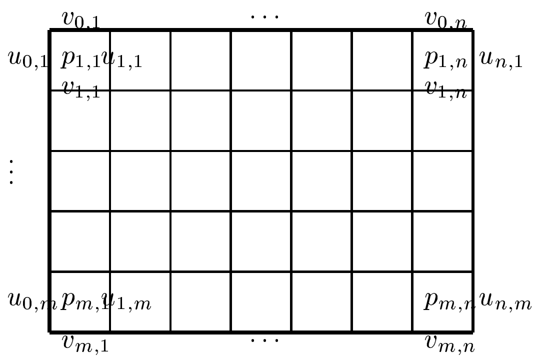
$$-\nu \int_c^d \frac{\partial v}{\partial y} dy + (d-c)\bar{p} = \nu(v(b,d) - v(b,c)) + (d-c)\bar{p}$$

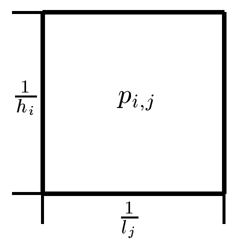
In [2] Hirt and Cook developed an iteration method modifying an idea of Chorin [1] to solve (1) using a time integration scheme of the following kind:

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\tau} + (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k = -\nabla p^{k+1} + \nu \Delta \mathbf{u}^k \tag{6}$$

$$\nabla \cdot \mathbf{u}^{k+1} = 0 \tag{7}$$

For the spatial discretization a staggered mesh with mesh sizes $\frac{1}{l_j}, \frac{1}{h_i}$ is used.





It is well known, that the size of the time-step τ is restricted by the following conditions

$$\tau\nu \cdot max\{h^2, l^2\} \leq 1 \tag{8}$$

$$\tau \cdot max\{u \cdot l\}, \tau \cdot max\{v \cdot h\} \leq 1 \tag{9}$$

using a scheme (6). Especially (8) seems to be the crucial restriction. To overcome (8) time integration schemes, which treat the diffusive term $\nu\Delta\mathbf{u}$ implicitly were developed. Recent investigations [13] have shown, that in turbulent flows $\tau max\{v \cdot h\}$ can be of the same order of magnitude as $\tau\nu \max\{l^2, h^2\}$.

2 The $p - \mathbf{u}$-iteration

To derive the discretization of (6) the following approximations are used:

$$div_{ij}\mathbf{u} = h_i(v_{ij} - v_{i-1j}) + l_j(u_{ij} - u_{ij-1}) \tag{10}$$

$$\frac{\partial p}{\partial x}_{ij} = L_j(p_{ij+1} - p_{ij}) \; ; \quad \frac{\partial p}{\partial y}_{ij} = H_i(p_{i+1j} - p_{ij}) \tag{11}$$

where $1/L_j := 1/2(1/l_{j+1} + 1/l_j)$ and $1/H_j := 1/2(1/h_{j+1} + 1/h_j)$. Let $\mathbf{u}^{k+1}, p^{k+1}$ be discretized by $\mathbf{u}, p$, then the linear system according to (6) reads

$$\begin{pmatrix} A & G \\ D & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ where } g \in Im(D) \tag{12}$$

A is a slight modification of I because of (3). This form is not the basis of PUI. For simplicity we denote the iteration sublevel for the computation of $\mathbf{u}^{k+1}, p^{k+1}$ by $\mathbf{u}^{,s}, p^{,s}$. Then PUI is given by

- start:

$$p^{,0} = \text{any initial guess, e.g. } p^k \tag{13}$$

$$\mathbf{u}^{,0} = \mathbf{u}^k - \tau(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k - \tau\nabla p^{,0} + \tau\nu\Delta\mathbf{u}^k \tag{14}$$

- do for each cell ${}_{ij}$

$$\delta p^{,s+1} =: p^{,s+1} - p^{,s} = -\omega \left(\frac{\partial div \mathbf{u}}{\partial p}_{ij} \right)^{-1} div_{ij} \mathbf{u}^{,s,s+1/2} \tag{15}$$

$$u_{ij}^{,s+**} = u_{ij}^{,s+*} - \delta p^{,s+1} \cdot L_j \; ; \; u_{ij-1}^{,s+**} = u_{ij-1}^{,s+*} + \delta p^{,s+1} \cdot L_{j-1}$$

$$v_{ij}^{,s+**} = v_{ij}^{,s+*} - \delta p^{,s+1} \cdot H_i \; ; \; v_{i-1j}^{,s+**} = v_{i-1j}^{,s+*} + \delta p^{,s+1} \cdot H_{i-1} \tag{16}$$

