Convergence analysis of pressure-velocity iteration methods for solving unsteady Navier-Stokes equations

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Abstract. Due to the implementation of numerical solution algorithms for the nonstationary Navier-Stokes equations of an incompressible fluid on massively parallel computers iterative methods are of special interest.

A red-black pressure-velocity-iteration which allows an efficient parallelization based on a domain decomposition [4] will be analyzed in this paper.

We prove the equivalence of the pressure-velocity-iteration (PUI) by Chorin/Hirt/Cook [2][3] with a SOR-iteration to solve a poisson equation for the pressure. We show this on a 2D rectangle with some special outflow boundary conditions and Dirichlet data for the velocity elsewhere. This equivalence allows us to prove the convergence of that iteration scheme. We also discuss the stablity of the occuring discrete Laplacian in discrete Sobolev spaces.

Keywords: Navier-Stokes equation, incompressible flow, FV discretisation, staggered grids, semi-implicit time integration

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INTRODUCTION

In the sequel the 2D consideration is used only for reasons of simplicity. The results of the paper can be generalised for the 3D case.

On the rectangle $\Omega = [a,b] \times [c,d]$ the nonstationary, incompressible Navier-Stokes equations (NSE) are given:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} \ , \ \nabla \cdot \mathbf{u} = 0 \tag{1}$$

On $\Gamma_D = \{a\} \times [c,d] \bigcup \{c,d\} \times [a,b]$ we have Dirichlet-data for $\mathbf{u}$ and on the outflow $\Gamma_{out} = \{b\} \times]c,d[$ we use a natural boundary condition of [5] or [6]:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_D \text{ on } \Gamma_D && (2)\\ \nu \frac{\partial u}{\partial x} &= p - \bar{p} \text{ on } \Gamma_{out} && (3)\\ \frac{\partial v}{\partial x} &= 0 \text{ on } \Gamma_{out} && (4) \end{aligned}$$

If a solution $p \in H^1(\Omega)$ of (1) exists, it is unique, since

$$\int_c^d p(b,y)dy = \nu \int_c^d \frac{\partial u}{\partial x} dy + (d-c)\bar{p} = -\nu \int_c^d \frac{\partial v}{\partial y} dy + (d-c)\bar{p} = \nu(v(b,d) - v(b,c)) + (d-c)\bar{p} \tag{5}$$

Hirt and Cook developed in [3] an iteration method modifying an idea of Chorin [2] to solve (1) using a time integration scheme of the following kind:

$$\begin{aligned} \frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\tau} + (\mathbf{u}^k \cdot \nabla)\mathbf{u}^k &= -\nabla p^{k+1} + \nu \Delta \mathbf{u}^k && (6)\\ \nabla \cdot \mathbf{u}^{k+1} &= 0 && (7) \end{aligned}$$

For the spatial discretization a staggered mesh with mesh sizes $\frac{1}{l_j}, \frac{1}{h_i}$ is used. This inverse spelling of the spatial grid sizes was used for a better readability of the formulas below.

It is well known, that the size of the time-step τ is restricted by the following conditions

$$\tau\nu \cdot max\{h^2, l^2\} \leq 1 \tag{8}$$

$$\tau \cdot max\{u \cdot l\}, \tau \cdot max\{v \cdot h\} \leq 1 \tag{9}$$

using a scheme (6). Especially (8) seems to be the crucial restriction. To overcome (8) time integration schemes, which treat the diffusive term $\nu\Delta\mathbf{u}$ implicitly were developed. Recent investigations [9] have shown, that in turbulent flows $\tau max\{v \cdot h\}$ can be of the same order of magnitude as $\tau\nu \max\{l^2, h^2\}$. In the sequel we will only give the qualitative results as lemmata and theorems. Their proofs are noted in detail in [1].

THE $p-\mathbf{u}$-ITERATION

To derive the discretization of (6) the following approximations are used:

$$div_{ij}\mathbf{u} = h_i(v_{ij} - v_{i-1j}) + l_j(u_{ij} - u_{ij-1}) \tag{10}$$

$$\frac{\partial p}{\partial x}_{ij} = L_j(p_{ij+1} - p_{ij}) \; ; \quad \frac{\partial p}{\partial y}_{ij} = H_i(p_{i+1j} - p_{ij}) \tag{11}$$

where $1/L_j := 1/2(1/l_{j+1} + 1/l_j)$ and $1/H_j := 1/2(1/h_{j+1} + 1/h_j)$. Let $\mathbf{u}^{k+1}, p^{k+1}$ be discretized by $\mathbf{u}, p$, then the linear system according to (6) reads

$$\begin{pmatrix} A & G \\ D & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ where } g \in Im(D) \tag{12}$$

