

A Discrete Parametrized Surface Theory in \mathbb{R}^3

Tim Hoffmann¹, Andrew O. Sageman-Furnas^{2,*}, and Max Wardetzky²

¹Lehrstuhl für Geometrie und Visualisierung, Technische Universität München, Boltzmannstrasse 3, 85748 Garching bei München, Germany and ²Institute für Numerische und Angewandte Mathematik, Georg-August-Universität Göttingen, Lotzestrasse 16-18, 37083 Göttingen, Germany

*Correspondence to be sent to: e-mail: aosafu@mathematik.uni-goettingen.de

We propose a discrete surface theory in \mathbb{R}^3 that unites the most prevalent versions of discrete special parametrizations. Our theory encapsulates a large class of discrete surfaces given by a Lax representation and, in particular, the one-parameter associated families of constant curvature surfaces. Our theory is not restricted to integrable geometries, but extends to a general surface theory.

1 Introduction

A *quad net* is a map from a strongly regular polytopal cell decomposition of a surface with all faces being quadrilaterals into \mathbb{R}^3 with nonvanishing straight edges. A polytopal cell decomposition is *strongly regular* if each edge connects distinct vertices and meets at most two faces. Notice, in particular, that nonplanar faces are admissible. In *discrete differential geometry*, quad nets are understood as discretizations of parametrized surfaces [9, 10, 13]. In this agenda many classes of special surfaces have been discretized using algebro-geometric approaches for integrable geometry—originally using discrete analogues of soliton theory techniques (e.g., discrete Lax pairs and finite-gap integration [6]) to construct nets, but more recently using the notion of 3D consistency (reviewed

Received September 30, 2015; Revised January 27, 2016; Accepted January 28, 2016

in [10]). As in the smooth setting, these approaches have been successfully applied to space forms (see, e.g., [15–17, 22]). As an example consider the case of K-surfaces, i.e., surfaces of constant negative Gauß curvature. The integrability equations of classical surface theory are equivalent to the famous sine-Gordon equation [1, 20]. In an integrable discretization, the sine-Gordon equation becomes a finite difference equation for which integrability is encoded by a certain closing condition around a 3D cube. Both in the smooth and discrete settings, integrability is bound to specific choices of parameterizations, such as asymptotic line parametrizations for K-surfaces. In this way different classes of surfaces, such as minimal surfaces or surfaces of constant mean curvature (CMC), lead to different partial differential equations and give rise to different parametrizations. In the discrete case, this is reflected by developments that treat different special surfaces by disparate approaches. These integrable discretizations maintain characteristic properties of their smooth counterparts (e.g., the transformation theory of Darboux, Bäcklund, Bianchi, etc.). What has been lacking, however, is a unified discrete theory that lifts the restriction of special surface parametrizations. Indeed, different from the case of classical smooth surface theory, existing literature does not provide a general discrete theory for quad nets.

We propose a theory that encompasses the most prevalent versions of existing discrete special parametrizations (reviewed in [8]), such as discrete conjugate nets [39], discrete (circular) curvature line nets [5, 18, 32], discrete isothermic nets [7, 11], and discrete asymptotic line nets [38, 43]. Our approach provides a curvature theory that, in particular, yields appropriate curvatures for previously defined discrete minimal [7], discrete CMC [8, 23, 34], discrete constant negative Gauß curvature [6, 24, 35, 38, 43], and discrete developable surfaces [29]. This theory not only retrieves the curvature definitions given in [14, 40] in the case of planar faces but extends to the general setting of nonplanar quads. Moreover, for the first time, it provides a way to understand the one-parameter *associated families* of discrete surfaces of constant curvature, both in terms of discrete curvature and discrete conformality.

To each vertex of a quad net we associate a unit vector, which we view as a unit normal or Gauß map. Then the fundamental property of our approach is the following *edge-constraint* that couples discrete surface points and normals: the average normal along an edge is perpendicular to that edge. This condition arises from a Steiner-type (i.e., offset and mixed area) perspective on curvature and, while surprisingly elementary, has profound consequences for the theory. By introducing a Gauß map for general nonplanar quad nets, our theory builds on basic construction principles of the classical smooth setting.

The paper is organized as follows: after the definition of edge-constraint nets (Section 2) we introduce their curvatures, naturally extending the work of Schief [40] and Bobenko, Pottmann, and Wallner [14]. These curvatures are then shown to be consistent with first, second, and third fundamental forms for edge-constraint nets. We describe the classical discrete integrable surfaces of constant curvature (circular minimal, circular CMC, and asymptotic and circular K-nets) and show that they are indeed edge-constraint nets of constant curvature (Section 3). Even more, we show that they possess associated families that are also edge-constraint nets exhibiting constant curvature. In particular, the proof for CMC nets shows a rather unexpected connection between their 3D compatibility cube and the general Bianchi permutability cube for discrete curves [25, 42]. The section closes with a discussion of discrete developable nets. We then provide a short treatment on how discrete conformality is represented in our theory (Section 4), showing that the members of the associated family of minimal nets are conformally equivalent. We conclude (Section 5) by showing that a rather general class of nets generated by a Sym–Bobenko formula is in fact a subset of edge-constraint nets.

2 Edge-Constraint Nets

2.1 Setup

A natural discrete analogue of a parametrized surface patch is a map from $\mathbb{Z}^2 \rightarrow \mathbb{R}^3$ corresponding to a single chart. To consider discrete atlases, we relax the combinatorial restrictions and think more generally of maps from *quadrilateral graphs*. We will use the words quadrilateral and quad interchangeably.

Definition 2.1 (Quadrilateral net). A *quad graph* \mathbb{G} is a strongly regular polytopal cell decomposition of a surface with all faces being quadrilaterals. A map $f : \mathbb{G} \rightarrow \mathbb{R}^3$ with nonvanishing straight edges is called a (*quad*) *net*. \square

Remark 2.2 (Shift notation). As shown in Figure 1, we use *shift notation* [10] to describe the points of a quad net: when the underlying quad graph has the combinatorics of \mathbb{Z}^2 , we denote a point by $f = f_{k,l}$ for some $k, l \in \mathbb{Z}$ and define the shift operators $f_1 := f_{k+1,l}$ and $f_2 := f_{k,l+1}$. The point diagonal to f is given by a shift in each direction, $f_{12} := f_{k+1,l+1}$. In what follows we do not restrict ourselves to the combinatorics of \mathbb{Z}^2 , but continue to use shift notation, as there is no ambiguity when the discussion is restricted to a point f ; oriented edge $f_i - f$ with $i = 1, 2$; or quad (f, f_1, f_{12}, f_2) . \square

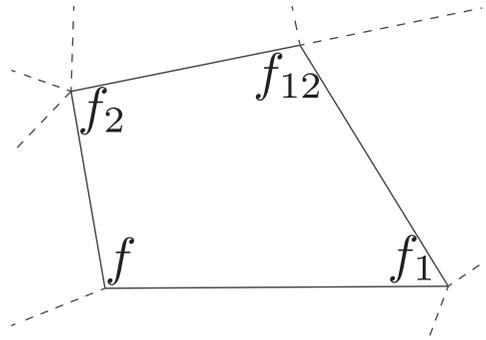


Fig. 1. Shift notation used to describe the points of a quad net f .

Immersed parametrized surfaces in the smooth setting can be thought of either as a smooth family of points or as the envelope of a family of tangent planes. One defines a *contact element* at a point $p \in \mathbb{R}^2$ of a parametrized surface $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as the pair $(f(p), P(p))$, consisting of a point $f(p)$ and the oriented tangent plane $P(p)$ passing through it. $P(p)$ is completely determined by its unit normal $n(p)$ when anchored at $f(p)$; considering $n(p)$ at the origin defines the *Gauß map* $n : D \subset \mathbb{R}^2 \rightarrow \mathbb{S}^2$. Using this perspective we consider parametrized surfaces as the pair of maps (f, n) , an immersion together with its Gauß map.

Analogously we consider discrete parametrized surfaces not as a single quad net, but as a pair of nets that are weakly coupled, mimicking the relationship between an immersion and its Gauß map in the smooth setting. We call this pair an *edge-constraint net*, which is our main object of study.

Definition 2.3 (Edge-constraint net). Let \mathbb{G} be a quad graph. We call a pair of quadrilateral nets $(f, n) : \mathbb{G} \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ a *contact element net*. A contact element net is called an *edge-constraint net* if it satisfies the following:

Edge-constraint: For each pair of points of f connected by an edge, the average of the normals at those points is perpendicular to the edge, i.e. for $i = 1, 2$ we have $f_i - f \perp \frac{1}{2}(n_i + n)$.

We further assume that f contains no vanishing edges, i.e., that $f_i - f$ is always nonzero.

The maps $f : \mathbb{G} \rightarrow \mathbb{R}^3$ and $n : \mathbb{G} \rightarrow \mathbb{S}^2$ are called the (*discrete*) *immersion* and *Gauß map*, respectively. \square

Remark 2.4. We refer to edge-constraint nets using the combinatorics of the underlying quad graph, i.e., an *edge-constraint net quad* is four immersion points together with their Gauß map normals. \square

The edge-constraint discretizes a coupling between the Gauß map and immersion; in the smooth setting, it is generic in the following sense.

Lemma 2.5. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ parametrize a smooth surface patch with Gauß map $n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$. For every point $p \in \mathbb{R}^2$ and unit vector $v \in \mathbb{R}^2$, let the images of the line $p + tv$ where $t \in \mathbb{R}$ be given (with a slight abuse of notation) by $f(t) := f(p + tv)$ and $n(t) := n(p + tv)$, respectively. Then the central and one-sided difference approximations to the edge-constraint along $f(t)$ are satisfied up to second order, i.e.,

$$\begin{aligned} (f(\epsilon) - f(-\epsilon)) \cdot (n(\epsilon) + n(-\epsilon)) &= 0 + O(\epsilon^3) \text{ and} \\ (f(\epsilon) - f(0)) \cdot (n(\epsilon) + n(0)) &= 0 + O(\epsilon^3). \end{aligned} \tag{1}$$

\square

Proof. $f'(t) \cdot n(t) = 0$ by construction, so in particular $f''(t) \cdot n(t) + f'(t) \cdot n'(t) = 0$. The statement then follows by Taylor expanding $f(t)$ and $n(t)$ around $t = 0$. \blacksquare

The simplest class of edge-constraint nets is that given by quad nets in spheres.

Lemma 2.6. Let $f : \mathbb{G} \rightarrow r\mathbb{S}^2$ be a quad net in the sphere of radius $r > 0$. Then f together with the Gauß map $n = f/r$ is an edge-constraint net. \square

Edge-constraint nets naturally exhibit offset nets by adding multiples of the Gauß map to the original immersion while keeping the Gauß map fixed. This observation is the foundation of their curvature theory.

Lemma 2.7 (Offset nets). For any $t \in \mathbb{R}$ and contact element net $(f, n) : \mathbb{G} \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$, the contact element net $(f + tn, n)$, where linear combinations are taken on vertices, is an edge-constraint net if and only if (f, n) is an edge-constraint net. \square

2.2 Curvatures from offsets

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a smooth parametrized surface with Gauß map $n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$. For each $t \in \mathbb{R}$ and $p = (x, y) \in \mathbb{R}^2$, we define the offset surface by $f^t(x, y) := f(x, y) + tn(x, y)$. We only consider smooth parametrizations f which give rise to smooth offsets f^t for small enough $|t|$. It is easily seen that n is also the Gauß map for the offset surface f^t .

For all $p = (x, y) \in \mathbb{R}^2$, the area element at $f^t(x, y)$ can be expressed in terms of the area element, mean, and Gauß curvatures at $f(x, y)$. This relationship is known as the *Steiner formula* and is best understood through the *mixed area form*.

Definition 2.8 (Mixed area form). Let $g, h : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ parametrize two smooth surfaces that share a Gauß map $N : \mathbb{R}^2 \rightarrow \mathbb{S}^2$. For every $p = (x, y) \in \mathbb{R}^2$, we define the *mixed area form* in the tangent plane $P \perp N(x, y)$ by

$$A(g, h) := \frac{1}{2}(\det(g_x, h_y, N) + \det(h_x, g_y, N)), \quad (2)$$

where subscripts denote partial derivatives. When $g = h$, the mixed area form reduces to the area element of g . \square

Remark 2.9. To define the mixed area form we switched notation to capital N , as opposed to little n , for the Gauß map. While in the smooth setting these two objects coincide, the discrete Gauß map for an edge-constraint net (also denoted by n) lives on vertices, whereas we define the mixed area form for an edge-constraint net on faces. We define a new unit vector per face, which we call the *projection direction* (denoted by N), that defines the tangent plane P where this mixed area form lives. \square

We now state the Steiner formula and consequently define the mean and Gauß curvature functions on a smooth parametrized surface. The same definitions carry over to edge-constraint nets.

