Mathematical Modeling and Simulation of the COVID-19 Pandemic

Günter Bärwolff*
Technische Universität Berlin

Abstract

The actual pandemic is a great challenge for quite different research areas. Beside the virology research mathematical models and simulations can be a valuable contribution to the understanding of the dynamics of the pandemic and can give recommendations to physicians and politicians for their decisions.

Based on actual data about COVID-19 infected people from the European Centre for Disease Prevention and Control [2] (ECDC) parameters will be determined and applied in mathematical models. Parameters for several countries like UK, USA, Italy, Spain, Germany and China will be estimated and used in a SIR-type model following the fundamental paper of W.O. Kermack and A.G. McKendrick [1]. Strategies for the commencing and ending of social and economic shutdown measures are discussed.

We will not be overbearing regarding the results of our numerical simulations, but we understand the results no more and no less as a contribution to the actual controversial discussion of the different scientific communities.

We disclaim qualitative mathematical consideration like existence and uniqueness of solutions and concentrate our interest on the practical application and numerical experiments. The numerical solution of the ordinary differential equation system of the modified SIR model is being done with a Runge-Kutta integration method of fourth order [4].

Suggestions about appropriate points in time at which to commence with lockdown measures based on the acceleration rate of infections conclude the paper. At the end the applicability of the SIR model could be shown, but weakness and strength of the modeling will be discussed also. This paper is an improved sequel of [5].

1 Introduction

The dynamic development of susceptible, infected and recovered people in a certain region, for example the population of a country or a part of a federation, is the aim of our modeling. At first we note one important presupposition for our modeling. We suppose that the distribution of the susceptible people is equal, i.e. the density is approximately constant. This is a very strict supposition, but this is acceptable for example for cities and congested urban areas like New York or the Ruhr area in Germany. At the beginning of the pandemic an exponential growth of infected people is supposed.

*mailto:baerwolf@math.tu-berlin.de

2 The mathematical SIR model

In the so called SIR-model of Kermack and McKendrick [1] I denotes the infected people, S stands for the susceptible and R denotes the recovered people. There are also multitudes of generalizations of the SIR-model but we constrain our investigations to the species I, S and R only, because the basic behavior of SIR-type models can be described by the following simple one. The dynamics of infections and recoveries can be approximated by the system of ordinary differential equations

$$\frac{dS}{dt} = -\beta\frac{S}{N}I \tag{1}$$

$$\frac{dI}{dt} = \beta\frac{S}{N}I - \gamma I \tag{2}$$

$$\frac{dR}{dt} = \gamma I\ . \tag{3}$$

We understand β as the number of others that one infected person encounters per unit time (per day). γ is the reciprocal value of the typical time from infection to recovery. N is the total number of people involved in the epidemic disease and there is $N = S + I + R$.

The empirical data currently available suggests that the corona infection typically lasts for some 14 days. This means $\gamma = 1/14 \approx 0{,}07$.

The choice of β is more complicated and will be considered in the following.

3 The estimation of β based on real data

We use the European Centre for Disease Prevention and Control [2] as a data for the COVID-19 infected people for the period from December 31st 2019 to April 8th 2020.

At the beginning of the pandemic the quotient S/N is nearly equal to 1. Also, at the early stage no-one has yet recovered. Thus we can describe the early regime by the ordinary differential equation

$$\frac{dI}{dt} = \beta I$$

with the solution

$$I(t) = I_0 \exp(\beta t)\ . \tag{4}$$

Especially in medicine, psychology and other life sciences the logarithm behavior of data is usually considered. Thus, we consider the following logarithmic table.

day	log(number of infected people)
t_1	$\log I_1$
t_2	$\log I_2$
$\vdots$	$\vdots$
t_k	$\log I_k$

The logarithm of (4) leads to

$$\log I(t) = \log I_0 + \beta t$$

and based on the logarithmic table the functional

$$L(I_0, \beta) = \sum_{j=1}^{k} [\log I_0 + \beta t_j - \log I_j]^2 \ , \tag{5}$$

is to minimize. The solution of this linear optimization problem is trivial and it is available in most computer algebra systems as a "black box" of the logarithmic-linear regression.

The following figures show the results for the same periods as above for Spain, the UK, the USA and Italy.

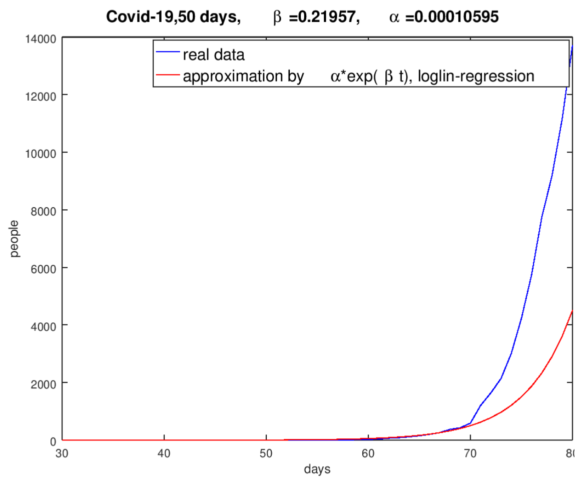


Figure 1: log-lin-result for Spain (January 31st 2020 to March 20th 2020)

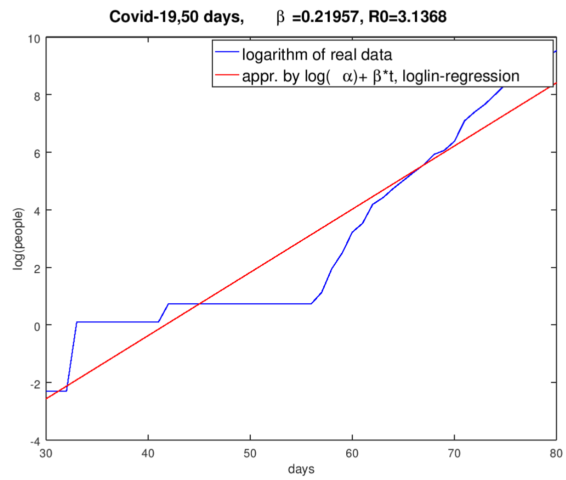


Figure 2: Logarithm of the Spanish result (January 31st 2020 to March 20th 2020)

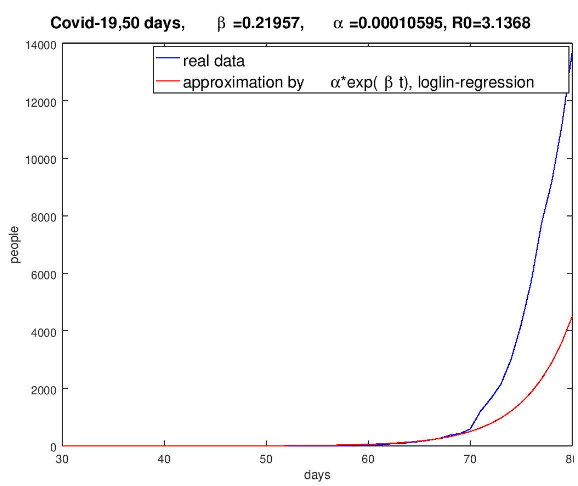


Figure 3: log-lin-result of the UK (January 30th 2020 to March 20th 2020)