- until $max\{div_{ij}\mathbf{u}^{,s+*}\} \leq \varepsilon_{tol}$

During one iteration each velocity value is updated twice. If there are no further informations about the order of the unknowns, the sublevel ${}^{,s,s+1/2,s+1}$ is not clear. This problem is solved in section 3. ω is a relaxation parameter and $B_{ij} := -\frac{\partial div \mathbf{u}}{\partial p}_{ij}$, which is derived in the following:
Substitution of $u_{ij}, u_{i,j-1}, v_{ij}, v_{i1-j}$ in (10) by the discrete version of (6) leads to

$$div_{ij}\mathbf{u} = h_i(\frac{\partial p}{\partial y}_{ij} + f_{v,ij} - \frac{\partial p}{\partial y}_{i-1j} - f_{v,i-1j}) + l_j(\frac{\partial p}{\partial x}_{ij} + f_{u,ij} - \frac{\partial p}{\partial x}_{ij-1} - f_{u,ij-1}) \tag{17}$$

for inner cells. Substitution of $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial x}$ by (11), one obtains

$$\frac{\partial div_{ij}\mathbf{u}}{\partial p_{ij}} = h_i(-H_i + 0 - (-H_{i-1}) - 0) + l_j(L_j + 0 - (-L_{j-1}) - 0) \tag{18}$$

3 Convergence of PUI

Now the cells ${}_{ij}$ and p_{ij} are ordered in a *red-black* manner. We denote the p-values of the black cells by p_0 and those of the red cells by p_1 respectivly.

Lemma 3.1. *The red-black version of PUI restricted to the p-values is a red-black SOR iteration for the solution of*

$$F \cdot p := DA^{-1}G \cdot p = -g + DA^{-1}f =: \tilde{f} \tag{19}$$

Proof. P_{01} is a permutation matrix, which transforms the p-values from the original order to the red-black order.

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = P_{01} \cdot p \tag{20}$$

Thus the restriction operators $R_i p = p_i$ become $R_0 = (I_0 \mid 0)P_{01}$, $R_1 = (0 \mid I_1)P_{01}$. Since each red cell is surrounded only by black cells, PUI reads as

$$\begin{aligned} p_0^{,s+1} &= p_0^{,s} + R_0\omega E^{-1}(D\mathbf{u}^{,s} - g) \\ \mathbf{u}^{,s+1/2} &= A^{-1}(f - GP_{01}^T(p_0^{,s+1}, p_1^{,s})) \\ p_1^{,s+1} &= p_1^{,s} + R_1\omega E^{-1}(D\mathbf{u}^{,s+1/2} - g) \\ \mathbf{u}^{,s+1} &= A^{-1}(f - GP_{01}^T(p_0^{,s+1}, p_1^{,s+1})) \end{aligned} \tag{21}$$

E is diagonal-matrix, which consists of B_{ij}. Because of (13) we have $A\mathbf{u}^{,0} + Gp^{,0} = f$, thus $\mathbf{u}^{,s}$; $\mathbf{u}^{,s+1/2}$ can be substituted in (21)

$$\begin{aligned} p_0^{,s+1} &= p_0^{,s} + R_0\omega E^{-1}(-DA^{-1}Gp^{,s} - g + DA^{-1}f) \\ p_1^{,s+1} &= p_1^{,s} + R_1\omega E^{-1}(-DA^{-1}GP_{01}^T(p_0^{,s+1}, p_1^{,s}) - g + DA^{-1}f) \end{aligned} \tag{22}$$

If $E \overset{!}{=} Diag(F)$, then (22) and

$$F^{01} := \begin{pmatrix} D_0 & M_1 \\ M_0 & D_1 \end{pmatrix} := P_{01} \cdot F \cdot P_{01}^T \tag{23}$$

especially

$$E_{01} := \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix} := P_{01} \cdot Diag(F) \cdot P_{01}^T \tag{24}$$

would imply

$$\begin{aligned} p_0^{,s+1} &= p_0^{,s} + \omega(I_0 \mid 0)P_{01}P_{01}^T E_{01}^{-1}(-P_{01}FP_{01}^T P_{01}p^{,s} + P_{01}\tilde{f}) \\ &= p_0^{,s} + \omega(I_0 \mid 0)E_{01}^{-1}(-F^{01}(p_0^{,s}, p_1^{,s}) + P_{01}\tilde{f}) \\ &= p_0^{,s} + \omega D_0^{-1}(-D_0 p_0^{,s} - M_1 p_1^{,s} + (P_{01}\tilde{f})_0) \end{aligned} \tag{25}$$

$$\begin{aligned} p_1^{,s+1} &= p_1^{,s} + \omega(0 \mid I_1)P_{01}P_{01}^T E_{01}^{-1}(-P_{01}FP_{01}^T(p_0^{,s+1}, p_1^{,s}) + P_{01}\tilde{f}) \\ &= p_1^{,s} + \omega(0 \mid I_1)E_{01}^{-1}(-F^{01}(p_0^{,s+1}, p_1^{,s}) + P_{01}\tilde{f}) \\ &= p_1^{,s} + \omega D_1^{-1}(-M_1 p_0^{,s+1} - D_1 p_1^{,s} + (P_{01}\tilde{f})_1) \end{aligned} \tag{26}$$

- the red-black SOR iteration for (19). The proof for $E = Diag(DA^{-1}G)$ is presented at the end of the next section.