Theorem 2.10 (Steiner formula). Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a smooth surface parametrization with Gauß map $n : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ and offset surface $f^t : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Then for each $p = (x, y) \in \mathbb{R}^2$ the following relationship holds

$$\begin{aligned} A(f^t, f^t) &= A(f, f) + 2tA(f, n) + t^2A(n, n), \\ &= (1 + 2t\mathcal{H} + t^2\mathcal{K})A(f, f), \end{aligned} \quad (3)$$

defining

$$\mathcal{H} := \frac{A(f, n)}{A(f, f)} \text{ and } \mathcal{K} := \frac{A(n, n)}{A(f, f)}, \quad (4)$$

as the mean and Gauß curvature functions, respectively. \square

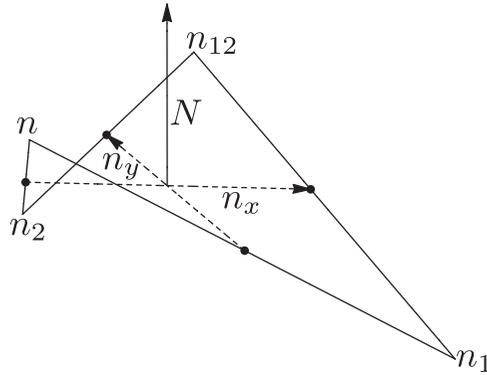


Fig. 2. The partial derivatives, n_x and n_y , of a (possibly nonplanar) quadrilateral from the Gauß map n of an edge-constraint net as defined by the midpoint connectors. A *projection direction*, $N \perp n_x, n_y$ is also shown, determining the face tangent plane P where the mixed area form is defined.

Definition 2.11. Consider a single quadrilateral from an edge-constraint net (f, n) . We define the partial derivatives as the midpoint connectors of the (possibly nonplanar) quadrilaterals for each of f and n , for example, for the Gauß map, as shown in Figure 2, we have

$$n_x := \frac{1}{2}(n_{12} + n_1) - \frac{1}{2}(n_2 + n), \quad n_y := \frac{1}{2}(n_{12} + n_2) - \frac{1}{2}(n_1 + n), \quad (5)$$

and likewise for f . Then the set of admissible *projection directions* is defined as

$$U := \{N \in \mathbb{S}^2 \mid N \perp \text{span}\{n_x, n_y\}\}. \quad (6)$$

Generically, the projection direction is unique (up to sign) and we choose

$$N := \frac{n_x \times n_y}{\|n_x \times n_y\|}. \quad (7)$$

In the (*quad*) *tangent plane* $P \perp N$ we define the (*discrete*) *mixed area form* via Equation (2), yielding a Steiner formula (Equation (3)). \square

Remark 2.12 (Degenerate Gauß maps). The set of admissible projection directions also generates a consistent curvature theory in the degenerate situation where the partial derivatives of the Gauß map are not linearly independent. Degenerate Gauß maps naturally arise in the theory of developable surfaces, so we defer this discussion to the theory of developable edge-constraint nets in Section 3.4. \square

Definition 2.13 (Edge-constraint curvatures). Using the Steiner formula (Equation (3)) we define *mean* and *Gauß curvatures* for edge-constraint nets via Equation (4). \square

Remark 2.14 (Degenerate immersions). Clearly, the curvatures are only well defined when the immersion has nonvanishing area $A(f, f) \neq 0$. Every statement we make about curvatures assumes the immersion quad has nonvanishing area. \square

Remark 2.15 (Sign of projection direction). The sign of the projection direction in the generic setting does not correspond to a change in local orientation. The mixed area form changes sign, but the mean and Gauß curvatures are invariant to this choice. However, flipping the Gauß map ($n \rightarrow -n$) changes the sign of the mean curvature, as expected. \square

Remark 2.16 (Choice of partial derivatives). The choice to define the partial derivatives as the midpoint connectors is to guide intuition. In fact, one has the freedom to choose *any linear combination* of the midpoint connectors to be the partial derivatives, as long as *the same combination* is chosen for both the immersion and the Gauß map. This corresponds to the freedom to locally reparametrize in the smooth setting. The mean and Gauß curvatures and forthcoming definitions of principal curvatures and curvature line fields are all invariant to this choice. As expected, the mixed area forms and fundamental forms will change as these are not invariant to a local reparametrization in the smooth setting either. \square

In the smooth setting, one can also derive the mean and Gauß curvatures at a point via the fundamental forms and shape operator living in the tangent plane to that point. The shape operator is a real self-adjoint linear operator whose eigenvalues and eigenvectors are the principal curvatures and curvature lines, respectively. From Definition 2.11, we can discretize the fundamental forms and shape operator in the plane perpendicular to the projection direction of each face of an edge-constraint net.

Definition 2.17 (Fundamental forms). Consider a single quad from an edge-constraint net (f, n) . Let π be the projection into the quad tangent plane P . Set $\hat{f}_x := \pi(f_x) = f_x - (f_x \cdot N)N$ and similarly for \hat{f}_y . Since $N \perp n_x, n_y$ by Definition 2.11, $\hat{n}_x = n_x$ and $\hat{n}_y = n_y$. We define the *fundamental forms* and *shape operator* by:

$$\begin{aligned} \text{I} &:= \begin{pmatrix} \hat{f}_x \cdot \hat{f}_x & \hat{f}_x \cdot \hat{f}_y \\ \hat{f}_y \cdot \hat{f}_x & \hat{f}_y \cdot \hat{f}_y \end{pmatrix}, & \text{II} &:= \begin{pmatrix} \hat{f}_x \cdot n_x & \hat{f}_x \cdot n_y \\ \hat{f}_y \cdot n_x & \hat{f}_y \cdot n_y \end{pmatrix}, \\ \text{III} &:= \begin{pmatrix} n_x \cdot n_x & n_x \cdot n_y \\ n_y \cdot n_x & n_y \cdot n_y \end{pmatrix}, & S &:= \text{I}^{-1}\text{II}. \end{aligned} \tag{8}$$

The eigenvalues (k_1, k_2) and eigenvectors of the shape operator S are the *principal curvatures* and *curvature line fields* of the quadrilateral. \square

The existence of principal curvatures and curvature line fields follows from the symmetry of the second fundamental form:

Lemma 2.18. Consider a single quad from an edge-constraint net (f, n) , then the second fundamental form is symmetric. \square

Proof. In the above notation, we need to show $\hat{f}_x \cdot n_y - n_x \cdot \hat{f}_y = 0$. As $N \perp n_x, n_y$ this quantity is the same when unprojected, i.e.,

$$\hat{f}_x \cdot n_y - n_x \cdot \hat{f}_y = f_x \cdot n_y - n_x \cdot f_y. \tag{9}$$

Expanding out $f_x \cdot n_y - n_x \cdot f_y$ one finds it is a constant multiple of the sum of the edge-constraint conditions once around the quadrilateral, which vanishes as it vanishes on each edge by assumption. \blacksquare

The mean and Gauß curvatures per quad defined via the Steiner formula are equal to the ones derived from the eigenvalues of the shape operator:

Lemma 2.19 (Curvature and fundamental form relationships). The following relations hold true in the smooth and discrete cases:

1. $\mathcal{K} = k_1 k_2,$
2. $\mathcal{H} = \frac{1}{2} (k_1 + k_2),$
3. $\text{III} - 2\mathcal{H} \text{II} + \mathcal{K} \text{I} = 0,$ and
4. $A(f, f)^2 = \det \text{I}.$ \square

Example 2.20 (Spherical edge-constraint nets). Let $(f, f/r)$ be an edge-constraint net in the sphere of radius $r > 0$ as determined by Lemma 2.6. Then every quadrilateral has the expected Gauß ($\mathcal{K} = \frac{1}{r^2}$) and mean curvature ($\mathcal{H} = \frac{1}{r}$). \square

Example 2.21 (Curvature line fields). Figure 3 shows the curvature line fields of an ellipsoid in the smooth and discrete settings. \square

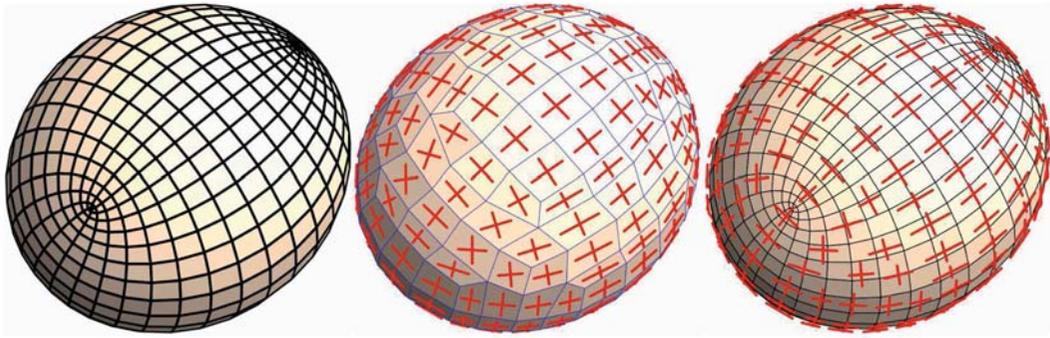


Fig. 3. Left: Curvature lines of a smooth ellipsoid. Middle: Curvature line field of an ellipsoid edge-constraint net. Right: Overlay of smooth and discrete curvature line fields.

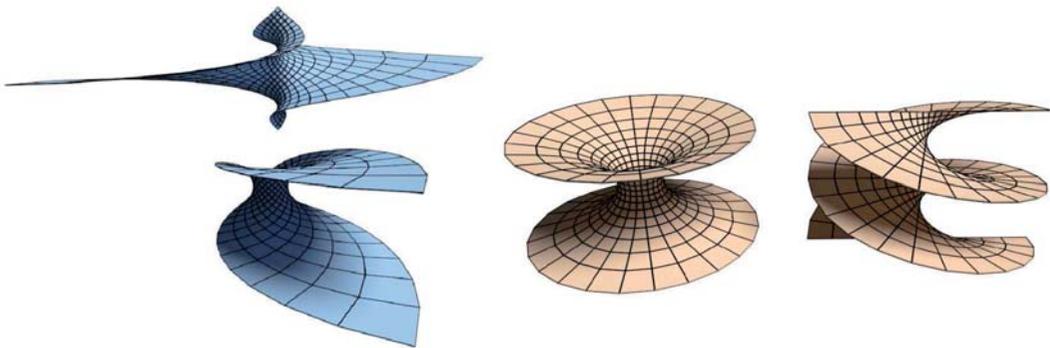


Fig. 4. Two pairs of minimal edge-constraint nets from the same associated family generated from Weierstrass data. Left: Circular net helicoid and an A-net catenoid. Right: Circular net catenoid and A-net helicoid.

3 Constant Curvature Nets

Edge-constraint nets and their curvature theory provide a unifying geometric framework through which to understand previously defined notions of discrete surfaces of constant curvature in special parametrizations. Due to their governing integrable structure, these surfaces naturally arise in one-parameter *associated families* that in the smooth setting fix the respective curvature, but change the type of parametrization. Previous notions of discrete curvature exist for each particular type of special parametrization, but have been difficult to reconcile with the corresponding (differently parametrized) associated families (examples of which are shown in Figure 4).

In what follows we rectify these discrepancies by showing that the algebraically constructed discrete isothermic minimal surfaces [7], discrete isothermic CMC surfaces

[7], discrete asymptotic line constant negative Gauß curvature surfaces [6], and discrete curvature line constant negative Gauß curvature surfaces [28], *together with each of their respective associated families* are in fact edge-constraint nets with their respective curvatures constant. To close we introduce a theory of developable edge-constraint nets, a non-integrable example.

3.1 Discrete minimal surfaces

Definition 3.1 (Minimal edge-constraint net). An edge-constraint net (f, n) is called *minimal* if every quad has vanishing mean curvature ($\mathcal{H} = 0$), i.e., the mixed area $A(f, n)$ vanishes. \square

In the smooth setting, minimal surfaces are often parametrized by isothermic (curvature line and conformal) coordinates arising naturally from their construction from holomorphic Weierstrass data: stereographically project a holomorphic function $g : \mathbb{C} \rightarrow \mathbb{C}$ on to the Riemann sphere to get a conformal map $n : \mathbb{C} \rightarrow \mathbb{S}^2$. Now, think of n as the Gauß map to a surface and construct the *Christoffel dual* isothermic surface $f : \mathbb{C} \rightarrow \mathbb{R}^3$ by integrating

$$f_x(x, y) = \frac{n_x(x, y)}{\|n_x(x, y)\|^2} \quad \text{and} \quad f_y(x, y) = -\frac{n_y(x, y)}{\|n_y(x, y)\|^2}. \quad (10)$$

The resulting f is an isothermic parametrization of a minimal surface in \mathbb{R}^3 with Gauß map given by the conformal map n . This process of generating a minimal surface is called the Weierstrass representation.

Bobenko and Pinkall defined discrete minimal surfaces as a special case of discrete isothermic surfaces and showed they exhibit a discrete Weierstrass representation [7]. These nets indeed have vanishing mean curvature in a curvature theory for nets with planar faces (that in the case of contact element nets is contained in the present theory) [14, 40].

In complete analogy to the smooth case, one can extend this representation into an associated family. This corresponds to locally rotating the frame, therefore changing the type of parametrization away from being curvature line (while staying conformal in the smooth setting). While this is an algebraic way to define the discrete nets of the associated family there has been no notion through which one can understand their minimality. The goal of this section is to rectify this by showing that every member of the associated family is an edge-constraint net and that its mean curvature vanishes on every quad.

Formulating the discrete Weierstrass representation requires discrete analogues of curvature line parametrizations, Christoffel duals, and isothermic parametrizations. We briefly introduce these notions, but emphasize that each of these discrete objects is interesting in its own right (see the book by Bobenko and Suris [10]).