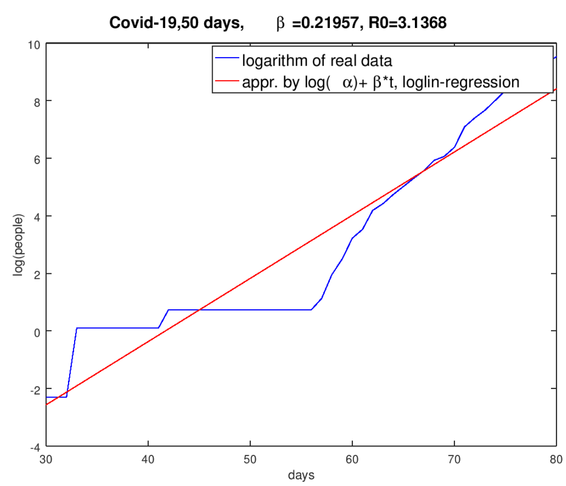


Figure 4: Logarithm of the UK result (January 30th 2020 to March 20th 2020)

Figures 1-8 show that the logarithmic-linear regression implies poor results. It must be said that evaluated β-values are related to the stated period. For the logarithmic-linear regression method we guessed the respective periods for every country by a visual inspection of the graphs of the infected people over days.

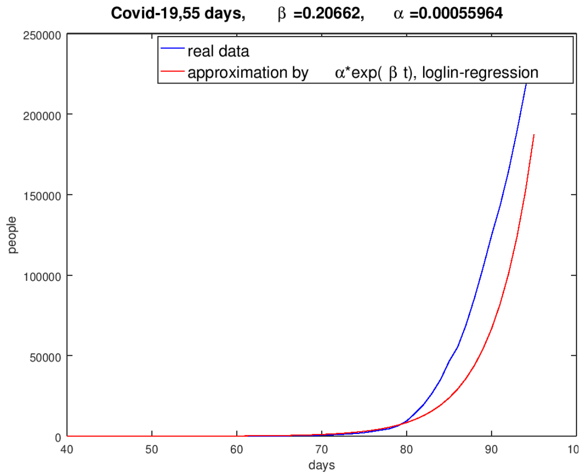


Figure 5: log-lin-result of the USA (February 10th 2020 to April 4th 2020)

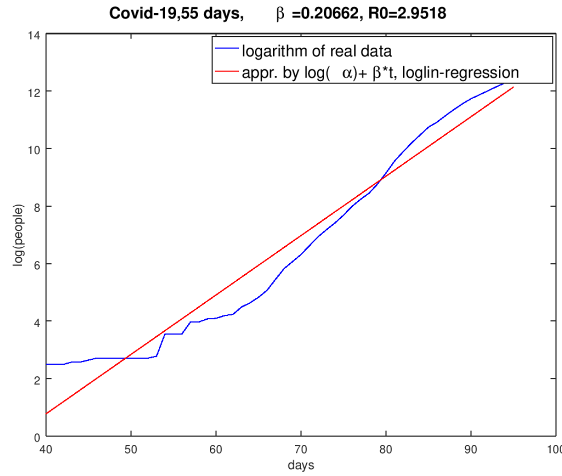


Figure 6: Logarithm of the USA result (February 10th 2020 to April 4th 2020)

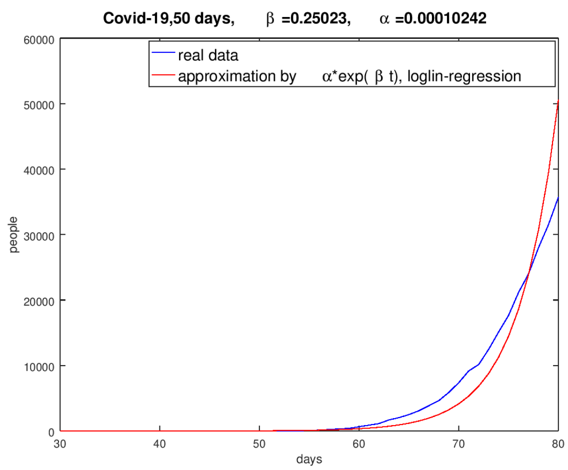


Figure 7: log-lin-result of Italy (January 31st 2020 to March 20th 2020)

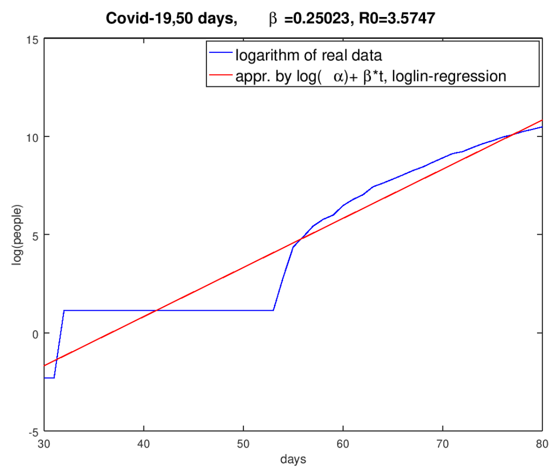


Figure 8: Logarithm of the Italian result (January 31st 2020 to March 20th 2020)

Instead of the above table of logarithmic values the following one is used with the aim of a better approximation. We are looking for periods in the spreadsheets of infected people per day where the course can be described by a function of type (4). Starting with a spreadsheet like

day	number of infected people
t_1	I_1
t_2	I_2
$\vdots$	$\vdots$
t_k	I_k

for a certain country and a chosen period $[t_1, t_k]$ we search for the minimum of the functional

$$F(I_0, \beta) = \sum_{j=1}^{k} [I_0 \exp(\beta t_j) - I_j]^2 \ ,$$

i.e.

$$\min_{(I_0,\beta)\in\mathbb{R}^2} F(I_0,\beta)\ . \tag{6}$$

We solved this non-linear minimum problem with the damped Gauss-Newton method (see [4]). The results of the above discussed logarithmic-linear method for β and α proved as good start iterations for the Gauss-Newton method. We found the subsequent results for the considered countries. Thereby we chose different periods for the countries with the aim to approximate the infection course in a good quality. The following figures show the graphs and the evaluated parameter.