□

3.1 Evaluation of $F := DA^{-1}G$ and $DA^{-1}f$

For simplicity the Dirichlet-values $v_{m,.}$ are treated together with $u_{m,.}$ as unknowns. Then the definition of A, G, f ensures $v_{m,.} = v_D(x_j, d)$. The natural boundary condition is discretized by

$$u_{.,n} - u_{.,n-1} - 1/(l_n\nu)p_{.,n} = -1/(l_n\nu)\bar{p} \tag{27}$$

To keep the matrices A, G, D managable, their components are equiped whith the following indices: for example D_{ij}, u_{kl} denotes the coefficient of u_{kl} for computation of div_{ij} and $G_{u_{ij},kl}$ is the coefficient of p_{kl} in the momentum equation for u_{ij}. Thus:

$$\begin{aligned}
D_{ij,u_{kl}} &= \delta_{ik}(\delta_{jl} - \delta_{j-1l})l_j \\
D_{ij,v_{kl}} &= (\delta_{ik} - \delta_{i-1k})h_i\delta_{jl} \\
\\
A_{u_{ij},u_{kl}} &= \delta_{ik}(\delta_{jl} - \delta_{jn}\delta_{ln-1}) \\
A_{v_{ij},v_{kl}} &= \delta_{jl}\delta_{ik} \\
A_{u_{ij},v_{kl}} &= A_{v_{ij},u_{kl}} = 0 \\
\\
1/\tau G_{u_{ij},kl} &= \delta_{ik}((\delta_{j+1l} - \delta_{jl})(1-\delta_{jn})L_j + \alpha\delta_{jl}\delta_{jn}) \\
1/\tau G_{v_{ij},kl} &= (\delta_{i+1k} - \delta_{ik})(1-\delta_{im})H_i\delta_{jl}
\end{aligned} \tag{28}$$

B where $\alpha = -\frac{1}{\tau\nu l_n}$ from (27).
We start with the computation of A^{-1}.

$$\begin{aligned}
A^{-1}_{u_{ij},u_{kl}} &= \delta_{ik}(\delta_{jl} + \delta_{jn}\delta_{ln-1}) \\
A^{-1}_{v_{ij},v_{kl}} &= \delta_{jl}\delta_{ik} \\
A^{-1}_{u_{ij},v_{kl}} &= A_{v_{ij},u_{kl}} = 0
\end{aligned} \tag{29}$$

Proof:

$$\begin{aligned}
(A^{-1}A)_{u_{ij},u_{kl}} &= \sum_{r,s}\left[A^{-1}_{u_{ij},u_{rs}}A_{u_{rs},u_{kl}} + A^{-1}_{u_{ij},v_{rs}}A_{v_{rs},u_{kl}}\right] \\
&= \sum_{r,s}\left[\delta_{ir}(\delta_{js} + \delta_{jn}\delta_{sn-1})A_{u_{rs},u_{kl}} + 0 * 0\right] \\
&= \sum_{s}(\delta_{js} + \delta_{jn}\delta_{sn-1})A_{u_{is},u_{kl}} \\
&= A_{u_{ij},u_{kl}} + \delta_{jn}A_{u_{in-1},u_{kl}} \\
&= \delta_{ik}\left[\delta_{jl} - \delta_{jn}\delta_{ln-1} + \delta_{jn}(\delta_{n-1l} - \delta_{n-1n}\delta_{ln-1})\right] \\
&= \delta_{ik}\delta_{jl}
\end{aligned} \tag{30}$$

The computations for

$$(A^{-1}A)_{u_{ij},v_{kl}} = (A^{-1}A)_{v_{ij},u_{kl}} = 0 \quad \text{and} \quad (A^{-1}A)_{v_{ij},v_{kl}} = \delta_{ik}\delta_{jl}$$

are of the same kind.
Now $C = DA^{-1}$ is given by:

$$\begin{aligned}
(DA^{-1})_{ij,u_{kl}} &= \sum_{r,s}(D_{ij,u_{rs}}A^{-1}_{u_{rs},u_{kl}} + D_{ij,v_{rs}}A^{-1}_{v_{rs},u_{kl}}) \\
&= \sum_{r,s}(\delta_{ir}(\delta_{js} - \delta_{j-1s})l_j A^{-1}_{u_{rs},u_{kl}} + 0) \\
&= l_j \sum_{s}(\delta_{js} - \delta_{j-1s})A^{-1}_{u_{is},u_{kl}} \\
&= l_j(A^{-1}_{u_{ij},u_{kl}} - (1-\delta_{j1})A^{-1}_{u_{ij-1},u_{kl}}) \\
&= l_j\delta_{ik}(\delta_{jl} + \delta_{jn}\delta_{ln-1} - (1-\delta_{j1})\delta_{j-1l}) \qquad (31)
\end{aligned}$$

$$\begin{aligned}
(DA^{-1})_{ij,v_{kl}} &= \sum_{r,s}(D_{ij,u_{rs}}A^{-1}_{u_{rs},v_{kl}} + D_{ij,v_{rs}}A^{-1}_{v_{rs},v_{kl}}) \\
&= \sum_{r,s}(0 + \delta_{js}(\delta_{ir} - \delta_{i-1r})h_i\delta_{rk}\delta_{jl}) \\
&= h_i(\delta_{ik} - \delta_{i-1k})\delta_{jl} \qquad (32)
\end{aligned}$$