Definition 3.2 (Circular net). A contact element net (f, n) is called a *circular net* or *discrete curvature line net* if:

1. every quad of the immersion f is *circular*, its vertices lie on a circle; and
2. the Gauß map along each edge is found by reflection through the immersion edge perpendicular bisector plane, i.e., for $i = 1, 2$ we have

$$n = n_i - 2 \frac{n_i \cdot (f_i - f)}{\|f_i - f\|^2} (f_i - f). \quad (11)$$

Lemma 3.3. Let (f, n) be a circular net, then it is an edge-constraint net. □

Proof. Equation (11) gives: $f_i - f \parallel n_i - n \perp n_i + n$. ■

The symmetry imposed by the second property implies that the Gauß map and all offset nets $(f + tn, n)$ are also circular nets, with corresponding quads lying in parallel planes.

Definition 3.4 (Isothermic net). Let (f, n) be a circular net. Then (f, n) is a *discrete isothermic net* if there exists a second circular net (f^*, n) with the same Gauß map such that $A(f, f^*) = 0$. The net (f^*, n) is unique (up to scaling and translation) and called the *discrete Christoffel dual net* of (f, n) . □

We now state a few important properties of discrete isothermic nets that we require in the following [11].

Lemma 3.5. Let (f, n) be a discrete isothermic net with Christoffel dual (f^*, n) . Then the following hold:

1. There exist real values β per edge that coincide for opposite edges on each quad and the cross-ratio of every quad factorizes, i.e.,

$$\frac{(f_1 - f)(f_{12} - f_2)}{(f_{12} - f_1)(f_2 - f)} = \frac{\beta_1}{\beta_2}, \quad (12)$$

with $\beta_1 = \beta_{f_1-f}$ and $\beta_2 = \beta_{f_2-f}$ associated with shifts in the first and second lattice directions, respectively.

2. Corresponding edges of f and f^* are parallel and satisfy:

$$f_i^* - f^* = \beta_i \frac{f_i - f}{\|f_i - f\|^2} \text{ for } i = 1, 2, \quad (13)$$

while non-corresponding diagonals are parallel and satisfy:

$$f_{12}^* - f^* = (\beta_2 - \beta_1) \frac{f_2 - f_1}{\|f_2 - f_1\|^2} \text{ and } f_2^* - f_1^* = (\beta_2 - \beta_1) \frac{f_{12} - f}{\|f_{12} - f\|^2}. \quad (14) \quad \square$$

For the rest of this section we restrict the discussion to the special case of discrete isothermic nets whose immersion quads have cross-ratio minus one, in particular, we assume that $\beta_1 = 1$ and $\beta_2 = -1$ for every quad. The reason for doing this is that if we think of the cross-ratio as a discrete analog of f_x^2/f_y^2 , then cross-ratio minus one corresponds to $f_x^2 = -f_y^2$, the square of the defining property of conformal maps $f_x = if_y$; the more general notion of factorizing cross-ratio allows for reparametrizations of the parameter lines.

It is essential to emphasize that the restriction to cross-ratio minus one solely serves the purpose of simplifying the algebra. Every result that follows also holds with the more general definition, with the pre-factors β_1 and β_2 cropping up in the expected places.

The notion of a discrete holomorphic function just restricts the notion of discrete isothermicity to the plane.

Definition 3.6. A complex function $g : \mathbb{Z}^2 \rightarrow \mathbb{C}$ is called a *discrete holomorphic function* if every quad of its image has cross-ratio minus one. \square

We now define the discrete Weierstrass representation by following the same procedure as in the smooth case.

Definition 3.7 (Weierstrass representation of discrete isothermic minimal nets). Let $g : \mathbb{Z}^2 \rightarrow \mathbb{C}$ be a discrete holomorphic function and consider the discrete isothermic net (f, n) where:

1. the Gauß map n is given by the stereographic projection of the holomorphic data g , i.e.,

$$\begin{aligned}
 n &:= \frac{1}{1 + |g|^2} \left(g + \bar{g}, \frac{1}{i}(g - \bar{g}), |g|^2 - 1 \right), \text{ so in particular} \\
 n_i &:= \frac{1}{1 + |g_i|^2} \left(g_i + \bar{g}_i, \frac{1}{i}(g_i - \bar{g}_i), |g_i|^2 - 1 \right) \text{ for } i = 1, 2;
 \end{aligned}
 \tag{15}$$

2. and f is given (up to translations) as the discrete isothermic dual immersion of n , i.e., for a shift in either direction

$$\begin{aligned}
 f_1 - f &:= \frac{n_1 - n}{\|n_1 - n\|^2} = \Re \left(\frac{1}{2(g_1 - g)} (1 - g_1 g, i(1 + g_1 g), g_1 + g) \right) \text{ and} \\
 f_2 - f &:= -\frac{n_2 - n}{\|n_2 - n\|^2} = -\Re \left(\frac{1}{2(g_2 - g)} (1 - g_2 g, i(1 + g_2 g), g_2 + g) \right).
 \end{aligned}
 \tag{16}$$

We call the arising net a *discrete isothermic minimal net*. □

Lemma 3.8. Let (f, n) be a discrete isothermic minimal net. Then it is a minimal edge-constraint net. □

Proof. Discrete isothermic nets are circular nets, so they are edge-constraint nets and by construction (f, n) is the Christoffel dual of (n, n) , so $A(f, n) = 0$. ■

In the smooth setting the Weierstrass representation provides a way to compute the Gauß curvature explicitly and for isothermic surfaces is given by $\mathcal{K} = \frac{-4|g'|^4}{(1+|g|^2)^4}$, see [33]. This has a discrete analogue:

Lemma 3.9. Let (f, n) be a discrete isothermic minimal net arising from a discrete holomorphic function $g : \mathbb{Z}^2 \rightarrow \mathbb{C}$. Then the Gauß curvature of a quad is given in terms of g by:

$$\mathcal{K} = \frac{-4(|g_{12} - g||g_2 - g_1|)^2}{(1 + |g|^2)(1 + |g_1|^2)(1 + |g_{12}|^2)(1 + |g_2|^2)}.
 \tag{17}$$

□

Proof. Again we use the diagonals as the discrete partial derivatives. By Equation (14) we have $A(f, f) = -\frac{4}{\|n_x\|^2 \|n_y\|^2} A(n, n)$. Recall that the chordal distance between the stereographic projection of two points w and z in \mathbb{C} is

$$\frac{2}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} |z - w|.
 \tag{18}$$

Applying this to $\|n_x\|^2$ and $\|n_y\|^2$ explicitly recovers the result. ■

The discrete Weierstrass representation naturally gives rise to an associated family. We think of the discrete Weierstrass representation in terms of the corresponding *discrete complex Weierstrass vectors*

$$\omega_i = \frac{1}{2(g_i - g)}(1 - g_i g, i(1 + g_i g), g_i + g) \tag{19}$$

for each lattice direction $i = 1, 2$ so that the immersion edges can be concisely written: $f_1 - f = \Re\omega_1$ and $f_2 - f = -\Re\omega_2$.

Definition 3.10 (Associated family of discrete isothermic minimal net). Let $g : \mathbb{Z}^2 \rightarrow \mathbb{C}$ be a discrete holomorphic function, $n : \mathbb{Z}^2 \rightarrow \mathbb{S}^2$ be the stereographic projection of g , ω_i be the discrete complex Weierstrass vectors of g , and $\lambda = e^{i\alpha}$ for $\alpha \in [0, 2\pi]$. Then, the family of contact element nets (f^α, n) given by:

$$f_1^\alpha - f^\alpha := \Re(\lambda\omega_1) \text{ and } f_2^\alpha - f^\alpha := -\Re(\lambda\omega_2) \tag{20}$$

is called the *associated family* of the discrete isothermic minimal net (f^0, n) and λ is called the *spectral parameter*. □

Lemma 3.11. Let (f^0, n) be a discrete isothermic minimal net. Any member (f^α, n) of its associated family is an edge-constraint net as well. □

Proof. By formally extending the dot product to complex 3-vectors (i.e., $(z_1, z_2, z_3) \cdot (w_1, w_2, w_3) := \sum_{i=1}^3 z_i w_i$), and noting that the Gauß map $n \in \mathbb{R}^3$ is real valued we have for $i = 1, 2$ that $\Re(\lambda\omega_i) \cdot n = \Re(\lambda\omega_i \cdot n)$ and $\Re(\lambda\omega_i) \cdot n_i = \Re(\lambda\omega_i \cdot n_i)$. We easily compute $\Re(\lambda\omega_1 \cdot n) = -\Re(\frac{\lambda}{2})$ and $\Re(\lambda\omega_1 \cdot n_1) = \Re(\frac{\lambda}{2})$. Similarly, $\Re(\lambda\omega_2 \cdot n) = \Re(\frac{\lambda}{2})$ and $\Re(\lambda\omega_2 \cdot n_2) = -\Re(\frac{\lambda}{2})$. So $(f_i^\alpha - f^\alpha) \cdot n = -(f_i^\alpha - f^\alpha) \cdot n_i$. ■

The previous lemma provides us with a way to interpret the rotation generated by the multiplication of λ directly in \mathbb{R}^3 :

Lemma 3.12. Let (f^α, n) be the associated family of a discrete isothermic minimal net (f^0, n) . Then for each $\alpha \in [0, 2\pi]$ and $i = 1, 2$ we have:

$$f_i^\alpha - f^\alpha = (-1)^{(i-1)} \|f_i^0 - f^0\|^2 (\cos \alpha (n_i - n) - \sin \alpha (n_i \times n)). \tag{21}$$

In other words $f_i^\alpha - f^\alpha$ is given by $f_i^0 - f^0$ rotated in the plane perpendicular to $(n_i + n)$ by angle α , as shown in Figure 5. □

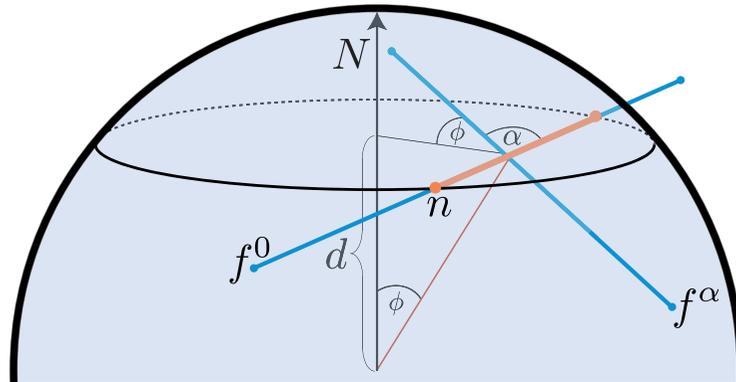


Fig. 5. The geometry of an edge in the associated family Equation (21) along with its original immersion edge (i.e., parallel to its Gauß map edge); the circle is the circumcircle of an adjacent Gauß map quad.

Proof. We show the result for the first lattice direction $i = 1$, the other lattice direction follows similarly.

For every $\alpha \in [0, 2\pi]$ the immersion edge $f_1^\alpha - f^\alpha$ is perpendicular to $(n_1 + n)/2$, so it is a linear combination of $(n_1 - n)$ and $(n_1 \times n)$. Furthermore, in terms of the complex discrete Weierstrass vector, this immersion edge is the linear combination $\Re(\lambda\omega_1) = \cos\alpha\Re\omega_1 - \sin\alpha\Im\omega_1$. Since (f^0, n) and (n, n) are dual discrete isothermic nets we immediately have $\Re\omega_1 = \|f_1^0 - f^0\|^2(n_1 - n)$, so we only have to show $\Im\omega_1 = \|f_1^0 - f^0\|^2(n_1 \times n)$.

A simple computation yields that the formal complex 3-vector dot product $\omega_1 \cdot \omega_1$ is real and equal to $1/4$. In particular, this implies that the real dot product $\Re\omega_1 \cdot \Im\omega_1 = 0$ and that $\|\Im\omega_1\|^2 = \|\Re\omega_1\|^2 - 1/4$. Hence $\Im\omega_1$ is parallel to $n_1 \times n$ (since it is perpendicular to both $(n_1 - n)$ and $(n_1 + n)$). Now, since the Gauß map vectors are unit length we have $\|n_1 \times n\|^2 = \|\frac{n_1+n}{2}\|^2\|n_1 - n\|^2$, which we use to conclude

$$\|\Im\omega_1\|^2 = \frac{1}{\|n_1 - n\|^2} - \frac{1}{4} = \frac{\|n_1 + n\|^2}{4\|n_1 - n\|^2} = (\|f_1^0 - f^0\|^2\|n_1 \times n\|^2)^2. \tag{22}$$

■

This more geometric construction of the associated family highlights the following important relationship (shown in Figure 6), which will lead to vanishing mean curvature.

Lemma 3.13 (Quad geometry of the associated family). Every immersion quad of a member of the associated family (f^α, n) when projected into its corresponding Gauß

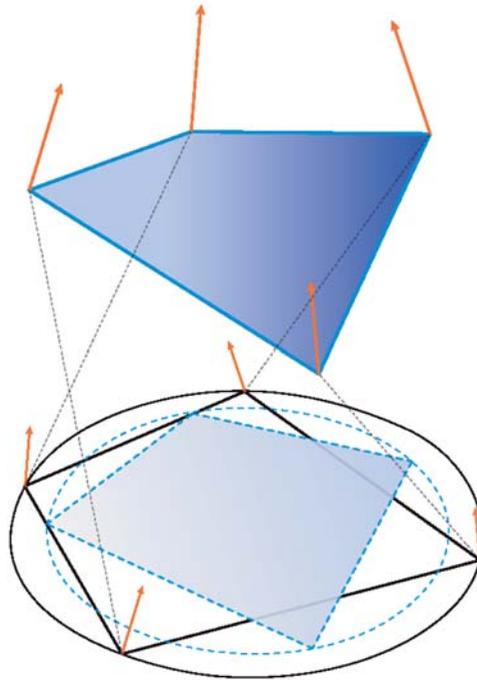


Fig. 6. A nonplanar quad (top) from the associated family of a discrete isothermic minimal net and its projection, which is a rotated and scaled version of the original circular quad (bottom, solid line). The Gauß map is constant.

map plane is a scaled and rotated version of its corresponding circular immersion quad of (f^0, n) . □

Proof. We show that an immersion edge of (f^α, n) when projected into one of its neighboring Gauß map planes is a scaled and rotated version of its corresponding immersion edge of (f^0, n) ; the setup is given in Figure 5. In particular, the scaling factor and rotation angle are independent of the lattice direction, so the result extends to the quads, proving the lemma.