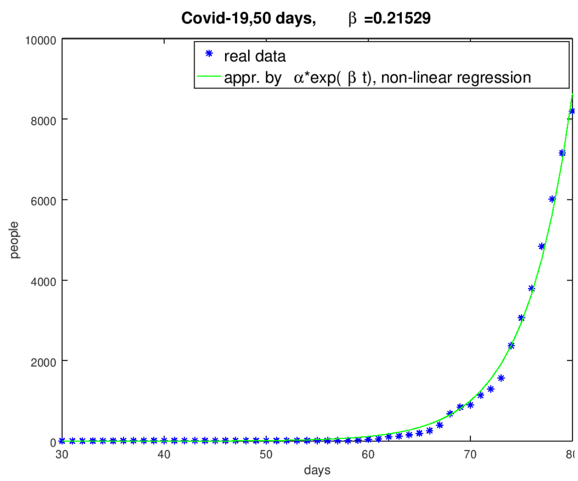


Figure 9: German course from January 31st 2020 to March 20th 2020

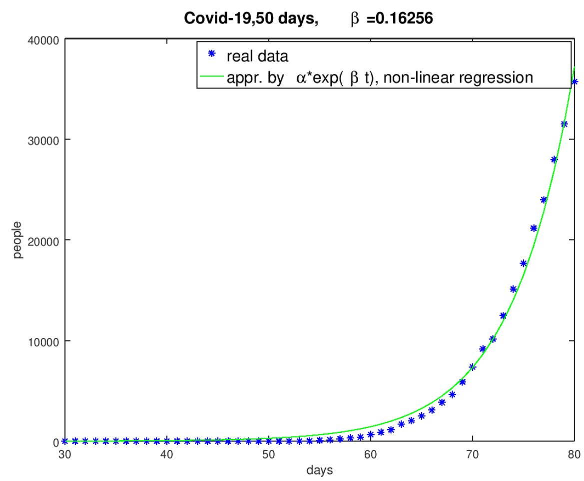


Figure 10: Italian course from January 31st 2020 to March 20th 2020

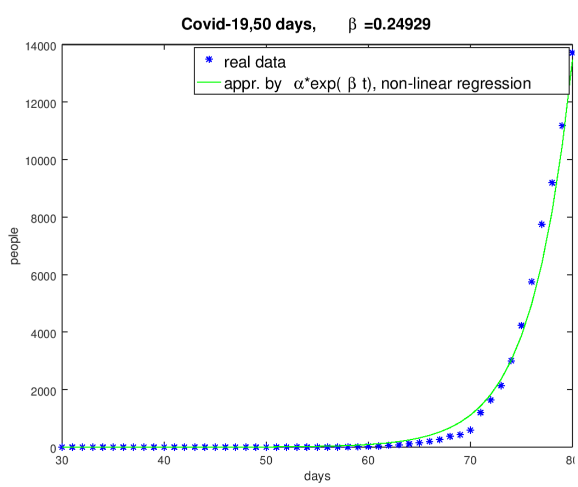


Figure 11: Spanish course from January 31st 2020 to March 20th 2020

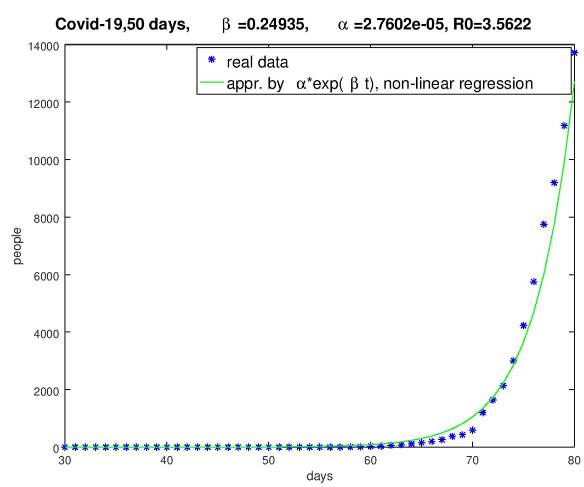


Figure 12: UK course from January 30th 2020 to March 20th 2020

We found some information on the parameters of Italy in the literature, for example $\beta = 0{,}25$, and we are afraid that this is a result of the logarithmic-linear regression.

A deeper look at the real data shows that the exponential behavior of the dynamic of the infected people we found only in the very beginning of the pandemic. Let us have a look at the data of New-York-City. The results are shown in the figures 15, 16 and 19

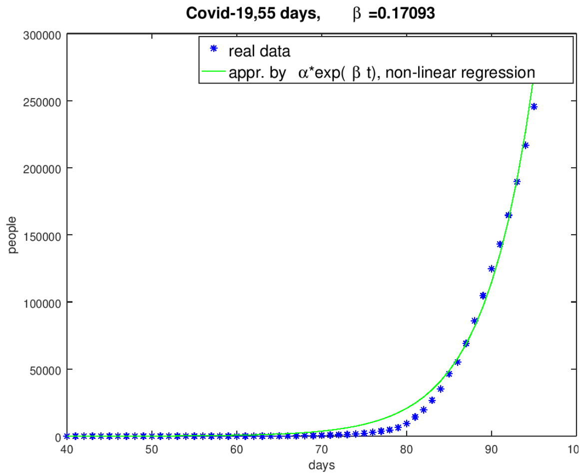


Figure 13: USA course from February 10th 2020 to April 4th 2020

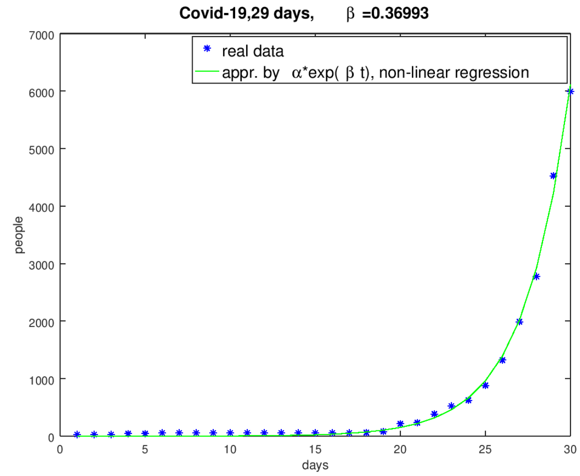


Figure 14: Chines course from December 31st 2019 to January 28th 2020

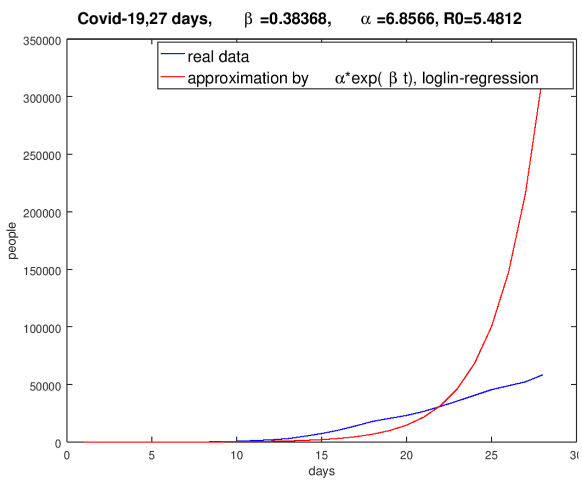


Figure 15: log-lin-result of NY-City (March 3rd to April 18th 2020)