Thus F becomes:

$$\frac{1}{\tau}F_{ij,kl} = \frac{1}{\tau}\sum_{r,s}\left[C_{ij,u_{rs}}G_{u_{rs},kl} + C_{ij,v_{rs}}G_{v_{rs},kl}\right]$$

$$= \sum_{r,s}\left[l_j\delta_{ir}(\delta_{js} - \delta_{j-1s} + \delta_{jn}\delta_{sn-1})G_{u_{rs},kl} + h_i(\delta_{ir} - \delta_{i-1r})\delta_{js}G_{v_{rs},kl}\right]$$

$$= l_j\sum_{s}(\delta_{js} - \delta_{j-1s} + \delta_{jn}\delta_{sn-1})G_{u_{is},kl} + h_i\sum_{r}(\delta_{ir} - \delta_{i-1r})G_{v_{rj},kl}$$

$$= l_j(G_{u_{ij},kl} - (1-\delta_{j1}-\delta_{jn})G_{u_{ij-1},kl}) + h_i(G_{v_{ij},kl} - (1-\delta_{i1})G_{v_{i-1j},kl}) \qquad (33)$$

For $1 \le i,k \le m$ and $1 \le j,l \le n$ we have $\frac{1}{\tau}F_{ij,kl} =$

$$\begin{aligned}
&-l_j(L_j + (1-\delta_{j1})L_{j-1}) - h_i((1-\delta_{im})H_i + (1-\delta_{i1})H_{i-1}) \\
&\qquad \text{for } i = k; j = l < n \qquad (34) \\
&l_n\alpha - h_i((1-\delta_{im})H_i + (1-\delta_{i1})H_{i-1}) \text{ for } i = k; j = l = n \qquad (35) \\
&l_jL_j \text{ for } i = k; l = j+1 \\
&(1-\delta_{jn}-\delta_{j1})l_jL_{j-1} \text{ for } i = k; l = j-1 \\
&h_iH_i \text{ for } k = i+1; l = j \\
&h_iH_{i-1} \text{ for } k = i-1; l = j \\
&0 \text{ elsewhere.}
\end{aligned}$$

To get an idea of F , we consider the lexicographic order of the p-values $p = (p_{1,1}, .., p_{1,n}; ....; p_{m,1}, .., p_{m,n})$

$$F^{lex} = \begin{pmatrix} B_1 & h_1 H_1 I & & & \\ h_2 H_1 I & B_2 & h_2 H_2 I & & \\ & \ddots & \ddots & \ddots & \\ & & h_{m-1} H_{m-2} I & B_{m-1} & h_{m-1} H_{m-1} I \\ & & & h_m H_{m-1} I & B_m \end{pmatrix} \tag{36}$$

with

$$B_i = \begin{pmatrix} B_{i,11} & l_1 L_1 & & & \\ l_2 L_1 & B_{i,22} & l_2 L_2 & & \\ & \ddots & \ddots & \ddots & \\ & & l_n L_{n-1} & B_{i,n-1,n-1} & l_{n-1} L_{n-1} \\ & & & 0 & B_{i,nn} \end{pmatrix} \tag{37}$$

Remark 3.2. The asserted identity of E and $Diag(F^{lex})$ is clear. For $j < n$ one has to substitute the momentum equation (4-#boundaries of the cell with Dirichlet-data)-times , which leads to (34). For $j = n$ (27) and (2-#boundaries of the cell with Dirichlet-data)-times the momentum equation is used, which coincides with (35).

3.2 Analysis of the red-black SOR Iteration

To show the convergence we need some auxiliary results, which are proved in the sequel. We denote by $J(A)$, $H(A, \omega)$ the iteration matrices of the Jacobi and SOR iteration as far as they exist for A.

Lemma 3.3. *Let A be, such that $J(A)$ exists, P be a permutation matrix for π ($Pe_i = e_{\pi(i)}$) and E be a nonsingular diagonal matrix. Then $J(PAP^T) = PJ(A)P^T$ and $J(EAE^{-1}) = EJ(A)E^{-1}$. Especially $J(PAP^T)$, $J(EAE^{-1})$ exist and have the same spectrum as $J(A)$.*

Proof. Let $A = L + D + R$, $PAP^T = \tilde{L} + \tilde{D} + \tilde{R}$ be the classical decompositions. Then $\tilde{D} = PDP^T$, since $PDP^T = P(d_1 e_1, .., d_n e_n)P^T =$

$$(d_1 e_{\pi(1)}, .., d_n e_{\pi(n)})P^T = \begin{pmatrix} d_{\pi^{-1}(1)} e^T_{\pi^{-1}(1)} \\ \vdots \\ d_{\pi^{-1}(n)} e^T_{\pi^{-1}(n)} \end{pmatrix} P^T = (d_{\pi^{-1}(1)} e_1, .., d_{\pi^{-1}(n)} e_n) \tag{38}$$

The diagonal D is mapped to a diagonal, which has to be the diagonal of PAP^T. Thus

$$\begin{aligned} J(PAP^T) = -(PDP^T)^{-1}(PAP^T - PDP^T) = \\ -PD^{-1}P^T P(A - D)P^T = PJ(A)P^T \end{aligned} \tag{39}$$

The proof for $\tilde{D} = EDE^{-1} = D$ is of the same type.