A is a slight modification of I because of (3). This form is not the basis of PUI. For simplicity we denote the iteration sublevel for the computation of $\mathbf{u}^{k+1}, p^{k+1}$ by $\mathbf{u}^{,s}, p^{,s}$. Then PUI is given by

- start:

$$p^{,0} = \text{any initial guess, e.g. } p^k \tag{13}$$

$$\mathbf{u}^{,0} = \mathbf{u}^k - \tau(\mathbf{u}^k \cdot \nabla)\mathbf{u}^k - \tau\nabla p^{,0} + \tau\nu\Delta\mathbf{u}^k \tag{14}$$

- do for each cell $_{ij}$

$$\delta p^{,s+1} =: p^{,s+1} - p^{,s} = -\omega \left(\frac{\partial div\mathbf{u}}{\partial p}_{ij} \right)^{-1} div_{ij}\mathbf{u}^{,s,s+1/2} \tag{15}$$

$$u_{ij}^{,s+**} = u_{ij}^{,s+*} - \delta p^{,s+1} \cdot L_j \; ; \; u_{ij-1}^{,s+**} = u_{ij-1}^{,s+*} + \delta p^{,s+1} \cdot L_{j-1}$$

$$v_{ij}^{,s+**} = v_{ij}^{,s+*} - \delta p^{,s+1} \cdot H_i \; ; \; v_{i-1j}^{,s+**} = v_{i-1j}^{,s+*} + \delta p^{,s+1} \cdot H_{i-1} \tag{16}$$

- until $max\{div_{ij}\mathbf{u}^{,s+*}\} \leq \varepsilon_{tol}$

During one iteration each velocity value is updated twice. If there are no further informations about the order of the unknowns, the sublevel $^{,s,s+1/2,s+1}$ is not clear. This problem will be solved in section . ω is a relaxation parameter and $B_{ij} := -\frac{\partial div\mathbf{u}}{\partial p}_{ij}$, which is derived as follows:
The substitution of $u_{ij}, u_{i,j-1}, v_{ij}, v_{i1-j}$ in (10) by the discrete version of (6) leads to

$$div_{ij}\mathbf{u} = h_i(\frac{\partial p}{\partial y}_{ij} + f_{v,ij} - \frac{\partial p}{\partial y}_{i-1j} - f_{v,i-1j}) + l_j(\frac{\partial p}{\partial x}_{ij} + f_{u,ij} - \frac{\partial p}{\partial x}_{ij-1} - f_{u,ij-1}) \tag{17}$$

for inner cells. The substitution of $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$ by (11) gives

$$\frac{\partial div_{ij}\mathbf{u}}{\partial p_{ij}} = h_i(-H_i + 0 - (-H_{i-1}) - 0) + l_j(L_j + 0 - (-L_{j-1}) - 0) \tag{18}$$

CONVERGENCE OF PUI

Now the cells $_{ij}$ and p_{ij} are ordered in a *red-black* manner. We denote the p-values of black cells by p_0 and those of red cells by p_1 respectivly.

Lemma 1. *The red-black version of PUI restricted to the p-values is a red-black SOR iteration for the solution of*

$$F\cdot p := DA^{-1}G\cdot p = -g + DA^{-1}f =: \tilde{f} \tag{19}$$

Evaluation of $F := DA^{-1}G$ and $\tilde{f} = DA^{-1}f$

For simplicity the Dirichlet data $v_{m,.}$ are treated together with $u_{m,.}$ as unknowns. Then the definition of A, G, f ensures $v_{m,.} = v_D(x_j, d)$. The natural boundary condition is discretized by

$$u_{.,n} - u_{.,n-1} - 1/(l_n\nu)p_{.,n} = -1/(l_n\nu)\overline{p} \tag{20}$$

To keep the matrices A, G, D managable, their components are equiped with the following indices: for example $D_{ij,u_{kl}}$ denotes the coefficient of u_{kl} for computation of div_{ij} and $G_{u_{ij},kl}$ is the coefficient of p_{kl} in the momentum equation for u_{ij}. Thus:

$$\begin{aligned} &D_{ij,u_{kl}} = \delta_{ik}(\delta_{jl} - \delta_{j-1l})l_j, D_{ij,v_{kl}} = (\delta_{ik} - \delta_{i-1k})h_i\delta_{jl}, A_{u_{ij},u_{kl}} = \delta_{ik}(\delta_{jl} - \delta_{jn}\delta_{ln-1}), A_{v_{ij},v_{kl}} = \delta_{jl}\delta_{ik}, \\ &A_{u_{ij},v_{kl}} = A_{v_{ij},u_{kl}} = 0, 1/\tau G_{u_{ij},kl} = \delta_{ik}((\delta_{j+1l} - \delta_{jl})(1-\delta_{jn})L_j + \alpha\delta_{jl}\delta_{jn}), 1/\tau G_{v_{ij},kl} = (\delta_{i+1k} - \delta_{ik})(1-\delta_{im})H_i\delta_{jl}, \end{aligned} \tag{21}$$

where $\alpha = -\frac{1}{\tau\nu l_n}$ from (20).
A detailed evaluation of A^{-1} gives

$$A^{-1}_{u_{ij},u_{kl}} = \delta_{ik}(\delta_{jl} + \delta_{jn}\delta_{ln-1})\ ,\ A^{-1}_{v_{ij},v_{kl}} = \delta_{jl}\delta_{ik}\ ,\ A^{-1}_{u_{ij},v_{kl}} = A_{v_{ij},u_{kl}} = 0 \quad . \tag{22}$$

Now $C = DA^{-1}$ is given by:

$$\begin{aligned} (DA^{-1})_{ij,u_{kl}} &= \sum_{r,s}(D_{ij,u_{rs}}A^{-1}_{u_{rs},u_{kl}} + D_{ij,v_{rs}}A^{-1}_{v_{rs},u_{kl}}) \\ &= \sum_{r,s}(\delta_{ir}(\delta_{js} - \delta_{j-1s})l_jA^{-1}_{u_{rs},u_{kl}} + 0) = l_j\sum_s(\delta_{js} - \delta_{j-1s})A^{-1}_{u_{is},u_{kl}} \\ &= l_j(A^{-1}_{u_{ij},u_{kl}} - (1-\delta_{j1})A^{-1}_{u_{ij-1},u_{kl}}) = l_j\delta_{ik}(\delta_{jl} + \delta_{jn}\delta_{ln-1} - (1-\delta_{j1})\delta_{j-1l}) \end{aligned} \tag{23}$$

$$\begin{aligned} (DA^{-1})_{ij,v_{kl}} &= \sum_{r,s}(D_{ij,u_{rs}}A^{-1}_{u_{rs},v_{kl}} + D_{ij,v_{rs}}A^{-1}_{v_{rs},v_{kl}}) \\ &= \sum_{r,s}(0 + \delta_{js}(\delta_{ir} - \delta_{i-1r})h_i\delta_{rk}\delta_{jl}) = h_i(\delta_{ik} - \delta_{i-1k})\delta_{jl} \end{aligned} \tag{24}$$

Thus F becomes:

$$\begin{aligned} \frac{1}{\tau}F_{ij,kl} &= \frac{1}{\tau}\sum_{r,s}\left[C_{ij,u_{rs}}G_{u_{rs},kl} + C_{ij,v_{rs}}G_{v_{rs},kl}\right] \\ &= \sum_{r,s}\left[l_j\delta_{ir}(\delta_{js} - \delta_{j-1s} + \delta_{jn}\delta_{sn-1})G_{u_{rs},kl} + h_i(\delta_{ir} - \delta_{i-1r})\delta_{js}G_{v_{rs},kl}\right] \\ &= l_j\sum_s(\delta_{js} - \delta_{j-1s} + \delta_{jn}\delta_{sn-1})G_{u_{is},kl} + h_i\sum_r(\delta_{ir} - \delta_{i-1r})G_{v_{rj},kl} \\ &= l_j(G_{u_{ij},kl} - (1-\delta_{j1} - \delta_{jn})G_{u_{ij-1},kl}) + h_i(G_{v_{ij},kl} - (1-\delta_{i1})G_{v_{i-1j},kl}) \end{aligned} \tag{25}$$

For $1 \le i,k \le m$ and $1 \le j,l \le n$ we have $\frac{1}{\tau}F_{ij,kl} =$

$$\begin{aligned}
-l_j(L_j+(1-\delta_{j1})L_{j-1})-h_i((1-\delta_{im})H_i+(1-\delta_{i1})H_{i-1})& \\
\text{for } i=k; j=l<n & \quad (26)\\
l_n\alpha-h_i((1-\delta_{im})H_i+(1-\delta_{i1})H_{i-1}) \text{ for } i=k; j=l=n & \quad (27)\\
l_jL_j \text{ for } i=k; l=j+1 &\\
(1-\delta_{jn}-\delta_{j1})l_jL_{j-1} \text{ for } i=k; l=j-1 &\\
h_iH_i \text{ for } k=i+1; l=j &\\
h_iH_{i-1} \text{ for } k=i-1; l=j &\\
0 \text{ elsewhere,} &
\end{aligned}$$