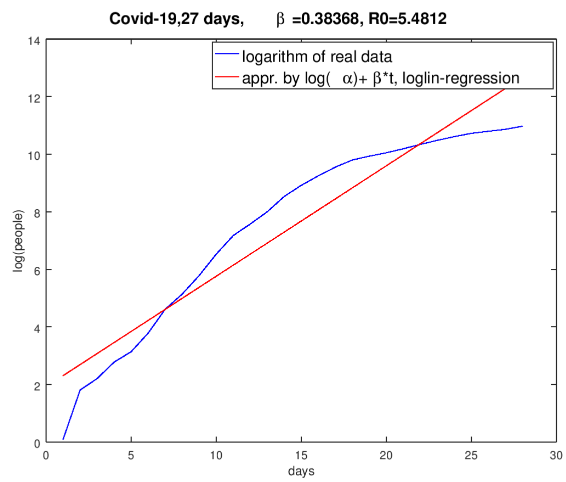


Figure 16: Logarithm of the NY-City results (March 3rd to April 18th 2020)

In the German hotspot Bavaria we found the following results for the period from February 24th to April 20th 2020 (see figures 17, 18 and 20). The attempt to fit an exponential behavior with the NY-City data is shown in fig. 19.

At the end we can state that estimation of the parameter β is complicated, but successful in most of the considered countries and regions. The results of the solution of the minimum-problem (6) to evaluate β are in most of the cases with respect to the fitting of the real data better than the results of the minimization of functional (5).

4 Some numerical computations for Germany and Spain

With the choice of β-value $0{,}215$ (see fig. 9) which is evaluated on the basis of the real data of ECDC and $\gamma = 0{,}07$ one gets the course of the pandemic dynamics pictured in fig. 21[1]. R_0 is the basis reproduction number of persons, infected by

[1] $I0$ denotes the initial value of the I species, that is January 31th 2020. $Imax$ stands for the maximum of I. The total number N for Germany is guessed to be 70 millions.

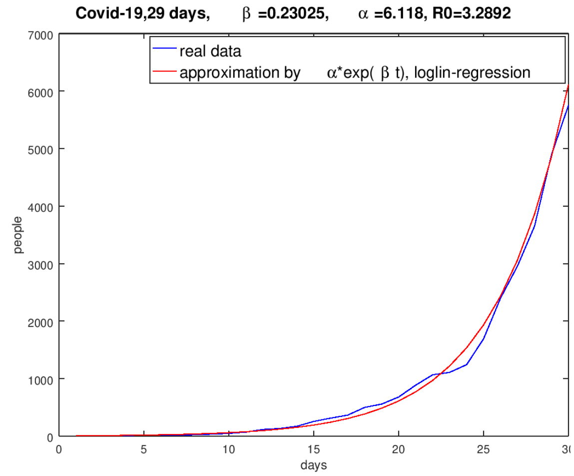


Figure 17: log-lin-result of Bavaria (February 24th to April 20th 2020)

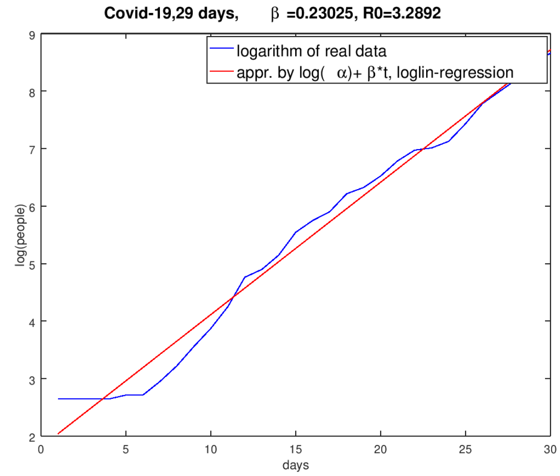


Figure 18: Logarithm of the Bavarian results (February 24th to April 20th 2020)

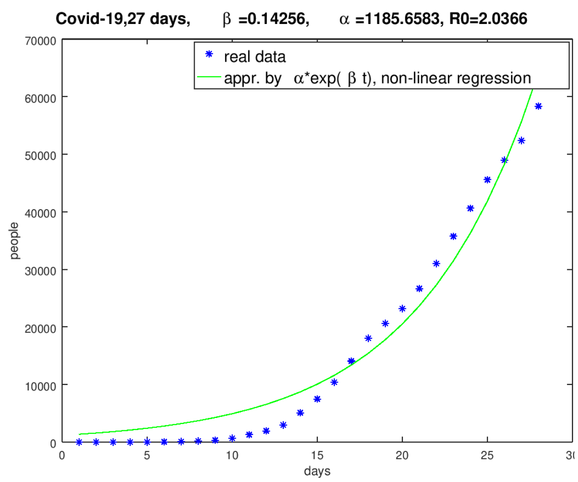


Figure 19: NY-City course (March 3rd to April 18th 2020)

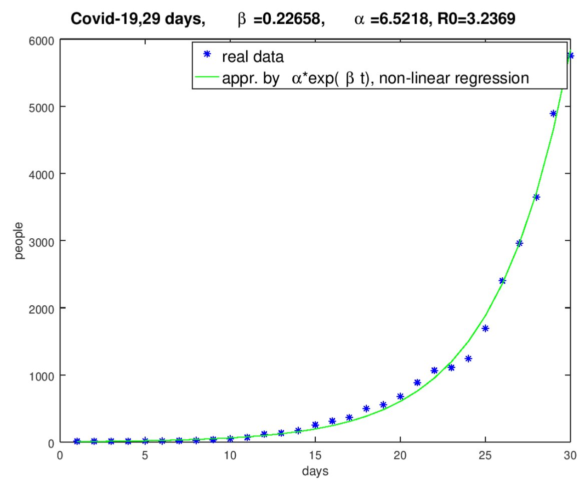


Figure 20: Bavarian data from February 24th to April 20th 2020

the transmission of a pathogen from one infected person during the infectious time ($R_0 = \beta/\gamma$) in the following figures[2] .

Neither data from ECDC nor the data from the German Robert-Koch-Institut and the data from the Johns Hopkins University are correct, for we have to reasonably assume that there are a number of unknown cases. It is guessed that the data covers only 15% of the real cases. Considering this we get slightly changed results and in the subsequent computations we will include estimated number of unknown cases to the initial values of I.

For Spain we use the β-value $0{,}249$ (see fig. 10) and $\gamma = 0{,}07$ we get the course pictured in fig. 22. N is set to 40 millions.