What follows is identical for both lattice directions, so we work with the first one. The Gauß map quad is circular so the projection direction N anchored at the origin passes through its circumcenter (at height d) and is normal to its plane; let π be the projection into this plane. Furthermore, $N \perp (n_1 - n)$ so from Equation (21) we see that

$$\pi(f_1^\alpha - f^\alpha) = \|f_1^0 - f^0\|^2 (\cos \alpha (n_1 - n) - \sin \alpha \cos \phi (n_1 \times n)), \tag{23}$$

where ϕ is the angle between $f_1^\alpha - f^\alpha$ and the Gauß map plane. Observe that the angle between N and $\frac{n_1+n}{2}$ is also ϕ , so we can calculate:

$$\begin{aligned}
 \|\pi(f_1^\alpha - f^\alpha)\|^2 &= \|f_1^0 - f^0\|^4(\cos^2 \alpha \|n_1 - n\|^2 + \sin^2 \alpha \cos^2 \phi \|n_1 \times n\|^2) \\
 &= \|f_1^0 - f^0\|^2 \left(\cos^2 \alpha + \sin^2 \alpha \cos^2 \phi \left\| \frac{n_1 + n}{2} \right\|^2 \right). \\
 &= \|f_1^0 - f^0\|^2(\cos^2 \alpha + \sin^2 \alpha d^2).
 \end{aligned}
 \tag{24}$$

Therefore, the projected edge length is the original edge length scaled by a factor ς that is independent of the lattice direction and rotated by angle θ , i.e.,

$$\varsigma = \sqrt{\cos^2 \alpha + \sin^2 \alpha d^2} \quad \text{and} \quad \cos \theta = \frac{\cos \alpha}{\sqrt{\cos^2 \alpha + \sin^2 \alpha d^2}}.
 \tag{25}$$

■

We now prove the main result of this section.

Theorem 3.14 (Minimality of the associated family). All contact element nets (f^α, n) for $\alpha \in [0, 2\pi]$ in the associated family of a discrete isothermic minimal net (f^0, n) are minimal edge-constraint nets. □

Proof. Choose an arbitrary $\alpha \in [0, 2\pi]$. Consider a single quad of (f^α, n) with projection direction N , projection map π , and partial derivatives given by the diagonals. By Lemma 3.13 there exist a rotation by angle θ in the plane perpendicular to N and a scaling factor ς (both depending on α) that bring the original immersion quad of f^0 into the projected associated family quad of f^α . Noticing that $\cos \alpha$ equals zero or one exactly when $\cos \theta$ is also zero or one, respectively, we write:

$$\begin{aligned}
 \pi(f_x^\alpha) &= \varsigma \|f_x^0\| \left(\cos \theta \frac{f_x^0}{\|f_x^0\|} + \sin \theta \frac{f_y^0}{\|f_y^0\|} \right) \quad \text{and} \\
 \pi(f_y^\alpha) &= \varsigma \|f_y^0\| \left(-\sin \theta \frac{f_x^0}{\|f_x^0\|} + \cos \theta \frac{f_y^0}{\|f_y^0\|} \right).
 \end{aligned}
 \tag{26}$$

The original net (f^0, n) is discrete isothermic so Equation (14) implies that: $\frac{f_x^0}{\|f_x^0\|} = -\frac{n_y}{\|n_y\|}$ and $\frac{f_y^0}{\|f_y^0\|} = -\frac{n_x}{\|n_x\|}$, and that $\|f_x^0\| \|n_y\| = \|n_x\| \|f_y^0\| = 2$. Therefore, we compute twice the mixed area and see that it vanishes:

$$\begin{aligned}
 2A(f^\alpha, n) &= \det(\pi(f_x^\alpha), n_y, N) + \det(n_x, \pi(f_y^\alpha), N) \\
 &= -2\varsigma \left(\det \left(\sin \theta \frac{n_x}{\|n_x\|}, \frac{n_y}{\|n_y\|}, N \right) + \det \left(\frac{n_x}{\|n_x\|}, -\sin \theta \frac{n_y}{\|n_y\|}, N \right) \right) \\
 &= 0.
 \end{aligned}
 \tag{27}$$

■

In contrast to the smooth case, the Gauß curvature does not stay constant in the discrete associated family, but it changes in a controlled way:

Theorem 3.15 (Gauß curvature of the associated family). Let (f^α, n) be contact element net in the associated family of a discrete isothermic minimal net (f^0, n) and let \mathcal{K}^α and \mathcal{K}^0 be their respective Gauß curvatures. Then

$$\mathcal{K}^\alpha = \frac{\mathcal{K}^0}{\zeta^2}, \tag{28}$$

where $\zeta^2 = \cos^2 \alpha + \sin^2 \alpha d^2$ is the square of the scaling factor as above and d is the distance to the circumcenter of the corresponding Gauß map quad.

Additionally, ζ will approach one in the continuum limit. □

Proof. By Lemma 3.13 we have $A(f^\alpha, f^\alpha) = \zeta^2 A(f^0, f^0)$ and $A(n, n)$ is constant throughout the family. ■

As in the smooth case we can define the *conjugate discrete isothermic minimal net* of (f^0, n) as $(f^{\frac{\pi}{2}}, n)$. The members of the associated family are linear combinations of the discrete isothermic net and its conjugate net, since $(f^{\frac{\pi}{2}}, n)$ arises as the imaginary part of the complex Weierstrass vectors. The conjugate net is known to be in asymptotic line parametrization. Discrete analogues of such parametrizations are known as *A-nets* and were originally introduced by Sauer [38] and Wunderlich [43] to investigate surfaces of constant negative Gauß curvature (as we will do in Section 3.3).

Definition 3.16 (Discrete asymptotic net). An edge-constraint net is an *A-net* if its immersion has planar vertex stars (i.e., if all immersion edges meeting at a vertex lie in a common plane) and the Gauß map is given by choosing unit normals to these planes. □

This definition corresponds to the fact that the osculating planes of asymptotic lines are the tangential planes of the surface. A-nets always have nonpositive Gauß curvature. We close by showing that conjugate discrete isothermic minimal nets are indeed A-nets.

Lemma 3.17. The conjugate net $(f^{\frac{\pi}{2}}, n)$ of a discrete isothermic minimal net (f^0, n) is an A-net. □

Proof. From Equation (21) the edges of the conjugate net satisfy:

$$f_1^{\frac{\pi}{2}} - f^{\frac{\pi}{2}} = -\|f_1^0 - f^0\|(n_1 \times n) \text{ and } f_2^{\frac{\pi}{2}} - f^{\frac{\pi}{2}} = \|f_2^0 - f^0\|(n_2 \times n). \quad (29)$$

In particular, this means that all edges emanating from a generic vertex of $f^{\frac{\pi}{2}}$ are perpendicular to the corresponding Gauss map n , which is the definition of an A-net. ■

3.2 Discrete CMC surfaces

Definition 3.18 (CMC edge-constraint net). An edge-constraint net (f, n) is said to have *constant mean curvature* if every quad of the net has the same nonvanishing mean curvature $\mathcal{H} \in \mathbb{R}^*$. □

Like their simpler minimal cousins, smooth CMC surfaces are often described in isothermic parametrizations. In such coordinates the Gauß-Codazzi equation is given in terms of the conformal metric parameter u (defined by $ds^2 = e^u(dx^2 + dy^2)$), and reduces to the integrable elliptic sinh-Gordon equation,

$$\Delta u = \mathcal{H} \sinh u. \quad (30)$$

Techniques of soliton theory have been immensely successful in explicitly constructing and classifying CMC surfaces (e.g., tori [3, 36]) in the classical setting of Euclidean three-space and other space forms. The integrability condition of a suitably gauged frame can be identified with the Lax representation of the integrable equation [4, 41] which harnesses methods from soliton theory for geometry and allows for structure preserving discretizations [10]. Moreover, one can then explicitly describe the immersed surfaces in terms of the so-called Sym–Bobenko formula, which by construction simultaneously generates the *associated family*.

Using this method Bobenko and Pinkall [8] defined discrete CMC surfaces as a subclass of discrete isothermic surfaces, just as they did for discrete minimal surfaces. For smooth CMC surfaces, the DPW method [21] is a Weierstrass type method that allows the construction of all CMC surfaces from holomorphic/meromorphic data. A discrete version of this method giving rise to the same frame description as Bobenko and Pinkall can be found in [23].

As with the minimal case, these discrete *isothermic* surfaces arising from the frame description have previously been shown to have constant mean curvature [14],

but once again, the naturally arising associated family leaves the realm of the special isothermic parametrization (to more general conformal parametrizations in the smooth setting). Thus, there has been no notion of discrete mean curvature through which this family could be geometrically understood. Again, we rectify this by showing that the original discrete isothermic net and its entire algebraically generated associated family are in fact CMC edge-constraint nets.

Since the curvature theory of edge-constraint nets satisfies the Steiner formula Equation (3), the linear Weingarten relationship and its corollary come for free by calculating the curvatures of an offset surface.

Lemma 3.19 (Linear Weingarten relationship). Let (f, n) be an edge-constraint net. Then for any $t \in \mathbb{R}$ the edge-constraint net given by the offset $(f^t, n) = (f + tn, n)$ has curvatures

$$\mathcal{K}_{f^t} = \frac{\mathcal{K}}{1 + 2\mathcal{H}t + \mathcal{K}t^2} \text{ and } \mathcal{H}_{f^t} = \frac{\mathcal{H} + \mathcal{K}t}{1 + 2\mathcal{H}t + \mathcal{K}t^2}, \tag{31}$$

where \mathcal{K}, \mathcal{H} are the Gauß and mean curvatures of the original surface (f, n) . If (f, n) has constant mean curvature, then there exist $\alpha, \beta \in \mathbb{R}$ only depending on t, \mathcal{H} such that

$$\alpha\mathcal{K}_{f^t} + \beta\mathcal{H}_{f^t} = 1. \tag{32}$$

In other words, for each offset net, α, β are constant on all quads of the net. □

Proof. Choose $\alpha = -t(\frac{1}{\mathcal{H}} + t)$ and $\beta = \frac{1}{\mathcal{H}} + 2t$. ■

An important corollary is that CMC edge-constraint nets come in pairs, just like their smooth counterparts (see Figure 7).

Corollary 3.20. Let (f, n) be a CMC edge-constraint net with mean curvature $\mathcal{H} = -\frac{1}{h}$, for some $h \in \mathbb{R}^+$. Then the offset net $(f^*, n) := (f + hn, n)$ is also a CMC edge-constraint net with mean curvature $\mathcal{H}^* = -\mathcal{H}$ and the *middle* edge-constraint net $(\hat{f}, n) := (f + \frac{h}{2}n, n)$ has constant positive Gauß curvature $4\mathcal{H}^2$. □

Remark 3.21. If (f, n) is a discrete isothermic net of CMC $\mathcal{H} = -\frac{1}{h}$, then the offset net $(f + hn, n)$ is in fact the discrete Christoffel dual isothermic net (Definition 3.4). □

For simplicity for the rest of our discussion, we rescale to $\mathcal{H} = -1$.

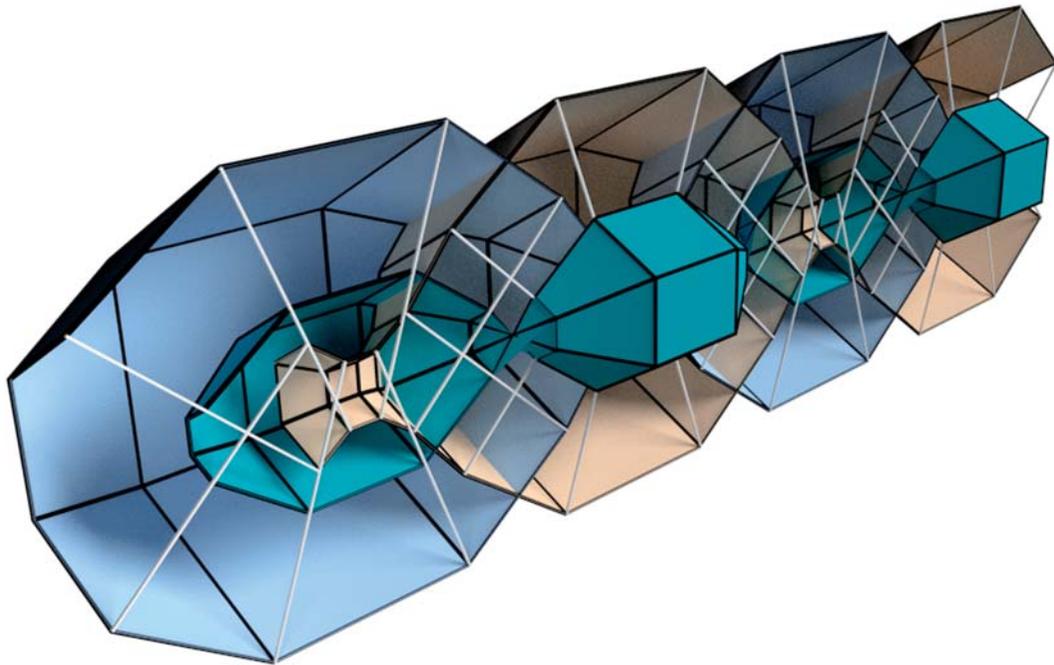


Fig. 7. Two dual Delaunay nets with the positive Gauß curvature net in between. The normal lines connecting them are shown as well.