□

Another usefull statement is

Lemma 3.4. *Let A be a symmetric,real matrix with $A_{ii} < 0$. Then the spectrum of $J(A)$ is real.*

Proof. For $A = L + D + R$ we have $J(A) = -D^{-1}(L+R) \stackrel{!}{=} (-D)^{-1}(L+R)$. Since $-D \geq 0$, $W := (-D)^{1/2}(-D)^{-1}(L+R)(-D)^{-1/2} =$ $(-D)^{-1/2}(L+R)(-D)^{-1/2}$ is a real, symmetric matrix. □

Now the main theorem of this section is given:

Theorem 3.5. *For F^{lex} we have*

1. *F^{lex} is reducible*
2. *F^{lex} has only real eigenvalues*
3. *$J(F^{lex})$ has only real eigenvalues*
4. *$\rho(J(F^{lex})) < 1$*

Proof. To start the proof we transform F^{lex} by a permutation matrix to F^{α}. Let $F^{lex} = (F^1, ..., F^{m*n})$, then P is that permutation matrix, that 'moves' the last columns of each block column to the right. Multiplication of P^T from the left moves the last row of each block row down.

$F^{\alpha} := P^T F^{lex} P =$

$$\begin{pmatrix} \tilde{B}_1 & h_1 H_1 I & & & & M_1 \\ h_2 H_1 I & \tilde{B}_2 & h_2 H_2 I & & & M_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & & \tilde{B}_{m-1} & h_{m-1} H_{m-1} I & M_{m-1} \\ & & & h_m H_{m-1} I & \tilde{B}_m & M_m \\ 0 & & \dots & & 0 & C \end{pmatrix} \tag{40}$$

and

$$C = \begin{pmatrix} C_{11} & h_1 H_1 & & \\ h_2 H_1 & C_{22} & \ddots & \\ & \ddots & \ddots & h_{m-1} H_{m-1} \\ & & h_m H_{m-1} & C_{mm} \end{pmatrix} \tag{41}$$

where $C_{ii} = l_n\alpha - (1-\delta_{i1})h_iH_{i-1} - (1-\delta_{im})h_iH_i$ and $\tilde{B}_i \in M(n-1)$; $M_i \in M(n-1,m)$

$$\tilde{B}_i = \begin{pmatrix} \tilde{B}_{i,11} & l_1L_1 & & \\ l_2L_1 & \tilde{B}_{i,22} & \ddots & \\ & \ddots & \ddots & l_{n-2}L_{n-2} \\ & & l_{n-1}L_{n-2} & \tilde{B}_{i,n-1,n-1} \end{pmatrix} ; \; M_i = l_{n-1}L_{n-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ e_i^T \end{pmatrix} \tag{42}$$

where $\tilde{B}_{i,jj} = -l_j(L_j + (1-\delta_{j1})L_{j-1}) - h_i((1-\delta_{im})H_i + (1-\delta_{i1})H_{i-1})$ see (34)
For simplicity we set

$$F^\alpha =: \begin{pmatrix} \tilde{F} & M \\ 0 & C \end{pmatrix} \tag{43}$$

A later remark dicusses the reducibility. Since

$$det(F^\alpha - \lambda I) = det(\tilde{F} - \lambda I)det(C - \lambda I) \tag{44}$$

and

$$J(F^\alpha) = \begin{pmatrix} J(\tilde{F}) & M^* \\ 0 & J(C) \end{pmatrix}, \tag{45}$$

the eigenvalues of F^α $\{,J(F^\alpha)\}$ and thus F^{lex} $\{,J(F^{lex})\}$ are the eigenvalues of $\tilde{F}$ $\{,J(\tilde{F})\ \}$ or C $\{,J(C)\}$.
Next we show that, $\tilde{F}$, C can be transformed to symmetric matrices via positive diagonal matrices. The lemmas 3.3 and 3.4 then show that the eigenvalues have to be real. Since the matrices $\tilde{B}_i$ differ only on the diagonal entries, there is one diagonal matrix $D = diag(d_1, .., d_{n-1})$, which transforms all $\tilde{B}_i$ to symmetric matrices.

$$D\tilde{B}_iD^{-1} = \begin{pmatrix} * & \frac{d_1l_1L_1}{d_2} & & \\ \frac{d_2l_2L_1}{d_1} & * & \frac{d_2l_2L_2}{d_3} & \\ \ddots & & \ddots & \\ & & & \frac{d_{n-2}l_{n-2}L_{n-2}}{d_{n-1}} \\ & & \frac{d_{n-1}l_{n-1}L_{n-2}}{d_{n-2}} & * \end{pmatrix} \tag{46}$$

We set $d_1 = 1$, d_i ; $i > 1$ are defined recursivly. Since $D\tilde{B}_iD^{-1}$ shall be symmetric, for $n-1 \geq j > 1$

$$d_jl_jL_{j-1}/d_{j-1} = d_{j-1}l_{j-1}L_{j-1}/d_j \Leftrightarrow d_j^2 = d_{j-1}^2\frac{l_{j-1}}{l_j} \Leftrightarrow d_j^2 = \frac{l_1}{l_j} \tag{47}$$

$D \geq 0$ transforms all $\tilde{B}_i$ to symmetric matrices. C is of the same form as $\tilde{B}_i$, thus it can be symmetrized in the same way.