Let us now discuss the case of strict social distancing. To do this we reduce the β-values after a few days to $\beta = 0{,}14$ for both, Germany and Spain.

The results in figures 23 and 24 compared to the results without the reduction of β (21 and 22) show the consequences. The climax of the number of infected people

[2]The values of R_0 in all of the following figures are applied to the β-value of the beginning of the pandemic.

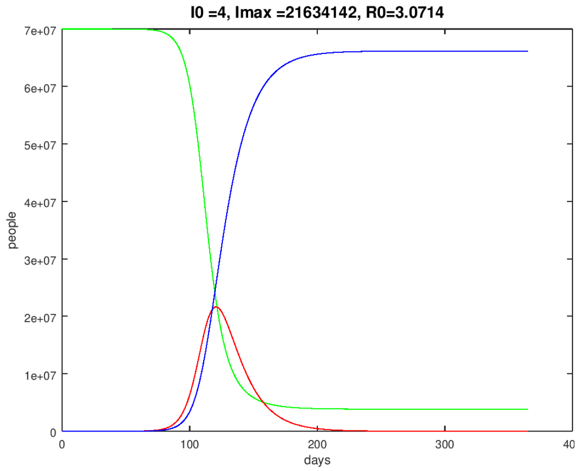


Figure 21: German course of one year, starting end of January 2020, S-green, I-red, R-blue

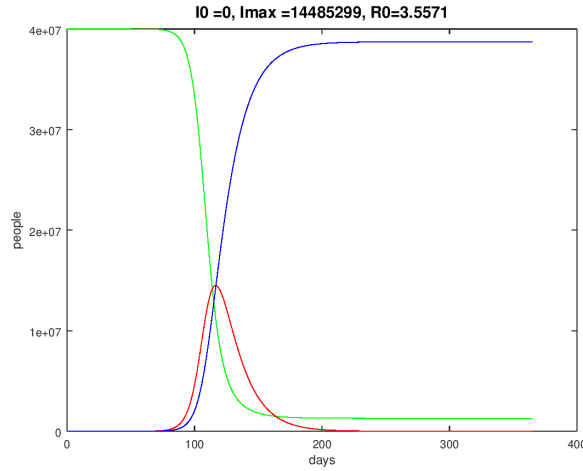


Figure 22: Spanish course of one year, starting end of January 2020, S-green, I-red, R-blue

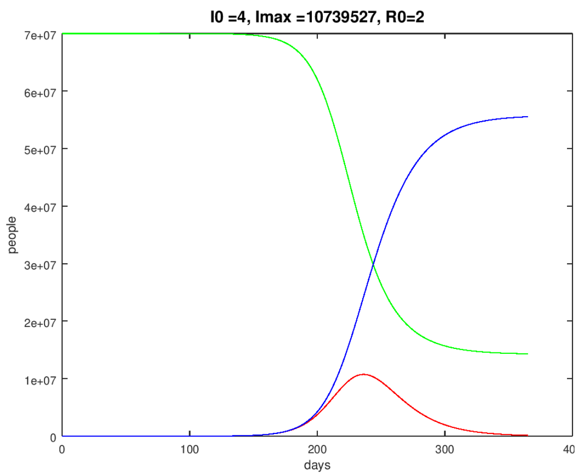


Figure 23: German course of one year with reduced β, starting end of January 2020, S-green, I-red, R-blue

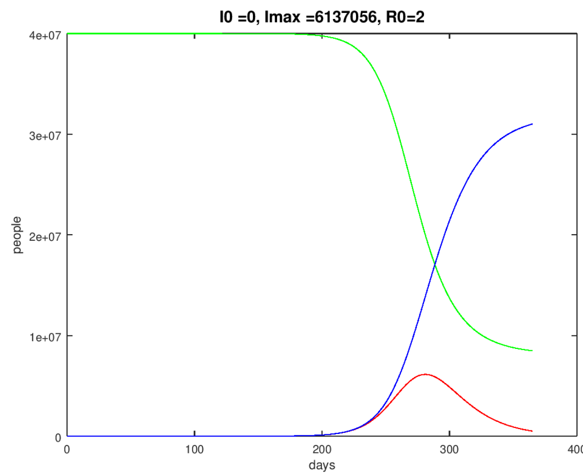


Figure 24: Spain course of one year with reduced β, starting end of January 2020, S-green, I-red, R-blue

moved to the autumn of the year with hard inconveniences for the population, but the wanted flattening is achieved.

To investigate the influence and sensitivity of the simulation results by the parameter β and the number N (sum of infected, susceptible and restored people) we used the German data and variation of it. In fig. 25 we see, that variation of the amount N leads more or less to a proportional scaling[3]. The variation of β showed a non-monotone and non-linear influence of β on the results, pictured in fig. 25.

[3] $N = 12$ millions is the population of Bavaria.

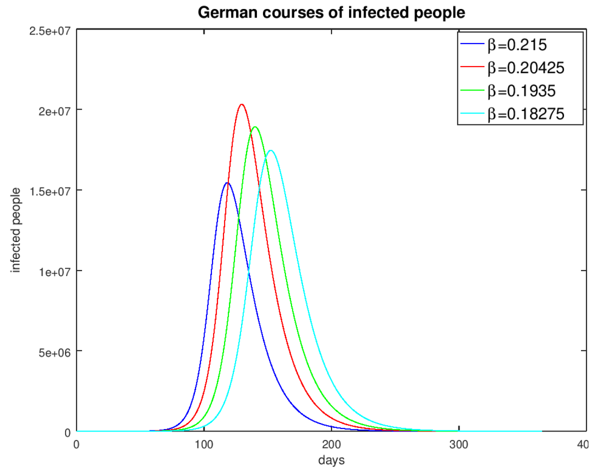


Figure 25: German course of one year depending on a β-variation

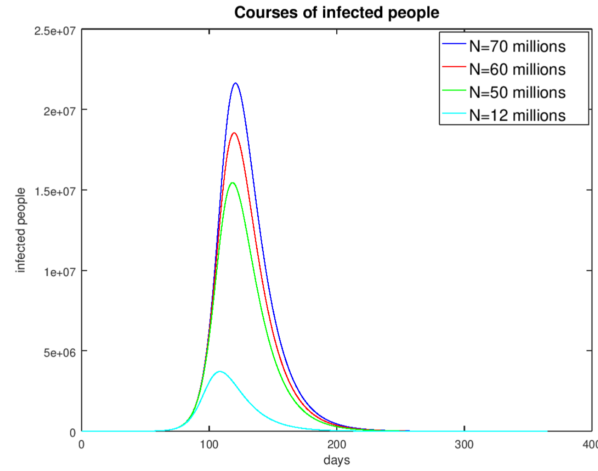


Figure 26: Course of one year depending on N ($\beta = 0\ 215$)