Lemma 3.22. Let (f, n) be an edge-constraint net with unit offset $(f^*, n) := (f + n, n)$. Consider a single quad, then

$$A(f, f^*) = 0 \iff \mathcal{H} = A(f, n)/A(f, f) = -1. \tag{33}$$

□

In other words, vanishing mixed area between a net and its unit offset for every quad is equivalent to both nets having constant mean curvature. The condition

$$A(f, f^*) = \det(f_{12} - f, f_2^* - f_1^*, N) + \det(f_{12}^* - f^*, f_2 - f_1, N) = 0 \tag{34}$$

can be understood geometrically as the vanishing sum of the (projected) areas of the curves formed by f, f_1^*, f_{12}, f_2^* and f^*, f_1, f_{12}^*, f_2 which we denote g and g^* , respectively:

$$\begin{aligned} g &= f, g_1 = f_1^*, g_{12} = f_{12}, \text{ and } g_2 = f_2^*; \text{ and} \\ g^* &= f^*, g_1^* = f_1, g_{12}^* = f_{12}^*, \text{ and } g_2^* = f_2. \end{aligned} \tag{35}$$

Therefore, to prove that an edge-constraint net has constant mean curvature we switch between the two combinatorial cubes C_f and C_g formed by f, f^* and g, g^* , respectively. They share the same vertex set but the edges of one are the diagonals of the other.

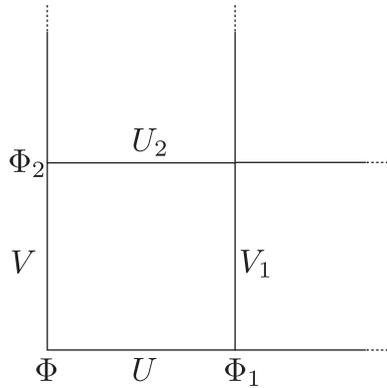


Fig. 8. Shift notation for the frame description. The frame of a net Φ lives on vertices, while the corresponding *Lax matrices* U and V live on edges; for both the frame and Lax matrices, one can understand shifts in either lattice direction.

By showing that the algebraically generated associated family of discrete CMC nets of Bobenko and Pinkall [8] are CMC edge-constraint nets, we find that their C_g cubes are built from skew parallelograms, yielding an unexpected connection to the 3D compatibility cube for discrete curves from the theory of integrable systems [25, 42]. We now briefly recapitulate the moving frame description of these nets.

We identify Euclidean three-space \mathbb{R}^3 with the imaginary part of the quaternions $\mathbb{H} = \text{span}\{\mathbb{1}, i, j, k\}$, i.e., $\mathbb{R}^3 \cong \text{span}\{i, j, k\} \cong \text{span}\{-i\sigma_1, -i\sigma_2, -i\sigma_3\}$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{36}$$

are the 2×2 complex *Pauli matrices* generating the Lie algebra $\mathfrak{su}(2)$. Using this representation one describes the surface via a moving frame $\Phi \in \mathfrak{S}\mathbb{H}$ which rotates the standard orthonormal frame of \mathbb{R}^3 into the surface tangent plane and normal vector. Specifically, conjugation by a quaternion corresponds to a rotation and the frame encodes the Gauß map directly by rotating k , i.e., $n = -i\Phi^{-1}\sigma_3\Phi = \Phi^{-1}k\Phi$. These frame descriptions are the natural language of integrable systems related to surface theory both smooth and discrete [4, 8, 10].

With the notation of Figure 8, we introduce the frame Φ of interest by initially setting Φ equal to the identity and then defining the vertex shifts

$$\Phi_1 := U\Phi \text{ and } \Phi_2 := V\Phi, \tag{37}$$

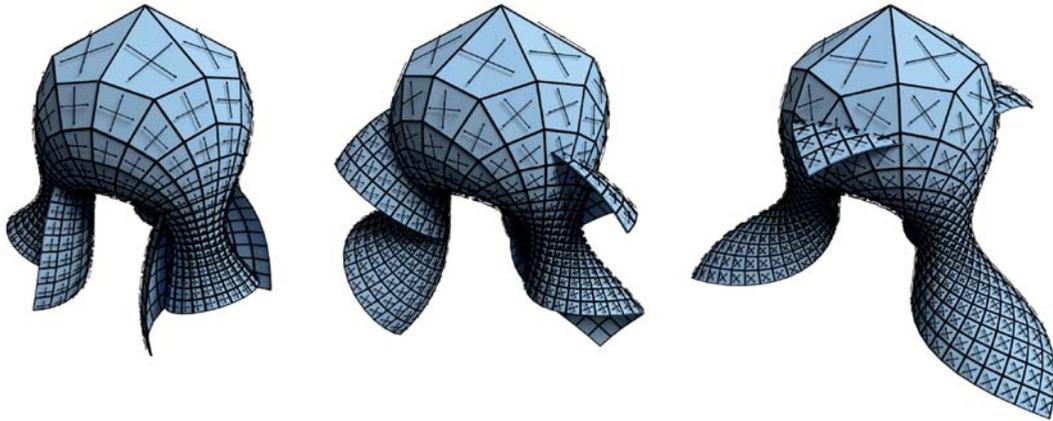


Fig. 9. Three members of the associated family of a discrete CMC edge-constraint net Smyth surface, together with their curvature line fields. For smooth Smyth surfaces, the associated family is known to be a reparametrization. Note how the curvature line fields keep their directions in the family.

where

$$U := \begin{pmatrix} a & -\lambda u - \frac{1}{\lambda u} \\ \frac{u}{\lambda} + \frac{\lambda}{u} & \bar{a} \end{pmatrix} \text{ and } V := \begin{pmatrix} b & -i\lambda v + \frac{i}{\lambda v} \\ i\frac{\lambda}{v} - \frac{i}{\lambda}v & \bar{b} \end{pmatrix} \quad (38)$$

are the *Lax matrices* with spectral parameter $\lambda = e^{i\alpha}$ for $\alpha \in [0, 2\pi]$, a, b complex-valued functions living on vertices (and \bar{a}, \bar{b} their complex conjugates), and u, v positive real-valued functions living on vertices [8]. To guarantee that each quad closes, i.e., $\Phi_{12} = \Phi_{21}$, the Lax matrices and their shifts V_1 and U_2 must satisfy the compatibility condition

$$V_1 U = U_2 V, \text{ with } \det(U) = \det(U_2) \text{ and } \det(V) = \det(V_1). \quad (39)$$

For every value of the spectral parameter the net is then generated by taking the imaginary part of the Sym–Bobenko formula [8] (an example is shown in Figure 9),

$$\begin{aligned} n &:= \Phi^{-1} \mathbb{k} \Phi \\ \mathfrak{f} &:= \left(-\Phi^{-1} \frac{\partial}{\partial \alpha} \Phi \Big|_{\lambda=e^{i\alpha}} + \frac{1}{2} \Phi^{-1} \mathbb{k} \Phi \right), \\ f &:= \Im \mathfrak{f}. \end{aligned} \quad (40)$$

Note that unlike in [8] we purposefully *do not* normalize the transport matrices U and V to have determinant 1. Thus Φ is not in $SU(2)$, necessitating taking the imaginary part.

Definition 3.23 (Associated family). Let $(f, n) := (f^\alpha, n^\alpha)$ be a net generated from the frame Φ (with spectral parameter $\lambda = e^{i\alpha}$) given by Equations (38–40). We call (f, n) a member of the associated family of a discrete isothermic CMC net. \square

Lemma 3.24. Every member of the associated family of a discrete isothermic CMC net is an edge-constraint net. Furthermore, for $i = 1, 2$ the edge-constraint is expressed in the equation

$$n_i = -(\mathbb{f}_i - \mathbb{f})n(\mathbb{f}_i - \mathbb{f})^{-1}. \tag{41}$$

\square

We defer the proof to Section 5 where a general discussion of edge-constraint nets arising from Lax pairs is provided.

In general, the edge-constraint can be understood along a shift in either lattice direction $i = 1, 2$ as first negating n and then rotating it along the edge $f_i - f$ to find n_i , which, when written quaternionically, gives rise to the following definition.

Definition 3.25 (Normal transport quaternions). Consider a quad from an edge-constraint net (f, n) . The quaternions given by $\phi := \tau + (f_1 - f), \tau \in \mathbb{R}$ and $\psi = \eta + (f_2 - f), \eta \in \mathbb{R}$ such that

$$n_1 = -\phi^{-1}n\phi \text{ and } n_2 = -\psi^{-1}n\psi, \tag{42}$$

are called *normal transport* quaternions. (Although inverses naturally arise on the right (Equation (41)) from the Sym–Bobenko formula, we prefer to define normal transports with inverses on the left; this simply corresponds to an opposite sign convention for the real part of the normal transport.) \square

This perspective yields insight into the geometry of the cubes C_f and C_g for an arbitrary edge-constraint net.

Lemma 3.26 (Edge-constraint as a skew parallelogram). Let (f, n) be an edge-constraint net with offset net $(f^*, n) = (f + n, n)$. For each quad, consider the combinatorial cubes C_f and C_g formed by it and its offset (as given in Equation (35)). Then the four sides of C_f and C_g are skew trapezoids and skew parallelograms, respectively. \square

Moreover, we see that if (f, n) is a member of the associated family of a discrete isothermic CMC net then for every quad we have the following three facts that

together imply that (f, n) has constant mean curvature: (i) the top and bottom of C_g are also parallelograms; (ii) all six sides are parallelograms of the same “folding parameter”, so C_g forms an “equally folded parallelogram cube”; and (iii) every equally folded parallelogram cube has vanishing (projected) mixed area between its top and bottom.

Let g, g_1, g_{12}, g_2 be a skew parallelogram built from the edge lengths ℓ_1 and ℓ_2 . It is straightforward to see that the dihedral angles δ_1 and δ_2 (measured between 0 and $\frac{\pi}{2}$) along the edges of this skew parallelogram (understood as edges of the enclosing tetrahedron) are equal for opposite edges and satisfy $\frac{\sin \delta_1}{\ell_1} = \frac{\sin \delta_2}{\ell_2}$.

Definition 3.27. The *folding parameter* of a skew parallelogram with the above notation is defined as

$$\sigma := \frac{\sin \delta_1}{\ell_1} = \frac{\sin \delta_2}{\ell_2}. \tag{43}$$

Lemma 3.28. Every skew parallelogram can be written in terms of two edges $g_1 - g$ and $g_2 - g$ with lengths ℓ_1 and ℓ_2 , respectively, and a folding parameter σ :

$$g_{12} - g_2 = (\rho \mathbb{1} + (g_2 - g_1))(g_1 - g)(\rho \mathbb{1} + (g_2 - g_1))^{-1},$$

$$\text{where } \rho = \frac{1}{\sigma} \left(\sqrt{1 - \sigma^2 \ell_1^2} - \sqrt{1 - \sigma^2 \ell_2^2} \right). \tag{44}$$

Proof. The real part ρ is the same as $\Re v$ in Equation (3.15) of [25], with $k = \tan \frac{\delta_1}{2} \cot \frac{\delta_2}{2}$ and $s = \ell_1$. Using Jacobi elliptic functions one can rewrite this expression to find the above equation. ■

This construction can be extended to three edges and a fixed folding parameter, yielding the known combinatorial 3D compatibility cube of skew parallelograms [25, 42]:

Theorem 3.29 (Darboux transform for parallelograms). Let g be a skew parallelogram with edge lengths ℓ_1, ℓ_2 and folding parameter σ . For every vector $\tilde{n} \in \mathbb{R}^3$, there exists a unique skew parallelogram g^* at constant distance $\|\tilde{n}\|$ from g such that:

1. g^* also has edge lengths ℓ_1, ℓ_2 and folding parameter σ ; and
2. every face of the combinatorial cube C_g formed by g and g^* is a skew parallelogram of folding parameter σ .

We call this object an *equally folded parallelogram cube*. (Instead of fixing the folding parameter σ one can also hold the real part ρ constant; this is also 3D compatible as shown in [37].) □

Proof. Recall that g itself can be generated from the two vectors $w_1 := g_1 - g$ and $w_2 := g_2 - g$ and the folding parameter σ . Therefore, we can rephrase the theorem statement as: given w_1, w_2, \tilde{n} and the folding parameter σ , show that completing the skew parallelogram twice in every direction forms a closed combinatorial cube. This is precisely the Bianchi Permutability Theorem for a single edge in the Darboux (Bäcklund) transformation of a discrete arc-length parametrized curve, a proof of which is given in [25]. ■

We now come back to the viewpoint that one can switch between the combinatorial cubes C_g and C_f as introduced in Equation (35).