withe the diagonal entries

$$B_{i,j} = \frac{1}{\tau}F_{ij,ij} \tag{28}$$

To get an idea of F , we consider the lexicographic order of the p-values
$p = (p_{1,1},..,p_{1,n};....;p_{m,1},..,p_{m,n})$

$$F^{lex} = \begin{pmatrix} B_1 & h_1H_1I & & & \\ h_2H_1I & B_2 & h_2H_2I & & \\ & \ddots & \ddots & \ddots & \\ & & h_{m-1}H_{m-2}I & B_{m-1} & h_{m-1}H_{m-1}I \\ & & & h_mH_{m-1}I & B_m \end{pmatrix} \tag{29}$$

with

$$B_i = \begin{pmatrix} B_{i,1} & l_1L_1 & & & \\ l_2L_1 & B_{i,2} & l_2L_2 & & \\ & \ddots & \ddots & \ddots & \\ & & l_nL_{n-1} & B_{i,n-1} & l_{n-1}L_{n-1} \\ & & & 0 & B_{i,n} \end{pmatrix} \tag{30}$$

Remark 2. The asserted identity of E and $Diag(F^{lex})$ is clear. For $j<n$ one has to substitute the momentum equation (4-#boundaries of the cell with Dirichlet-data)-times , which leads to (26). For $j=n$ (20) and (2-#boundaries of the cell with Dirichlet-data)-times the momentum equation is used, which coincides with (27).

To start the evaluation of $DA^{-1}f$ a representation of f is given. From (6) we have $\tilde{\mathbf{u}} = \mathbf{u}^k + \tau(\mathbf{u}\cdot\nabla)\mathbf{u}^k + \tau\nu\Delta\mathbf{u}^k$. Thus f reads

$$f_{u_{ij}} = (1-\delta_{jn})\tilde{u}_{ij} - \delta_{jn}l_n/\nu\bar{p}\,,\ f_{v_{ij}} = (1-\delta_{im})\tilde{v}_{ij} + \delta_{im}\cdot 0 \tag{31}$$

Now we obtain

$$\begin{aligned}
(DA^{-1}f)_{ij} &= \sum_{r,s}((DA^{-1})_{ij,u_{rs}}f_{u_{rs}} + (DA^{-1})_{ij,v_{rs}}f_{v_{rs}}) \\
&= \sum_{r,s}\Big[l_j\delta_{ir}(\delta_{js}+\delta_{jn}\delta_{sn-1}-(1-\delta_{j1})\delta_{j-1s})f_{u_{rs}} + h_i\delta_{js}(\delta_{ir}-\delta_{i-1r})f_{v_{rs}}\Big] \\
&= l_j\sum_s(\delta_{js}+\delta_{jn}\delta_{sn-1}-(1-\delta_{j1})\delta_{j-1s})f_{u_{is}} + h_i\sum_r(\delta_{ir}-\delta_{i-1r})f_{v_{rj}} \\
&= l_j(f_{u_{ij}}+\delta_{jn}f_{u_{in-1}}-(1-\delta_{j1})f_{u_{ij-1}}) + h_i(f_{v_{ij}}-(1-\delta_{i1})f_{v_{i-1j}})\,. \qquad (32)
\end{aligned}$$

Analysis of the red-black SOR Iteration

To show the convergence we need some auxiliary results, which are proved in the sequel. We denote by $J(A)$, $H(A,\omega)$ the iteration matrices of the Jacobi and SOR iteration as far as they exist for A.

Lemma 3. *Let A be, such that $J(A)$ exists, P be a permutation matrix for π ($Pe_i = e_{\pi(i)}$) and E be a nonsingular diagonal matrix. Then $J(PAP^T) = PJ(A)P^T$ and $J(EAE^{-1}) = EJ(A)E^{-1}$. Especially $J(PAP^T)$, $J(EAE^{-1})$ exist and have the same spectrum as $J(A)$.*

Another usefull statement is

Lemma 4. *Let A be a symmetric,real matrix with $A_{ii} < 0$. Then the spectrum of $J(A)$ is real.*

Now the main theorem of this section is given:

Theorem 5. *For F^{lex} we have*

1. *F^{lex} is reducible*
2. *F^{lex} has only real eigenvalues*
3. *$J(F^{lex})$ has only real eigenvalues*
4. *$\rho(J(F^{lex})) < 1$*

Theorem 6. *The red-black-SOR iteration for (19) converges for relaxation parameters $0 < \omega < 2$.*

Remark 7. F is the discrete operator of the 5-point stencil for the Laplacian with Neumann conditions on Γ_D. For Γ_{out} the boundary condition reads

$$-\nu\tau\frac{\partial^2 p}{\partial x^2} - p = r.h.s \tag{33}$$

Using the Poisson equation for the pressure we can substitute (33) by

$$+\nu\tau\frac{\partial^2 p}{\partial y^2} - p = r.h.s \text{ on } \Gamma_{out} \tag{34}$$

This is an ODE of Helmholtz type on Γ_{out} with homogeneous boundary conditions for $\frac{\partial p}{\partial y}$ in $\partial\Gamma_{out}$. This ODE has a unique solution under some assumptions for the right hand side.