Now $\tilde{F}$ is symmetrized by $\bar{D} = blockdiag(D_1, ..., D_m)$ where $D_i = \bar{d}_i D$, since $\bar{D}\tilde{F}\bar{D}^{-1} =$

$$\begin{pmatrix} D_1\tilde{B}_1D_1^{-1} & D_1h_1H_1D_2^{-1} & & & \\ D_2h_2H_1D_1^{-1} & D_2\tilde{B}_2D_2^{-1} & D_2h_2H_2D_3^{-1} & & \\ \ddots & \ddots & \ddots & & \\ & & D_{m-1}\tilde{B}_{m-1}D_{m-1}^{-1} & D_{m-1}h_{m-1}H_{m-1}D_m^{-1} \\ & & D_mh_mH_{m-1} & D_m\tilde{B}_mD_m^{-1} \end{pmatrix} \tag{48}$$

We again set $\bar{d}_1 = 1$ and obtain from

$$\bar{d}_ih_iH_{i-1}/\bar{d}_{i-1} = \bar{d}_{i-1}h_{i-1}H_{i-1}/\bar{d}_i \Leftrightarrow \bar{d}_i^2 = \bar{d}_{i-1}^2\frac{h_{i-1}}{h_i} \Leftrightarrow \bar{d}_i^2 = \frac{h_1}{h_i} \tag{49}$$

the matrix $\bar{D}$.
The forth assertion is easy to prove:
$\tilde{F}$, C have the same sparsity pattern as the matrices of the discrete 1-/2-dimensional Laplacian with Dirichlet data, which are well known to be unreducible. Additionally they posses in all rows a weak diagonal dominance and in some rows even a strong one. Thus

$$\rho(J(\tilde{F})), \rho(J(C)) < 1 \Rightarrow \rho(J(F^\alpha)) = \rho(J(F^{lex})) < 1 \tag{50}$$

□

Theorem 3.6. *The red-black-SOR iteration for (19) converges for relaxation parameters* $0 < \omega < 2$.

Proof. The red-black-SOR iteration for (19) is an ordinary lexicographic SOR iteration for the matrix F^{01} from lemma 3.1. Due to its special form, F^{01} is ordered consistently. Lemma 3.3 carries statements of 3.5 concerning the eigenvalues of $J(F^{lex})$ to the eigenavalues of $J(F^{01})$. The famous Young theorem ([8], Satz 5.6.5) then proves all assertions. □

Remark 3.7. F is the discrete operator of the 5-point stencil for the Laplacian with Neumann conditions on Γ_D. For Γ_{out} the boundary condition reads

$$-\nu\tau\frac{\partial^2 p}{\partial x^2} - p = r.h.s \tag{51}$$

Using the Poisson equation for the pressure we can substitute (51) by

$$+\nu\tau\frac{\partial^2 p}{\partial y^2} - p = r.h.s \text{ on } \Gamma_{out} \tag{52}$$

This is an ODE of Helmholtz type on Γ_{out} with homogeneous boundary conditions for $\frac{\partial p}{\partial y}$ in $\partial\Gamma_{out}$. This ODE has a unique solution under some assumptions for the right hand side.

3.3 Convergence for $\frac{\partial \mathbf{u}}{\partial x} = 0$ on Γ_{out}

In this section we consider another often used boundary condition on Γ_{out}

$$\frac{\partial \mathbf{u}}{\partial x} = 0 \text{ on } \Gamma_{out} \tag{53}$$

Using all notations of the previous sections we have $\alpha = 0$.

Lemma 3.8. *For $\alpha = 0$: $Ker(F^{lex}) = span\{\mathbf{1}\}$*

Proof. $F^{lex} \cdot \mathbf{1} = 0$ is easy to see. With the representation (43) of F^{α} follows

$$\begin{aligned} \begin{pmatrix} \tilde{F} & M \\ 0 & C \end{pmatrix} \cdot \begin{pmatrix} x_F \\ x_C \end{pmatrix} &= 0 \\ \Rightarrow C \cdot x_C = 0 \Rightarrow x_{C1} &= x_{C2} \Rightarrow x_{C2} = x_{C3} ... \Rightarrow x_{Cm-1} = x_{Cm} \end{aligned} \tag{54}$$

thus $dimKer(C) = 1$. $\tilde{F}$ is nonsingular, which proves this lemma.