5 Looking for other strategies of a temporary lockdown and extensive social distancing

In all countries concerned by the Corona pandemic a lockdown of the social life is discussed. In Germany the lockdown started on March 16th 2020. The effects of social distancing to decrease the infection rate can be modeled by a modification of the SIR model. The original differential equation system (1)-(3) is modified to

$$\frac{dS}{dt} = -\kappa\beta\frac{S}{N}I \tag{7}$$
$$\frac{dI}{dt} = \kappa\beta\frac{S}{N}I - \gamma I \tag{8}$$
$$\frac{dR}{dt} = \gamma I\ . \tag{9}$$

κ is a function with values in $[0{,}1]$. For example

$$\kappa(t) = \begin{cases} 0{,}5 & \text{for } t_0 \le t \le t_1 \\ 1 & \text{for } t > t_1,\ t < t_0 \end{cases}$$

means for example a reduction of the infection rate of 50% in the period $[t_0, t_1]$ ($\Delta_t = t_1 - t_0$ is the duration of the temporary lockdown in days). A good choice of t_0 and t_k is going to be complicated.

If we respect the chosen starting day of the German lockdown, March 16th 2020 (this conforms the 46th day of the concerned year), and we work[4] with

$$\kappa(t) = \begin{cases} 0{,}2 & \text{for } 46 \le t \le 76 \\ 1 & \text{for } t > 76,\ t < 46 \end{cases}$$

we got the result pictured in figures 27 and 28.

The numerical tests showed that a very early start of the lockdown resulting in a reduction of the infection rate β results in the typical Gaussian curve to be delayed by I; however, the amplitude (maximum value of I) doesn't really change.

[4]We will understand 20% of normality by a lockdown, this means $\kappa = 0\ 2$.

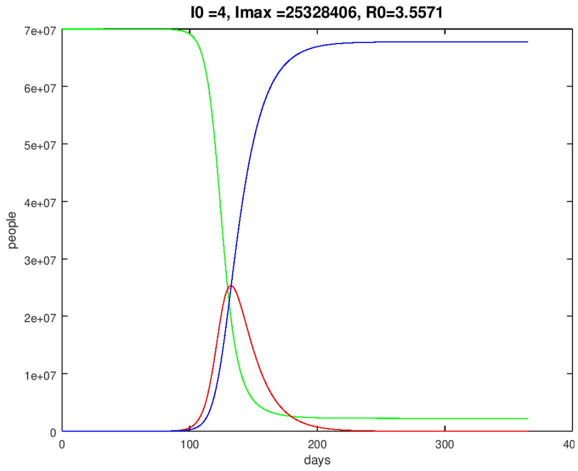


Figure 27: German course of one year, starting end of January 2020, S-green, I-red, R-blue, 30 days lockdown, starting time March 16th 2020

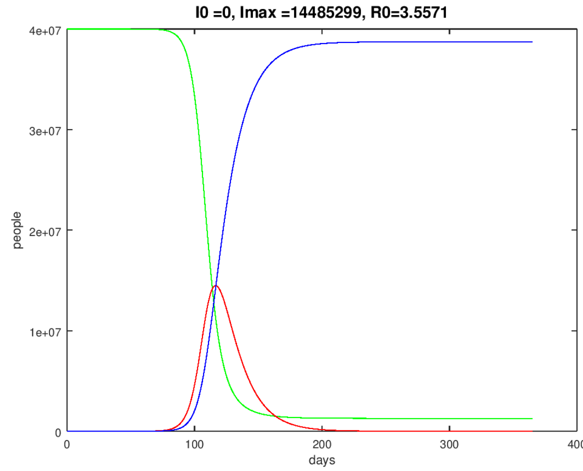


Figure 28: Spain course of one year, starting end of January 2020, S-green, I-red, R-blue, 30 days lockdown, starting time March 16th 2020

One knows that the development of the infected people looks like a Gaussian curve. The interesting points in time are those where the acceleration of the numbers of infected people increases or decreases, respectively.

These are the points in time where the curve of I is changing from a convex to a concave behavior or vice versa. The convexity or concavity can be controlled by the second derivative of $I(t)$.

Let us consider equation (2). By differentiation of (2) and the use of (1) we get

$$\begin{aligned}\frac{d^2I}{dt^2} &= \frac{\beta}{N}\frac{dS}{dt}I + \frac{\beta}{N}S\frac{dI}{dt} - \gamma\frac{dI}{dt} \\ &= -\frac{\beta}{N}^2 SI^2 + (\frac{\beta S}{N} - \gamma)(\frac{\beta S}{N} - \gamma)I \\ &= [(\frac{\beta S}{N} - \gamma)^2 - (\frac{\beta}{N})^2 SI]I \ .\end{aligned}$$

With that the I-curve will change from convex to concave if the relation

$$(\frac{\beta S}{N} - \gamma)^2 - (\frac{\beta}{N})^2 SI < 0 \Longleftrightarrow I > \frac{(\frac{\beta S}{N} - \gamma)^2 N^2}{\beta^2 S} \tag{10}$$

is valid. For the switching time follows

$$t_0 = \min_t\{t > 0,\ I(t) > (\frac{\beta S(t)}{N} - \gamma)^2 N^2)/(\beta^2 S(t))\} \ . \tag{11}$$

A lockdown starting at t_0 (assigning $\beta^* = \kappa\beta,\ \kappa \in [0{,}1[$) up to a point in time $t_1 = t_0 + \Delta_t$, with Δ_t as the duration of the lockdown in days, will be denoted as a **dynamical lockdown** (for $t > t_1$ β^* is reset to the original value β).

t_0 means the point in time up to which the growth rate increases and from which on it decreases. Fig. 29 shows the result of such a computation of a dynamical lockdown. We got $t_0 = 108$ ($\kappa = 0{,}2$). The result is significant. In fig. 31 a typical behavior of $\frac{d^2I}{dt^2}$ is plotted (in fig. 32 $\frac{d^2I}{dt^2}$ in the dynamical lockdown case).

The result of a dynamical 30 days lockdown for Spain is shown in fig. 30, where we found $t_0 = 106$ ($\kappa = 0{,}2$).