Theorem 3.30. Consider an equally folded parallelogram cube C_g with bottom and top g and g^* , respectively. Let $h \in \mathbb{R}$ be the constant distance between g and g^* . Then the bottom and top quads of the corresponding cube C_f with normals given by the vertical edges of C_f are edge-constraint net quads with mean curvatures $-\frac{1}{h}$ and $\frac{1}{h}$, respectively. □

Proof. Notice that C_g can be constructed from three vectors and a folding parameter using the simplified equation for the real part of the rotation quaternions, Equation (44). Using the quaternionic description and $\hat{G} := ((g_{12}^* - g_{12}) - (g^* - g)) \times ((g_2 - g_2^*) - (g_1 - g_1^*))$ one finds that

$$\det(g_{12}^* - g^*, g_2^* - g_1^*, \hat{G}) + \det(g_{12} - g, g_2 - g_1, \hat{G}) = 0. \tag{45}$$

Using that $N \parallel \hat{G}$, this is equivalent to $A(f, f^*) = A(f, f + hn) = 0$. Therefore, the edge-constraint quads (f, n) and $(f + hn, n)$ indeed have mean curvatures $-\frac{1}{h}$ and $\frac{1}{h}$. ■

Remark 3.31. The top and bottom faces in the previous theorem can be exchanged for any pair of opposite faces (i.e., front and back or left and right). It turns out that the direction of \hat{G} defined in the previous proof is independent of this choice (possibly up to sign). In other words, the quad tangent planes arising from every pair of opposite faces coincide. □

Theorem 3.32. Let (f, n) be a member of the associated family of a discrete isothermic CMC net (Definition 3.23) with spectral parameter $\lambda = e^{i\alpha}$. Then (f, n) is a CMC edge-constraint net. □

Proof. By Lemma 3.24 (f, n) is an edge-constraint net. Consider the unit offset net $(f^*, n) = (f + n, n)$. For every quad, we show that the corresponding combinatorial cube C_g is an equally folded parallelogram cube; the result then follows from Theorem 3.30.

We naturally extend the quaternionic description of $f = \Im f$ and $f^* = \Im(f + n)$ to $g := \Im g$ and $g^* := \Im g^*$, for example, $g_1^* - g = f_1 - f$. The non-unit edges of the parallelograms of the front and left sides of C_g are found to have squared lengths:

$$\begin{aligned} \ell_1^2 &:= \|g_1 - g\|^2 = 1 - 4 \left(\frac{\cos(2\alpha)}{\det(U)} + \frac{\sin^2(2\alpha)}{\det(U)^2} \right) \text{ and} \\ \ell_2^2 &:= \|g_2 - g\|^2 = 1 + 4 \left(\frac{\cos(2\alpha)}{\det(V)} - \frac{\sin^2(2\alpha)}{\det(V)^2} \right). \end{aligned} \tag{46}$$

Recall that by assumption $\det(U) = \det(U_2)$ and $\det(V) = \det(V_1)$, so the back and right sides of C_g are also parallelograms with non-unit edge lengths ℓ_1 and ℓ_2 , respectively. Therefore, the top and bottom of C_g are also parallelograms both built from the edge lengths ℓ_1 and ℓ_2 .

The transports (in the sense of Lemma 3.28) that yield the front, left, and top skew parallelograms of C_g are:

$$\begin{aligned} -n_1 &= (g_1^* - g)n(g_1^* - g)^{-1}, \\ -n_2 &= (g_2^* - g)n(g_2^* - g)^{-1}, \text{ and} \\ g_{12} - g_2 &= (g_2 - g_1)(g_1 - g)(g_2 - g_1)^{-1}. \end{aligned} \tag{47}$$

The real parts arising from the Sym–Bobenko formula for corresponding edges, for example, $\Re(f_1 - f) = \frac{1}{2} \operatorname{tr}(U^{-1} \frac{\partial}{\partial \alpha} U)$, can be computed directly using the derivative of the determinant:

$$\frac{\partial}{\partial \alpha} \det(U) = \det(U) \operatorname{tr} \left(U^{-1} \frac{\partial}{\partial \alpha} U \right). \tag{48}$$

Applying this formula to the “double transport”, for example, $V_1 U$, shows that the sum of the real parts of $f_1 - f$ and $f_{12} - f_1$ is in fact the real part of the diagonal transport furnished by $f_{12} - f$. We therefore find:

$$\begin{aligned} \Re(g_1^* - g) &= \frac{2 \sin(2\alpha)}{\det(U)}, \\ \Re(g_2^* - g) &= -\frac{2 \sin(2\alpha)}{\det(V)}, \text{ and} \\ \Re(g_2 - g_1) &= \Re(g_2 - g) - \Re(g_1 - g) = -\frac{2(\det(U) + \det(V)) \sin(2\alpha)}{\det(U) \det(V)}. \end{aligned} \tag{49}$$

Plugging the real parts from Equations (49) with the edge lengths from Equations (46) into the Equation (44) yields the folding parameter $\sigma = \sin(2\alpha)$ in every instance.

Therefore, C_g is an equally folded parallelogram cube. ■

3.3 Discrete constant negative Gauß curvature surfaces

Definition 3.33 (Constant negative curvature edge-constraint nets). An edge-constraint net (f, n) is said to have *constant negative (Gauß) curvature* if every quad of the net has the same negative nonvanishing Gauß curvature $\mathcal{K} \in \mathbb{R}^-$. □

A surface parametrized by asymptotic lines has constant negative Gauß curvature if and only if the asymptotic lines form a *Chebyshev net*, i.e., the parameter lines are parallel transports of each other in the sense of Levi-Civita. Thus, the directional derivative along each parameter line depends only on a single variable, so its integral curve exhibits a constant speed parametrization (possibly a different constant for each parameter line); Chebyshev nets are made up of “infinitesimal parallelograms” with side lengths $a, b \in \mathbb{R}^+$. Furthermore, the Gauß map of the asymptotic lines also forms a Chebyshev net on the sphere. In these coordinates, the Gauß–Codazzi equation reduces to the well-known sine-Gordon equation in the angle between the asymptotic lines $u(x, y)$:

$$-\mathcal{K}ab \sin u(x, y) = u_{xy}(x, y). \tag{50}$$

This equation is invariant to the transformation $a \rightarrow \lambda a$ and $b \rightarrow \lambda^{-1}b$ for all $\lambda = e^t$ with $t \in \mathbb{R}$; varying λ generates the *associated family* of a constant negative curvature asymptotic line parametrized surface, where the angles between the asymptotic lines are invariant.

From the above characterization, Sauer [38, 39] defined the following discrete analogue.

Definition 3.34 (K-nets). An edge-constraint net $(f, n) : D \subset \mathbb{Z}^2 \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ is a *K-net* if it is an A-net and there exist two lengths $a, b > 0$ such that every immersion quad is a skew parallelogram (*Chebyshev quad*) with edge lengths a and b . □

For K-nets, we restrict to regular combinatorics because having more than two asymptotic lines meet at a point is incompatible with having negative Gauß curvature. The Gauß map of a K-net is also built from Chebyshev quads.

The associated family for K-nets was defined geometrically by Wunderlich [43] using a transformation that, like in the smooth setting, preserves the interior angles of f while scaling its edges. These geometric constructions agree with an algebraic description (similar to that introduced for CMC nets in the previous section) in terms of a discrete sine-Gordon equation and its Lax pair that is then integrated via a Sym–Bobenko formula to construct the net [6, 19, 24]. However, due to the inherent nonplanarity of the quads in these nets, understanding their curvatures has remained elusive. The goal of this section is to show (using the geometric constructions) that K-nets indeed have constant negative Gauß curvature as edge-constraint nets.

K-nets can be constructed (up to global rotation and scale) directly from Cauchy data for their Gauß map [35]. Explicitly, a fourth Gauß map point n_{12} is determined by completing the skew parallelogram through three other points $n, n_1, n_2 \in \mathbb{S}^2$:

$$n_{12} := \frac{n \cdot (n_1 + n_2)}{1 + n_1 \cdot n_2} (n_1 + n_2) - n. \quad (51)$$

In other words, the Gauß map satisfies the discrete Moutard equation in \mathbb{S}^2 [31]. The immersion f is constructed from n via

$$f_1 - f := n_1 \times n \text{ and } f_2 - f := n \times n_2. \quad (52)$$

When $n_i \cdot n = \cos \Delta_i$ then the edge lengths of f are given by $\sin \Delta_i$ for $i = 1, 2$. Moreover, applying Napier’s analogies to the spherical parallelogram formed by n the two interior angles $\alpha, \beta \in (0, \pi)$ of a K-net immersion quad are related by $e^{i\alpha} = \frac{e^{i\beta k - 1}}{e^{i\beta - k}}$, where $k = \tan \frac{\Delta_1}{2} \tan \frac{\Delta_2}{2}$ [6].

The associated family can now be described by a family of pairs of Gauß map angles $(\Delta_1(\lambda), \Delta_2(\lambda))$ from which the K-nets are explicitly constructed.

Definition 3.35 (K-net associated family). Consider a K-net (f, n) with the above notation and let $\lambda = e^t$ for all $t \in \mathbb{R}$. We construct a new K-net from $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ defined by the transformation

$$\tan \frac{\Delta_1(\lambda)}{2} := \lambda \tan \frac{\Delta_1}{2}, \quad \tan \frac{\Delta_2(\lambda)}{2} := \lambda^{-1} \tan \frac{\Delta_2}{2}. \quad (53)$$

□

The interior angles of every quad are invariant to this transformation and the edge lengths transform as $\sin \Delta_1(\lambda)$ and $\sin \Delta_2(\lambda)$.

From this construction, one can compute the Gauß curvature of every quad directly yielding our main theorem.

Theorem 3.36. Every K-net has constant negative Gauß curvature. □

Proof. Direct computation using Equations (51) and (52) yields (up to global scaling)

$$\mathcal{K} = \frac{-2}{(n_1 + n_2) \cdot n} = \frac{-2}{\cos \Delta_1(\lambda) + \cos \Delta_2(\lambda)}. \tag{54}$$

\mathcal{K} depends on $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ but for a fixed $\lambda \in \mathbb{R}^+$ both of these angles are constant (by definition) for every quad of the K-net. To have the same negative constant for all members of the associated family, one must globally scale by a value dependent on λ . ■

Some authors define K-nets in the weakest sense where the $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ are allowed to vary along the parameter lines as single variable functions [10]. While still edge-constraint nets, they obviously do not have constant negative Gauß curvature.

Remark 3.37. Wunderlich gave an interpretation for curvatures of K-nets in the symmetric case of $\Delta_1 = \Delta_2$ [43]: He interprets the circles that touch pairs of incident triangles in opposite points in the quad symmetry planes as the curvature circles and shows that the product of their radii is constant (see Figure 10). This quantity is in fact the Gauß curvature as defined by Equation (4); the radii of the circles are the ratios of diagonals in the f and n quadrilaterals and since the diagonals are perpendicular in both quadrilaterals the product of their lengths is proportional to the (projected) area. □

Wunderlich’s ideas stem from the fact that in the smooth setting the angular bisectors of the asymptotic lines are the curvature lines. Observe that for K-nets the diagonals of each quad do indeed satisfy the edge-constraint with the same normals, for example, $f_{12} - f \perp n_{12} + n$ and for a K-net with all edge lengths equal we even have $n_{12} - n \parallel f_{12} - f$. This observation can be utilized to interpret the immersion edges of a circular net with constant negative Gauß curvature as diagonals in immersion quadrilaterals of K-nets (with equal lengths per K-net quadrilateral but in general different edge length for each circular net edge). Circular nets of constant negative Gauß curvature have been defined in [14, 28] and naturally carry over to edge-constraint nets.

Definition 3.38. An edge-constraint net $(f, n) : D \subset \mathbb{Z}^2 \rightarrow \mathbb{R}^3 \times \mathbb{S}^2$ that is a circular net with Gauß curvature $\mathcal{K} = -1$ is called a *cK-net*. □



Fig. 10. Wunderlich's curvature circles for a symmetric K-net quadrilateral.

It turns out that one can explicitly define cK-nets using the above observation that each edge of a cK-net is the diagonal of a K-net quad. In [26], the first two authors give a Lax pair representation for cK-nets based on the Lax pair for K-nets. The associated family of K-nets then naturally gives rise to an associated family for cK-nets.

Theorem 3.39. Each cK-net and all members of its associated family are edge-constraint nets of constant negative Gauß curvature. \square

Proof. The way we define cK-nets here makes the statement for cK-nets tautological. The construction of the associated family and the proof that each member has the same constant negative Gauß curvature can be found in [26]. \blacksquare

Figure 11 (left) shows a cK-net Kuen surface which arises as a Bäcklund transform of the pseudosphere shown in Figure 12 (right) (such cK-net pseudospheres can also be found in [14, 28]). Figure 11 (right) shows a member of its associated family. The immersion quadrilaterals are no longer circular; unlike for K-nets where all members of the associated family are themselves K-nets, the members of the associated family of cK-nets are no longer circular in general. This reflects the smooth setting where asymptotic but not curvature coordinates are preserved throughout the associated family of surfaces with constant negative Gauß curvature.

We can also create pseudospheres of revolution that have one asymptotic and one curvature line. The middle net of Figure 12 is generated by first constructing an asymptotic line with two degrees of freedom that are then used to impose rotational symmetry and constant negative Gauß curvature, respectively.