□

A conclusion of 3.8 is

Lemma 3.9. *1. $\rho(J(F^{01})) = 1$*

2. $J(F^{01})$ has only real eigenvalues

3. 1 and -1 are eigenvalues of $J(F^{01})$, the eigenspace of 1 is spanned by $\mathbf{1}$ and the eigenspace of -1 by $e_{01} := \begin{pmatrix} \mathbf{1}_0 \\ -\mathbf{1}_1 \end{pmatrix}$

Proof. Weak diagonal dominance of F^{01} and the Gerschgorin theorem gives $\rho(J(F^{01})) \leq 1$. Theorem 3.5 shows, that all eigenvalues are real.

$$F = L + D + R \ ; \ F \cdot x = \mathbf{0} \Leftrightarrow -D^{-1}(L + R) \cdot x = x \tag{55}$$

shows, that 1 is an eigenvalue and that the eigenspace is spanned by $\mathbf{1}$. F^{01} is ordered consistently which shows that -1 is an eigenvalue, too ([10], Satz 8.3.12).

$$-J(F^{01}) = -S^{-1} \begin{pmatrix} 0 & D_0^{-1} L_1 \\ D_1^{-1} L_0 & 0 \end{pmatrix} S \text{ where } S = \begin{pmatrix} I_0 & 0 \\ 0 & -I_1 \end{pmatrix} \tag{56}$$

Thus

$$Jx = -x \Leftrightarrow -Jx = x = S^{-1} JSx \Leftrightarrow JSx = Sx \Leftrightarrow Sx = c \cdot \mathbf{1} \tag{57}$$

$e_{01} := \begin{pmatrix} \mathbf{1}_0 \\ -\mathbf{1}_1 \end{pmatrix}$ spans the eigenspace of $\mu = -1$. □

Now we are ready to show that the SOR-iteration converges in the sense of damping all errors except the error that belongs to the global pressure level. That means

Lemma 3.10. *The iteration converges to a solution of (19).*

Proof. The consistent order of F^{01} implies that there is the following relation between the eigenvalues μ of $J(F^{01})$ and the eigenvalues λ of $H(F^{01}, \omega)$

$$\lambda + \omega - 1 = \pm\sqrt{\lambda}\omega\mu \tag{58}$$

From 3.8 and 3.9 follows that for all eigenvalues $\mu \neq \pm 1$, $|\mu| < 1$ yields. Using (58) one obtains for the corresponding eigenvalues of H $\lambda < 1$ for $0 < \omega < 2$. The corresponding eigenvalues of $\mu = \pm 1$ are

$$\lambda_1 = 1 \ ; \ \lambda_2 = (1-\omega)^2 < 1 \text{ for } 0 < \omega < 2 \tag{59}$$

A direct computation proves that only the pressure level remains undefined by the iteration

$$\begin{aligned} (D+\omega L)^{-1}((1-\omega)D - \omega R)x &= x \\ \Leftrightarrow ((1-\omega)D - \omega R)x &= (D+\omega L)x \\ \Leftrightarrow -\omega(L+D+R)x &= 0 \end{aligned} \tag{60}$$

□

4 Stability of the spatial discretization

This section deals with the stability of the discrete Laplacian F^α of the previous section. In order to obtain statements concerning the regularity in the discrete sobolev space H_h^1 (see [9], Chapter 9.2) we show the boundedness of $(F^\alpha)^{-1}$; $(F^\alpha)^{-T}$. This implies the stability in the discrete space L^2 $(=l^2)$.

Lemma 4.1. *If the grid stretching is bounded ($\frac{1}{q} \leq \frac{1}{l_{j+1}} / \frac{1}{l_j} \leq q$), then*

$$\|(F^\alpha)^{-1}\|_\infty \leq max\{\frac{q}{q+1}, \tau\nu\}. \tag{61}$$

Proof. (43) implies

$$(F^\alpha)^{-1} = \begin{pmatrix} \tilde{F}^{-1} & \tilde{F}^{-1}MC^{-1} \\ 0 & C^{-1} \end{pmatrix} \tag{62}$$

To prove the boundedness of $(F^{lex})^{-1}$ we show the boundedness of $\tilde{F}^{-1}$, $\tilde{F}^{-1}MC^{-1}$ and C^{-1}.
$-\tilde{F}, -C$ are M-matrices, since
$\tilde{F} - Diag(\tilde{F}), C - Diag(C), J(\tilde{F}), J(C) > 0$ and $\rho(J(-\tilde{F})), \rho(-J(C)) < 1$. Obviously

$$-C \cdot \mathbf{1} = -l_n\alpha\mathbf{1} = \frac{1}{\tau\nu}\mathbf{1}. \tag{63}$$

This proves $\|C^{-1}\|_\infty \leq \|\tau\nu\mathbf{1}\|_\infty = \tau\nu$.
The same type of proof is used for $\tilde{F}^{-1}$. Without loss of generality we assume $[a,b]=[0,1]$, then for

$$\mathbf{w}_{ij} = w(x_j) \text{ where } w(x) = (1-x^2)/2 \text{ and } x_j = \sum_{s=1}^{j-1} \frac{1}{l_s} + \frac{1}{2l_j} \tag{64}$$