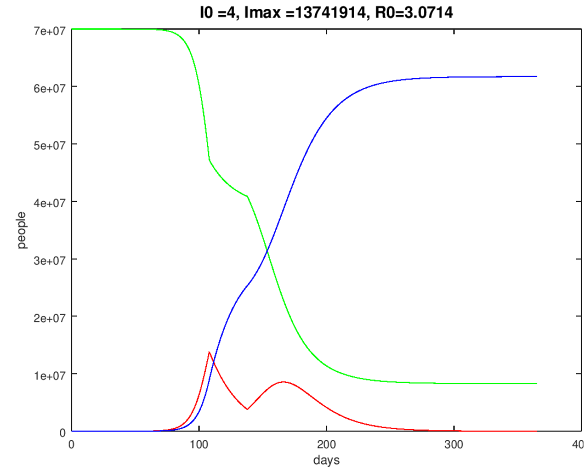


Figure 29: German course of one year, starting end of January 2020, dynamical lockdown, S-green, I-red, R-blue

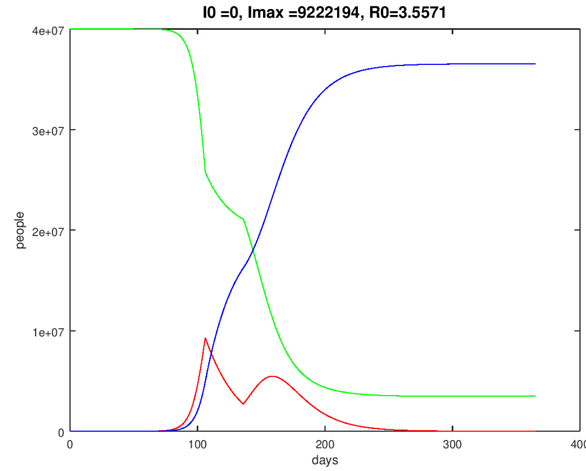


Figure 30: Spanish course of one year, starting end of March 2020, dynamical lockdown, S-green, I-red, R-blue

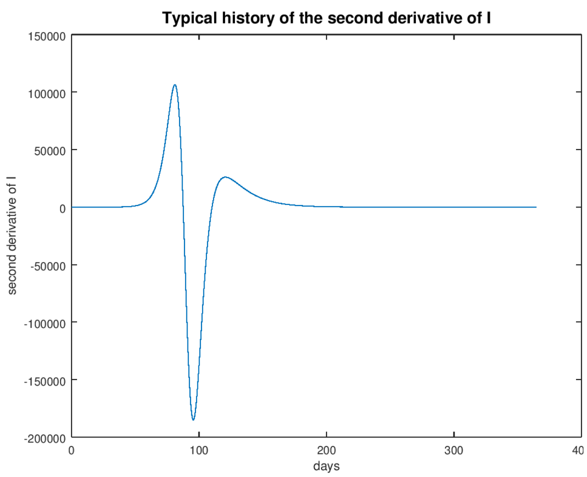


Figure 31: History of the second derivative of I (de)

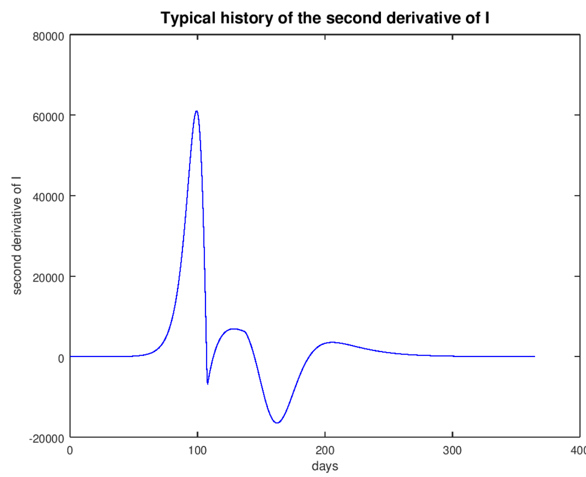


Figure 32: History of the second derivative of I with dynamical lockdown (de)

Data from China and South Korea suggest that the group of infected people with an age of 70 or more is of magnitude 10%. This group has a significant higher mortality rate than the rest of the infected people. Thus we can presume that α=10% of I must be especially sheltered and possibly medicated very intensively as a high-risk group.

Fig. 33 shows the German time history of the above defined high-risk group with a dynamical lockdown with $\kappa = 0{,}2$ compared to regime without social distancing. The maximum number of infected people decreases from approximately 1,7 million people to 0,8 million in the case of the lockdown (30 days lockdown).

This result proves the usefulness of a lockdown or a strict social distancing during an epidemic disease. We observe a flattening of the infection curve as requested by politicians and health professionals. With a strict social distancing for a limited time one can save time to find vaccines and time to improve the possibilities to help high-risk people in hospitals.

To see the influence of a social distancing we look at the Spanish situation without a lockdown and a dynamical lockdown of 30 days with fig. 34 ($\kappa = 0{,}2$) for the

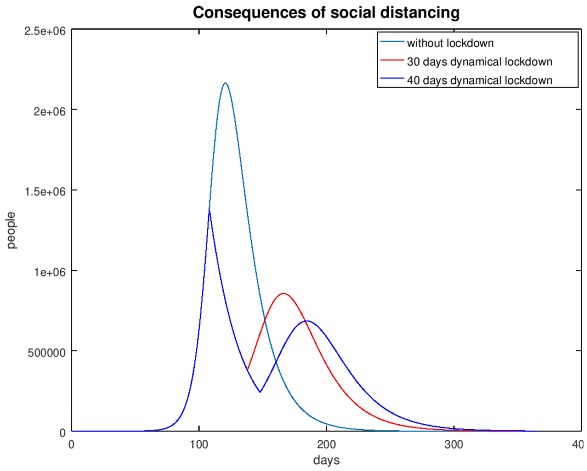


Figure 33: German history of the infected people of high-risk groups depending on a dynamical lockdown

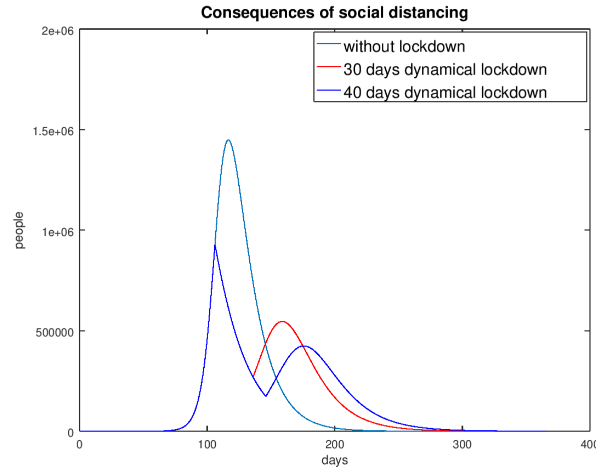


Figure 34: Spanish history of the infected people of high-risk groups depending on a dynamical lockdown

10% high-risk people.

The computations with the SIR model show, that the limited social distancing with a lockdown will be successful with a start behind the time greater or equal to t_0, found by the evaluation of the second derivative of I (formula (11)). If the limited lockdown is started at a time less then t_0 the effect of such a social distancing is not significant.

Bavaria is one of the origins of the German pandemic and is still a hot-spot. Therefore we will consider the simulation results for this German hot-spot. As parameters we use $\beta = 0\ 215$ and $N = 12$ millions. In the following figures the results for one year courses without and with lockdowns are shown.