Convergence for $\frac{\partial \mathbf{u}}{\partial x} = 0$ on Γ_{out}

In this section we consider another often used boundary condition on Γ_{out}

$$\frac{\partial \mathbf{u}}{\partial x} = 0 \text{ on } \Gamma_{out} \tag{35}$$

Using all notations of the previous sections we have $\alpha = 0$.

Lemma 8. *For $\alpha = 0$: $Ker(F^{lex}) = span\{\mathbf{1}\}$*

Lemma 9.
1. *$\rho(J(F^{01})) = 1$*
2. *$J(F^{01})$ has only real eigenvalues*
3. *1 and -1 are eigenvalues of $J(F^{01})$, the eigenspace of 1 is spanned by $\mathbf{1}$ and the eigenspace of -1 by $e_{01} := \begin{pmatrix} \mathbf{1}_0 \\ -\mathbf{1}_1 \end{pmatrix}$*

Now we are ready to show that the SOR-iteration converges in the sense of damping all errors except the error that belongs to the global pressure level. That means

Lemma 10. *The iteration converges to a solution of (19).*

Remark 11. For the solvability of the equation (1) it's necessary to show the property of the right hand side $\tilde{f}$:

$$\tilde{f} \cdot \mathbf{1} = 0\,. \tag{36}$$

The relation (36) means the mass conservation of the channel flow problem, i.e., the mass flux, coming in over Γ_D, must be equal to the mass flux, going over the outlet boundary Γ_{out}. Using the representation and evaluation (32) of the right hand side $DA^{-1}f$ the relation (36) is obvious.

STABILITY OF THE SPATIAL DISCRETIZATION

This section deals with the stability of the discrete Laplacian F^{α} of the previous section. In order to obtain statements concerning the regularity in the discrete sobolev space H_h^1 (see [8], Chapter 9.2) we show the boundedness of $(F^{\alpha})^{-1}$; $(F^{\alpha})^{-T}$. This implies the stability in the discrete space L^2 (=l^2).

Lemma 12. *If the grid stretching is bounded ($\frac{1}{q} \leq \frac{1}{l_{j+1}} / \frac{1}{l_j} \leq q$), then*

$$\|(F^{\alpha})^{-1}\|_{\infty} \leq max\{\frac{q}{q+1}, \tau\nu\}. \tag{37}$$

Lemma 13. *Using all notations of the previous lemma and the following notations* $p_h := max_{i,r} \frac{h_r}{h_i}$; $p_l := max_{j,s} \frac{l_s}{l_j}$, *then*

$$\|(F^{\alpha})^{-T}\|_{\infty} \leq max\{p_h p_l \frac{q}{q+1}, 2max_j\{l_j\}/l_1 \tau\nu p_h^2\}. \tag{38}$$

Remark 14. If condition (8) is satisfied, then one has for the L_h^2 stability

$$|(F^{\alpha})^{-1}|_{0\leftarrow 0} \leq \sqrt{p_h p_l}\frac{q}{q+1} \tag{39}$$

PRACTICAL APPLICATION OF THE DISCUSSED METHOD

The above dicussed pressure-velocity-iteration method is used in production codes for the numerical simulation of laminar and turbulent flow problems. A Navier-Stokes code developed in Munic at the university of the Bundeswehr[10] was parallelized for running on massively parallel computers of the Technische Universität Berlin and of the Konrad-Zuse-Zentrum Berlin. Together with engineers from the fluid mechanical department of the TU Berlin and the university of the Bundeswehr Munic investigations of a backward facing step flow problem are done [11]. With the aim of the decrease of the recirculation length the boundary layer will be manipulated by loudspeakers with certain frequencies und sound pressure. The loudspeakers are simulated in the mathematical model by non stationary Dirichlet boundary conditions.

With the presented strict proof of the convergence of the pressure-velocity-iteration method a gap between heuristic engineering development and numerical mathematics with respect to appropriate parallelization techniques was closed.

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