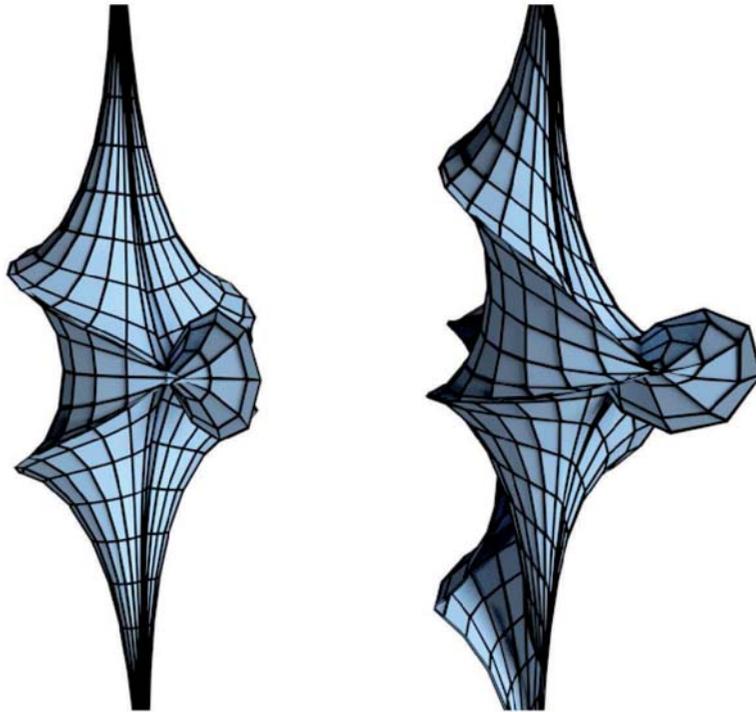


Fig. 11. A cK-net Kuen surface and a member of its associated family.

3.4 Discrete developable surfaces

Definition 3.40 (Developable edge-constraint net). An edge-constraint net (f, n) is called *developable* if every quad has vanishing Gauß curvature ($\mathcal{K} = 0$), i.e., the Gauß map has zero area. \square

As in the smooth setup, if the Gauß map is constant then either the shape operator vanishes or there exists exactly one nonzero principal curvature $2\mathcal{H}$ and corresponding curvature line, which is in fact even parallel to n_x and n_y .

Lemma 3.41. Consider a single quad from a developable edge-constraint net (f, n) with admissible projection direction $N \in \mathbb{S}^2$ such that $N \perp \text{span}\{n_x, n_y\}$. If they exist, the nonzero principal curvature and curvature line are invariant to the choice of N . \square

Proof. If n is nonconstant then $\text{span}\{n_x, n_y\}$ is one dimensional. Consider the reduced coordinates where $n_x, n_y \in \mathbb{R}^3$ are both multiples of e_1 , the first standard basis vector of \mathbb{R}^3 , and let $f_x, f_y \in \mathbb{R}^3$. The set of admissible projection directions is then parametrized

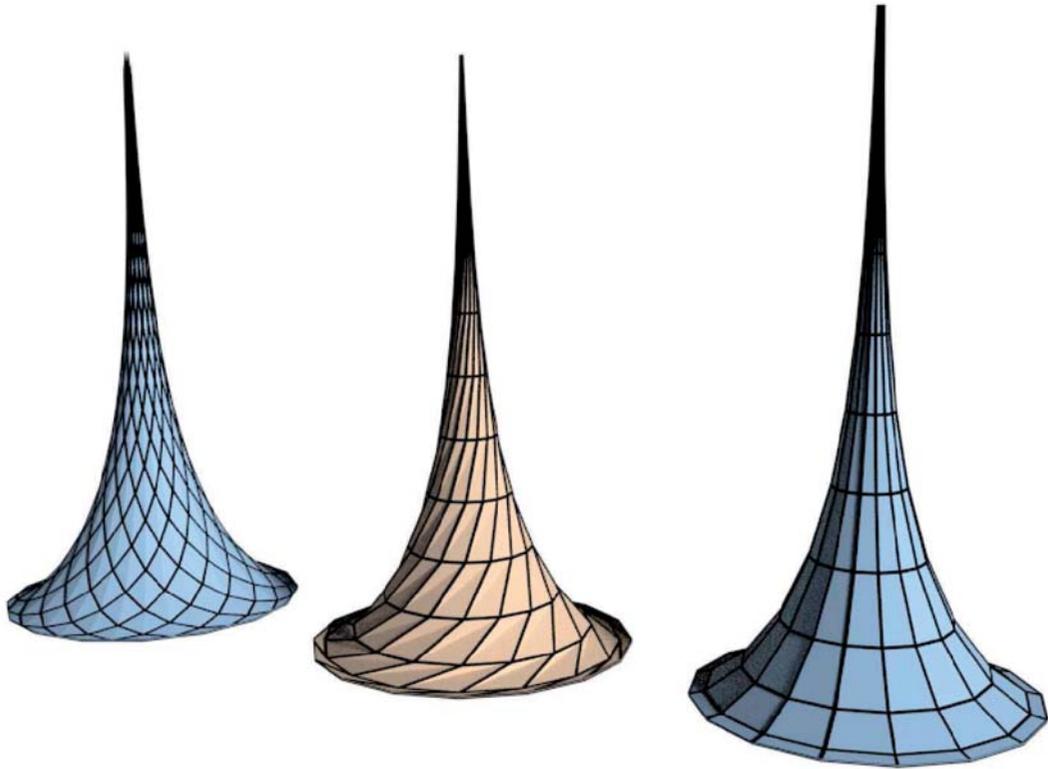


Fig. 12. Three pseudospheres of revolution which are constant negative curvature edge-constraint nets. Left: Discrete asymptotic line parametrization (K-net). Middle: One discrete asymptotic line and one discrete curvature line parametrization. Right: Discrete curvature line parametrization (cK-net).

by an \mathbb{S}^1 degree of freedom in e_2, e_3 . In these coordinates, direct computation (using that $f_x \cdot n_y = n_x \cdot f_y$) completes the proof. ■

Surfaces of planar strips have been considered as discrete developable as they can be unfolded into the plane [29]; such immersions correspond to developable curvature line edge-constraint nets, which are characterized by a discrete analogue of parallel framed curves [2] (e.g., see Figure 13 (left)).

A polygonal curve α with vertices $\alpha_0, \dots, \alpha_k$ and two orthonormal vectors (u_0, n_0) anchored at α_0 gives rise to a unique *discrete parallel frame* along α ; simply reflect through the perpendicular bisector planes of each edge of α . Then $u_{i+1} - u_i$ and $n_{i+1} - n_i$ are both parallel to $\alpha_{i+1} - \alpha_i$ for all $i = 0, \dots, k - 1$. This discrete parallel frame can also be understood as being generated from rotations about the curve binormal

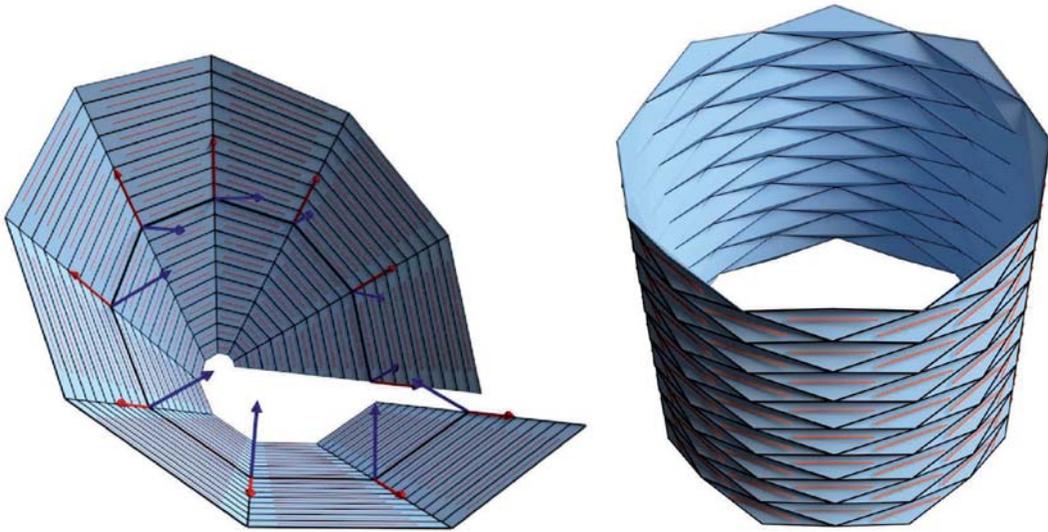


Fig. 13. Left: A circular developable edge-constraint net generated from a discrete parallel framed polygonal helix. Observe that the net is built from planar strips quadrangulated by isosceles trapezoids and that the curvature line field (thin line within each quad) is constant within each strip and parallel to the helix edge. The discrete rulings are the intersection lines of neighboring strips, along which the associated Gauß map is constant. Both ruling directions and Gauß map are shown with arrows along the generating polygon. Right: The Schwarz Lantern as a developable edge-constraint net with its curvature line field (thin line within each quad).

since the composition of two reflections is a rotation. To extend the polygonal curve α to a developable edge-constraint net (f, n) , fix a sampling y_j of the real line and define $f : (i, j) \mapsto \alpha_i + y_j u_i$ with Gauß map n_i . The resulting net (f, n) is then in fact circular.

Conversely, for a surface M built from a collection of planar strips with intersection lines u one can find a discrete parallel framed curve (α, u, n) giving rise to a developable circular edge-constraint net whose immersion realizes M : choose an initial point α_0 on an initial line u_0 and a unit normal $n_0 \perp u_0$. The reflection property then gives rise to unique α and n . It turns out that the curvature line is invariant to the initial choice of n_0 (it is always parallel to α), while the mean curvature is not (which can be interpreted as extra information on how the developable surface locally bends).

Examples of developable edge-constraint nets that are not in curvature line parametrization arise from the associated family of the CMC discrete isothermic cylinder; this family contains the well-known Schwarz Lantern [30] (see Figure 13 (right)) as an immersion with vertex normals that coincide with those of the smooth cylinder.

4 Towards a Conformal Perspective

In the smooth setting two conformal immersions $f, \tilde{f} : M \rightarrow \mathbb{R}^3 \cong \mathbb{S}\mathbb{H}$, for a manifold M are said to be spin-equivalent if there exists a *spin transformation* $\lambda : M \rightarrow \mathbb{H}^*$, such that $d\tilde{f} = \bar{\lambda}df\lambda$; the surface normal n transforms as $\tilde{n} = \lambda^{-1}n\lambda$. Geometrically, spin transformations correspond to stretch rotations of the tangent plane at every point. Therefore, they are conformal mappings and for simply connected domains any two surfaces which are conformally equivalent are related via a spin transformation. Kamberov, Pedit, and Pinkall [27] showed that one can classify all Bonnet pairs on a simply connected domain using spin transformations. Bonnet pairs are immersed surfaces that have the same metric and mean curvature but are not rigid body motions of each other.

We define a discrete spin transformation by “stretch-rotating” the normal transport quaternions (Definition 3.25) of an edge-constraint net.

Definition 4.1 (Discrete spin transformation). Let (f, n) be an edge-constraint net with quad graph \mathbb{G} . The *spin transformation* is a map $\lambda : \mathbb{G} \rightarrow \mathbb{H}^*$ which transforms (f, n) to (\tilde{f}, \tilde{n}) . The normal at each vertex and the normal transport quaternions transform by:

$$\tilde{n} = \lambda^{-1}n\lambda, \quad \tilde{\phi} = \bar{\lambda}\phi\lambda_1, \quad \text{and} \quad \tilde{\psi} = \bar{\lambda}\psi\lambda_2. \tag{55}$$

If the immersion of the spin transformed quadrilateral closes (i.e., $(\tilde{\phi} + \tilde{\psi}_1) - (\tilde{\phi}_2 + \tilde{\psi})$ is real), then one can construct a new edge-constraint net via

$$\tilde{f}_1 - \tilde{f} = \mathfrak{S}\tilde{\phi} \quad \text{and} \quad \tilde{f}_2 - \tilde{f} = \mathfrak{S}\tilde{\psi}. \tag{56}$$

□

Proof. The Gauß map \tilde{n} is still unit length since conjugation by a quaternion corresponds to a global rotation, so lengths are preserved. The edge constraint is satisfied since for an arbitrary edge (here denoted as first lattice shifts) we have

$$\tilde{\phi}^{-1}\tilde{n}\tilde{\phi} = \lambda_1^{-1}\phi^{-1}\bar{\lambda}^{-1}\lambda^{-1}n\lambda\bar{\lambda}\phi\lambda_1 = \lambda_1^{-1}\phi^{-1}n\phi\lambda_1 = \lambda_1^{-1}(-n_1)\lambda_1 = -\tilde{n}_1. \tag{57}$$

■

The spin transformation is invertible in the following sense, yielding a notion of discrete conformity.