we obtain $(-\tilde{F}\cdot\mathbf{w})_j =$

$$\begin{aligned}
= & \quad -\delta_{j>1} l_j L_{j-1}(1-x_{j-1}^2)/2 + (\delta_{j>1} l_j L_{j-1} + l_j L_j)(1-x_j^2)/2 \\
& \quad -\delta_{j<n-1} l_j L_j (1-x_{j+1}^2)/2 \\
= & \quad -\delta_{j>1} l_j L_{j-1}(1-x_{j-1}^2 - 1 + x_j^2)/2 - l_j L_j((1-x_{j+1}^2)\delta_{j<n-1} - 1 + x_j^2)/2 \\
= & \quad -\delta_{j>1} l_j L_{j-1}(x_j - x_{j-1})(x_{j-1}+x_j)/2 - l_j L_j((1-x_{j+1}^2)\delta_{j<n-1} - 1 + x_j^2)/2 \\
= & \quad -l_j/2(\delta_{j>1}(x_{j-1}+x_j) - l_j L_j((1-x_{j+1}^2)\delta_{j<n-1} - 1 + x_j^2)/2 \\
= & \quad l_j/2 \cdot \begin{cases} \frac{1}{2l_1} + \frac{1}{l_1} + \frac{1}{l_2} & \text{for } j=1 \\ \frac{1}{2l_{j+1}} + \frac{1}{l_j} + \frac{1}{2l_{j-1}} & \text{for } 1<j<n-1 \\ \geq 1 - x_{n-2} = \frac{1}{l_n} + \frac{1}{l_{n-1}} + \frac{1}{2l_{n-2}} & \text{for } j = n-1 \end{cases}
\end{aligned}$$

using the declaration $\delta_{j>1} = 1 - \delta_{j1}$. Thus

$$\|\tilde{F}^{-1}\|_\infty \leq \frac{2q}{q+1}\|\mathbf{w}\|_\infty \leq \frac{q}{q+1}. \tag{65}$$

Using (40,42) a direct computation yields $(-\tilde{F})\cdot\mathbf{1} = M\cdot\mathbf{1}$. The M-matrix property of $-\tilde{F}, -C$ shows $\tilde{F}^{-1}MC^{-1} \geq 0$. Now we are finished

$$\begin{aligned}
\|\tilde{F}^{-1}MC^{-1}\|_\infty = \|(-\tilde{F})^{-1}M(-C)^{-1}\cdot\mathbf{1}\|_\infty = \\
\|(-\tilde{F})^{-1}M\tau\nu\mathbf{1}\|_\infty = \tau\nu\|\mathbf{1}\|_\infty = \tau\nu
\end{aligned} \tag{66}$$

□

Lemma 4.2. *Using all notations of the previous lemma and the following notations* $p_h := max_{i,r}\frac{h_r}{h_i}$; $p_l := max_{j,s}\frac{l_s}{l_j}$, *then*

$$\|(F^\alpha)^{-T}\|_\infty \leq max\{p_h p_l \frac{q}{q+1}, 2max_j\{l_j\}/l_1\tau\nu p_h^2\}. \tag{67}$$

Proof. We use the matrices $\bar{D}$; D_h from lemma (3.5) which symmetrized $\tilde{F}$; C, with $\bar{D}_{ij,ij}^2 = \frac{h_1 l_1}{h_i l_j}$, $D_h^2 i,i = \frac{h_1}{h_i}$ and the following notations are used:
$\tilde{\mathbf{w}} := \frac{2q}{q+1}\mathbf{w}$; $pi := min_{i,j}\frac{h_1 l_1}{h_i l_j}$; $pa := max_{i,j}\frac{h_1 l_1}{h_i l_j}$.

$$\bar{D}\tilde{F}\bar{D}^{-1} = (\bar{D}\tilde{F}\bar{D}^{-1})^T \Leftrightarrow \bar{D}^2\tilde{F} = \tilde{F}^T\bar{D}^2 \Rightarrow (-\tilde{F})^T\bar{D}^2\tilde{\mathbf{w}} = \bar{D}^2(-\tilde{F})\tilde{\mathbf{w}} \geq \bar{D}^2\mathbf{1}$$

Thus

$$(-\tilde{F})\bar{D}^2\tilde{\mathbf{w}} \geq pi\mathbf{1} \Rightarrow \|\tilde{F}^{-T}\|_\infty \leq \frac{pa}{pi}\|\tilde{w}\|_\infty = p_h p_l \frac{q}{q+1}$$

The same steps for C^T with (63) lead to $\|C^{-T}\|_\infty \leq \tau\nu p_h$
The last estimate comes from

$$(-C)^{-T}M^T(-\tilde{F})^{-T}\mathbf{1} \leq (-C)^{-T}M^T\bar{D}^2\frac{1}{pi}\tilde{\mathbf{w}} =$$
$$\frac{w(x_{n-1})L_{n-1}}{pi}(-C)^{-T}\sum_{i=1}^{m}\frac{h_1}{h_i}e_i \leq \frac{w(x_{n-1})L_{n-1}}{pi}max_i\frac{h_1}{h_i}\tau\nu p_h\mathbf{1} \leq$$
$$\frac{w(x_{n-1})L_{n-1}}{pi}max_i\frac{h_1}{h_i}\tau\nu p_h\mathbf{1} \leq \frac{2}{pi}max_i\frac{h_1}{h_i}\tau\nu p_h\mathbf{1} = 2\max_j\frac{l_j}{l_1}\tau\nu p_h^2\mathbf{1} \tag{68}$$

$\square$

Remark 4.3. If condition (8) is satisfied, then one has for the L_h^2 stability

$$|(F^\alpha)^{-1}|_{0\leftarrow 0} \leq \sqrt{p_h p_l}\frac{q}{q+1} \tag{69}$$

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