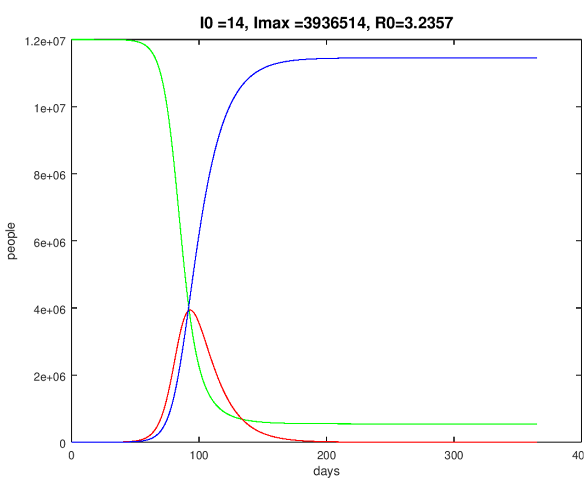


Figure 35: Bavarian one year course without lockdown

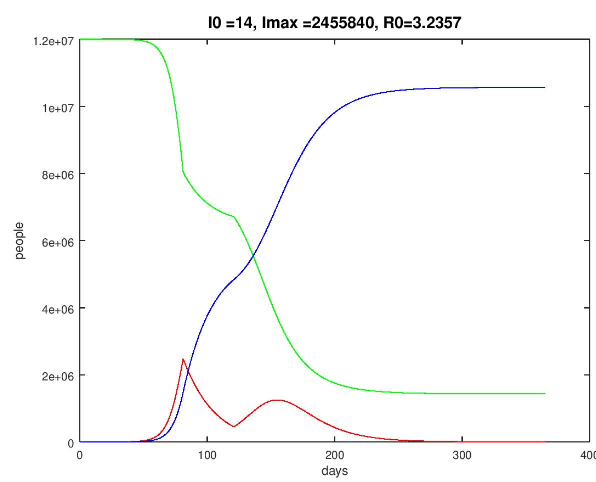
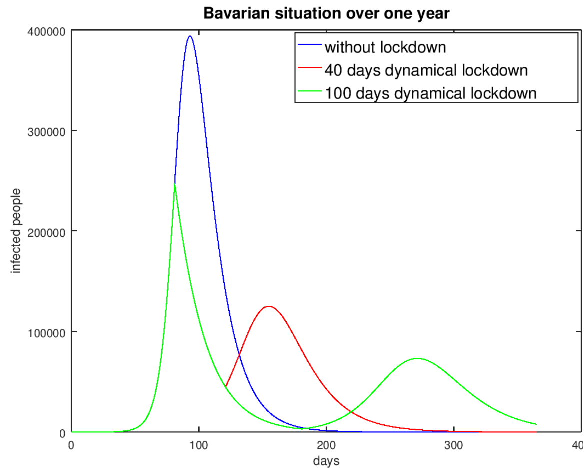


Figure 36: Bavarian one year course with lockdown

In fig. 37 the consequences of a 40-day social distancing/dynamical lockdown for the development of the high-risk infected people are shown. Because of the increasing number of infected people after the 40-day lockdown we simulated a step-wise return to normality. After the 40-day lockdown follow two 40-day periods with 60% and 80% respectively of normality. The result of this simulation is shown in fig. 38.



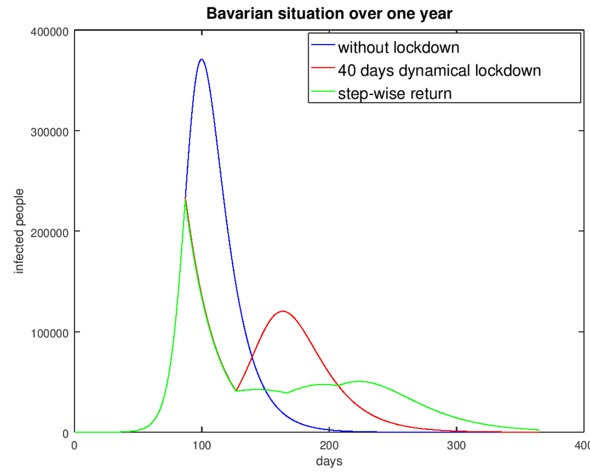


Figure 37: Bavarian one year course for the high-risk people

Figure 38: Bavarian one year course for the high-risk people, green curve for the step-wise return to normality

6 Closing remarks

If we write (2) or (8) resp. in the form

$$\frac{dI}{dt} = (\kappa\beta\frac{S}{N} - \gamma)I$$

we realize that the number of infected people decreases if

$$\kappa\beta\frac{S}{N} - \gamma < 0 \Longleftrightarrow S < N\frac{\gamma}{\kappa\beta} \tag{12}$$

is complied. The relation (12) shows that there are two possibilities for the rise of infected people to be inverted and the medical burden to be reduced.

a) The reduction of the stock of the species S. This can be obtained by immunization or vaccination. Another possibility is the isolation of high-risk people (70 years and older). Positive tests for antibodies reduce the stock of susceptible persons.

b) A second possibility is the reduction of the infection rate $\kappa\beta$. This can be achieved by strict lockdowns, social distancing at appropriate times, or rigid sanitarian moves.

The results are pessimistic in total with respect to a successful fight against the COVID-19-virus. Hopefully the reality is a bit more merciful than the mathematical model. But we rather err on the pessimistic side and be surprised by more benign developments.

Note again that the parameters β and κ are guessed very roughly. Also, the percentage α of the group of high-risk people is possibly overestimated. Depending on the capabilities and performance of the health system of the respective countries, those parameters may look different. The interpretation of κ as a random variable is thinkable, too.

At the end we have to point to a second bump of the course of infected people as an important issue of a limited lockdown. This must be respected in all decisions of physicians and politicians in connection with the handling of the pandemic. But the simulations for Bavaria pictured in fig. 38 show that there are step-wise return strategies, which can reduce second ramps of the course of infected people.

References

[1] W.O. Kermack and A.G. McKendrick, A contribution to the mathematical theory of epidemics. Proc. R. Soc. London A 115(1927)700.

[2] Bulletins of the European Centre for Disease Prevention and Control (https://www.ecdc.europa.eu/en/geographical-distribution-2019-ncov-cases) 2020.

[3] Bulletins of the John Hopkins University of world-wide Corona data (https://www.jhu.edu) 2020.

[4] G. Bärwolff, Numerics for engineers, physicists and computer scientists (3rd ed., in German). Springer-Spektrum 2020.

[5] G. Bärwolff, A Contribution to the Mathematical Modeling of the Corona/COVID-19 Pandemic. medRxiv.preprint 2020, doi: https://doi.org/10.1101/2020.04.01.20050229.

[6] Toshihisa Tomie, Understandig the present status and forcasting of COVID-19 in Wuhan. medRxiv.preprint 2020.