A local implicit pressure velocity iteration method for incompressible flow problems

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Abstract: The usage of the semi-implicit pressure velocity iteration method for the approximate solution of the non stationary NSE proposed by Chorin [1] and, for example, Harlow and Amsden [2] ,leads to stability restrictions of the form $2 \geq C + 2d$ with

$$\text{(i)} \quad C = \frac{\tau U}{h} \quad \text{and} \quad \text{(ii)} \quad d = \frac{\tau \nu}{h^2} \ ,$$

where τ denotes a time step, $\nu = \frac{1}{Re}$ stands for the viscosity of the fluid, Re denotes the Reynolds number, h represents a spatial discretization increment (Δx, Δy ...) and U is a characteristic velocity. C is called the Courant number. These restrictions are due to the explicit treatment of the convective (i) and the diffusive terms (ii), respectively. For problems with very small Reynolds numbers and/or fine (local) meshsizes h the term (ii) strongly restricts τ .

We present an improved p-v iteration method that corresponds to an implicit treatment of the diffusive terms, and thus overcomes restrictions caused by term (ii). Exploiting the features of Harlow's and Welch's staggered grids and solving the cell-local NSE leads to an improved pressure velocity iteration method without the nessecity of the solution of large equation systems.

Keywords: Navier-Stokes equations, Stokes equations, staggered grid finite volume method, pressure velocity iteration method

The governing equations and the discretization

The equations of momentum and continuity for an incompressible fluid are of the form

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, \tag{1}$$

$$div(\mathbf{u}) = 0, \tag{2}$$

together with suitable initial and boundary conditions for the velocity [5].

The classical pressure velocity iteration method for the solution of (1),(2) corresponds to the semi-implicit time-integration scheme

$$\frac{\mathbf{u}_m - \mathbf{u}_{m-1}}{\tau} - \nu\Delta\mathbf{u}_{m-1} + (\mathbf{u}_{m-1}\cdot\nabla)\mathbf{u}_{m-1} + \nabla p_m = 0, \tag{3}$$

$$div(\mathbf{u}_m) = 0. \tag{4}$$

The spatial discretization of the equations (3),(4) is done by a finite volume method on staggered grids for the velocity components and the pressure. The discretized equation system (3),(4) is solved with the pressure velocity iteration method described below.

The derivative of the discretized $div(\mathbf{u})$ with respect to the pressure p is approximated in a coarse way by a diagonal matrix $B := \frac{\partial div(\mathbf{u})}{\partial p}$ depending on discretization parameters Δx, Δy, ...,τ , which is derived from the momentum equation (3). The algorithm is then:

Until convergence, compute at each cell $(_{ijk})$:

a) $\delta p^s_{ijk} = -\omega B^{-1}_{ijk,ijk} div(\mathbf{u^s})_{ijk}$

$\quad p^{s+1}_{ijk} = p^s_{ijk} + \delta p^s_{ijk}$

b) obtain $(u, v, w)^{s+1}_{ijk}$, u^{s+1}_{i-1jk}, v^{s+1}_{ij-1k}, w^{s+1}_{ijk-1} from (3) using p^{s+1}_{ijk}, $\mathbf{u_{m-1}}$

The superscript s denotes an iteration level. Details are noted in [1] or [4].

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The viscous implicit pressure velocity iteration

A similar iteration scheme can be applied to the following Stokes-like time integration scheme

$$\frac{\mathbf{u}_m - \mathbf{u}_{m-1}}{\tau} - \nu\Delta\mathbf{u}_m + (\mathbf{u}_{m-1}\cdot\nabla)\mathbf{u}_{m-1} + \nabla p_m = 0, \tag{5}$$

$$div(\mathbf{u}_m) = 0. \tag{6}$$

The first idea was to replace step b) from the algorithm above by the coupled solution for the two velocity values for each coordinate direction. This is done by solving three 2×2 systems in the 3d case of the form

$$\begin{pmatrix} 1+\frac{6\tau\nu}{h^2} & -\frac{\tau\nu}{h^2} \\ -\frac{\tau\nu}{h^2} & 1+\frac{6\tau\nu}{h^2} \end{pmatrix} \cdot \begin{pmatrix} u^{s+1}_{ijk} \\ u^{s+1}_{i-1jk} \end{pmatrix} = \begin{pmatrix} r.h.s. \\ r.h.s. \end{pmatrix}.$$

The coupled solution for the two velocity values for each coordinate direction improves the convergence.

Currently we investigate a pressure velocity iteration method, where the velocity components and the pressure of the current cell are determined by the exact solution of momentum and continuity equation for this cell, to improve convergence again. This leads to local 7×7 systems of eqs. which, using the classical bordering algorithm, can be reduced to the solution of the same three cell-local 2×2 systems.

The developed modified viscous pressure velocity iteration methods are used to solve for example the non stationary flow problem of a crystal melt (very small Reynolds numbers) and for flow problems with moderate or higher Reynolds numbers.

The gain of using larger time steps dominates the growth of the effort per iteration trough the more sophisticated iteration process. Detailled comparisons of the efficience of the several developed and implemented methods will be presented.

There are also investigations of the implicit treatment of the convective terms of the momentum equations, which will be presented too.

Acknowledgment. This work is supported by the **Universitärer Forschungsschwerpunkt 8 - Kontrolle turbulenter Scherströmungen** at the TU Berlin and the authors are deeply indebted to H.-J. Kaltenbach and H.-G. Jeggle from the Fachbereich Mathematik of the TU Berlin for usefull discussions.

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