Lemma 4.2. If (\tilde{f}, \tilde{n}) is a spin transformation of (f, n) with λ , then (f, n) is a spin transformation of (\tilde{f}, \tilde{n}) with λ^{-1} . □

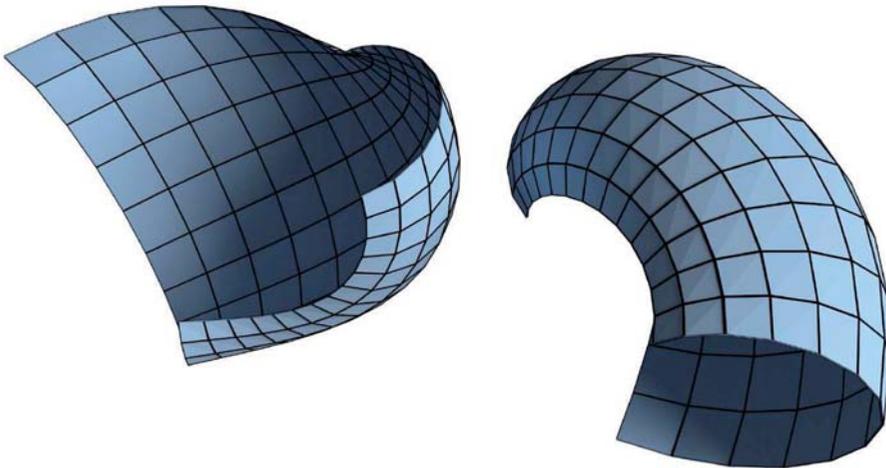


Fig. 14. A Bonnet pair: two edge-constraint nets with the same mean curvature per quad that are discretely conformally equivalent but not rigid body motions of each other.

Definition 4.3 (Discrete conformal). Two edge-constraint net quads are *discretely conformally equivalent* if they are spin transformations of each other. \square

This spin transformation can give rise to edge-constraint net Bonnet pairs, an example is shown in Figure 14. The Darboux transformations of discrete isothermic nets [22] are also spin transformations, the proof is along the same lines as that of the following theorem.

Theorem 4.4. The nets in the associated family of a discrete isothermic minimal net are conformal to each other. \square

Proof. Perform the spin transformation with $\lambda = \cos(\frac{\alpha}{2}) - \sin(\frac{\alpha}{2})n$ on the discrete isothermic minimal nets given by the Weierstrass representation (Definition 3.7) and see that one recovers the associated family edge (Definition 3.10—note the different use of the symbol λ therein). \blacksquare

5 Lax Pair Edge-Constraint Nets

In the smooth setting, several surfaces and their associated one-parameter families can be described in terms of a so-called Lax representation, for example, constant positive and negative Gauß curvature surfaces, CMC surfaces, Bianchi surfaces, and surfaces with harmonic inverse mean curvature [4]. In this representation \mathbb{R}^3 is identified with

the imaginary quaternions $\Im\mathbb{H}$ and, as was done in Section 3.2, we write elements in \mathbb{H} in terms of 2×2 complex matrices. Surfaces are constructed by integrating maps U, V, Φ from \mathbb{R}^2 into the quaternions \mathbb{H} satisfying the smooth compatibility conditions

$$\begin{aligned} U_y - V_x + [U, V] &= 0, \\ \Phi_x = U\Phi, \quad \Phi_y &= V\Phi. \end{aligned} \tag{58}$$

When U, V are in fact maps to $\Im\mathbb{H} \cong \mathfrak{su}(2)$ then Φ maps to $SU(2)$ and is the familiar orthonormal moving frame description of the surface.

In the discrete case, U, V, Φ map from \mathbb{Z}^2 to the invertible quaternions \mathbb{H}^* (with the notation of Figure 8) and satisfy the discrete compatibility conditions

$$\begin{aligned} V_1 U &= U_2 V, \\ \Phi_1 = U\Phi, \quad \Phi_2 &= V\Phi, \end{aligned} \tag{59}$$

starting from a fixed Φ (often the identity). A discrete theory of moving frames which allows these compatibility conditions to be integrated in general is not known.

Associated families of surfaces arise when U, V, Φ depend on a so-called spectral parameter $\lambda = \lambda(\alpha)$, defining a curve in \mathbb{C} (parametrized by a real parameter α), and satisfy the compatibility conditions for all $\lambda(\alpha)$. In this case, the compatibility equations are called a Lax representation for the family of surfaces. The spectral parameter dependence then allows for the surfaces to be found through differentiation (as opposed to integration) using the so-called Sym–Bobenko formula [4, 41].

Discrete Lax representations give rise to associated families of discrete nets f^α , together with their Gauß maps n^α , through the discrete Sym–Bobenko formula:

$$\begin{aligned} n^\alpha &:= \Phi^{-1} \mathbb{k} \Phi, \\ \mathbb{f}^\alpha &:= s \Phi^{-1} \frac{\partial}{\partial \alpha} \Phi|_{\lambda(\alpha)} + t n = s \Phi^{-1} \Phi_\alpha + t n, \\ f^\alpha &:= \Im \mathbb{f}, \end{aligned} \tag{60}$$

where $t \in \mathbb{R}$ is an offset constant.

We used this description explicitly to show that discrete isothermic CMC nets [8, 23, 34] and their associated families are CMC edge-constraint nets. By Corollary 3.20 discrete isothermic constant positive Gauß curvature nets and their associated families are constant positive Gauß curvature edge-constraint nets with the same Lax representation, but integrate with a different value for t in the Sym–Bobenko formula ($t = 0$ instead of $\frac{1}{2}$ as in Equation (40)). Although not explicitly used here, constant negative Gauß curvature edge-constraint nets in either asymptotic coordinates (K-nets) [6, 7] or curvature

line coordinates (cK-nets) [26], together with their respective associated families, have a Lax pair representation.

The next theorem characterizes those discrete Lax representations that give rise to edge-constraint nets, thus, making it easy to check if a particular family of surfaces arising from a Lax pair can be further understood using the geometry of edge-constraint nets.

Theorem 5.1. Let (f^α, n^α) be a family of pair of quad nets arising from a Lax representation $U(\lambda), V(\lambda), \Phi$ depending on a spectral parameter $\lambda(\alpha)$ satisfying Equation (59) by integrating (60). Then (f^α, n^α) is an edge-constraint net for each α if and only if the Lax matrices $U(\lambda)$ and $V(\lambda)$ depend on λ only in their off-diagonal entries. \square

Proof. We saw in Definition 3.25 that the edge-constraint can be phrased quaternionically in terms of the normal transport quaternions. For a net arising from the Sym–Bobenko formula Equation (60), the (inverse) of the normal transports are the \mathbb{H}^* -valued edges, for example, $\mathbb{f}_1 - \mathbb{f}$. A direct computation shows that satisfying the edge-constraint is equivalent to the relationships

$$n_1 = -(\mathbb{f}_1 - \mathbb{f})n(\mathbb{f}_1 - \mathbb{f})^{-1} \text{ and } n_2 = -(\mathbb{f}_2 - \mathbb{f})n(\mathbb{f}_2 - \mathbb{f})^{-1}. \tag{61}$$

The proof of the condition on Lax matrices is equivalent for both lattice directions, so we provide details for the first one, resulting in a condition on the $U(\lambda)$ matrices: up to a global rotation by Φ we find

$$\begin{aligned} (\mathbb{f}_1 - \mathbb{f})n &= (sU^{-1}U_\alpha + t(U^{-1}\mathbb{k}U - \mathbb{k}))\mathbb{k} \\ &= U^{-1}(sU_\alpha + t(\mathbb{k}U - U\mathbb{k}))\mathbb{k} \\ &= U^{-1}(sU_\alpha\mathbb{k} + t(\mathbb{k}U\mathbb{k} + U)) \\ &= U^{-1}(sU_\alpha\mathbb{k} + t((- \mathbb{k}U)(- \mathbb{k}) + (- \mathbb{k}U)(- \mathbb{k}U)^{-1}U)) \\ &\stackrel{!}{=} U^{-1}(- \mathbb{k}U)(\mathbb{f}_1 - \mathbb{f}) \\ &= -n_1(\mathbb{f}_1 - \mathbb{f}) \end{aligned} \tag{62}$$

precisely when:

$$U_\alpha\mathbb{k} = -\mathbb{k}U_\alpha, \tag{63}$$

which is equivalent to U having only off-diagonal entries dependent on $\lambda(\alpha)$ when written as a 2×2 complex matrix. \blacksquare

Funding

T.H. was supported by the Deutsche Forschungsgemeinschaft-Collaborative Research Center, TRR 109, “Discretization in Geometry and Dynamics”. M.W. was partially supported by the Deutsche Forschungsgemeinschaft-Collaborative Research Center, TRR 109, “Discretization in Geometry and Dynamics”.

Acknowledgements

We thank Julia Plehnert and Henrik Schumacher for helpful discussions, particularly with helping to resolve the definitions with a developable theory of edge-constraint nets.

References

- [1] Bianchi, L. *Lezioni di geometria differenziale*, vol. 1–2, seconda edizione, riveduta e considerevolmente aumentata edition, Pisa: Spoerri, 1902.
- [2] Bishop, R. L. “There is more than one way to frame a curve.” *American Mathematical Monthly* 82, no. 3 (1975): 246–51.
- [3] Bobenko, A. I. “All constant mean curvature tori in \mathbb{R}^3 , S^3 , H^3 in terms of theta-functions.” *Mathematische Annalen* 290, no. 1 (1991): 209–45.
- [4] Bobenko, A. I. “Surfaces in Terms of 2 by 2 Matrices: Old and New Integrable Cases.” In *Harmonic Maps and Integrable Systems*, edited by A. P. Fordy and J. C. Wood, 83–129. Braunschweig/Wiesbaden: Vieweg, 1994.
- [5] Bobenko, A. I. “Surfaces from Circles.” In *Oberwolfach Seminars*, edited by A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, 3–35. Basel: Birkhäuser, 2008.
- [6] Bobenko, A. I., and U. Pinkall. “Discrete surfaces with constant negative Gaussian curvature and the Hirota equation.” *Journal of Differential Geometry* 43, (1996): 527–611.
- [7] Bobenko, A. I., and U. Pinkall. “Discrete isothermic surfaces.” *Journal für die reine und angewandte Mathematik* (1996): 187–208.
- [8] Bobenko, A. I., and U. Pinkall. “Discretization of Surfaces and Integrable Systems.” In *Discrete Integrable Geometry and Physics*, edited by A. I. Bobenko and R. Seiler, 3–58. USA: Oxford University Press, 1999.
- [9] Bobenko, A. I., and R. Seiler. *Discrete Integrable Geometry and Physics*. Oxford Lecture Series in Mathematics and Its Applications. USA: Oxford University Press, 1999.
- [10] Bobenko, A. I., and Y. B. Suris. *Discrete Differential Geometry: Integrable Structure*. Graduate Studies in Mathematics, Vol. 98. USA: American Mathematical Society, 2008.
- [11] Bobenko, A. I., and Y. B. Suris. “Discrete Koenigs nets and discrete isothermic surfaces.” *International Mathematics Research Notices* 2009, no. 11 (2009): 1976–2012.
- [12] Bobenko, A. I., T. Hoffmann, and B. A. Springborn. “Minimal surfaces from circle patterns: geometry from combinatorics.” *Annals of Mathematics* 164, no. 1 (2006): 231–64.
- [13] Bobenko, A. I., J. M. Sullivan, P. Schröder, and G. M. Ziegler. *Discrete Differential Geometry*. Oberwolfach Seminars, Vol. 38. Germany: Springer, 2008.

- [14] Bobenko, A. I., H. Pottmann, and J. Wallner. "A curvature theory for discrete surfaces based on mesh parallelity." *Mathematische Annalen* 348, no. 1 (2010): 1–24.
- [15] Bobenko, A. I., U. Hertrich-Jeromin, and I. Lukyanenko. "Discrete constant mean curvature nets in space forms: Steiner's formula and Christoffel duality." *Discrete and Computational Geometry* 52, no. 4 (2014): 612–29.
- [16] Burstall, F., U. Hertrich-Jeromin, and W. Rossman. "Discrete linear Weingarten surfaces." (2014): preprint. arXiv:1406.1293.
- [17] Burstall, F., U. Hertrich-Jeromin, W. Rossman, and S. Santos. "Discrete special isothermic surfaces." *Geometriae Dedicata* 174, no. 1 (2014): 1–11.
- [18] Cieřliński, J., A. Doliwa, and P. M. Santini. "The integrable discrete analogues of orthogonal coordinate systems are multi-dimensional circular lattices." *Physics Letters A* 235, no. 5 (1997): 480–88.
- [19] Cieřliński, J. L. "Pseudospherical surfaces on time scales: a geometric definition and the spectral approach." *Journal of Physics A: Mathematical and Theoretical* 40, no. 42 (2007): 12525–538.
- [20] Darboux, G. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*. Paris: Gauthier-Villars, 1887.
- [21] Dorfmeister, J., F. Pedit, and H. Wu. "Weierstrass type representation of harmonic maps into symmetric spaces." *Communications and Analysis of Geometry* 6, (1998): 633–68.
- [22] Hertrich-Jeromin, U., T. Hoffmann, and U. Pinkall. "A Discrete Version of the Darboux Transform for Isothermic Surfaces." In *Discrete Integrable Geometry and Physics*, edited by A. I. Bobenko and R. Seiler, 59–81. USA: Oxford University Press, 1999.
- [23] Hoffmann, T. "Discrete CMC Surfaces and Discrete Holomorphic Maps." In *Discrete Integrable Geometry and Physics*, edited by A. I. Bobenko and R. Seiler, 97–112. Oxford University Press, 1999.
- [24] Hoffmann, T. "Discrete Amsler Surfaces and a Discrete Painlevé III Equation." In *Discrete Integrable Geometry and Physics*, edited by A. I. Bobenko and R. Seiler, 83–96. Oxford University Press, 1999.
- [25] Hoffmann, T. "Discrete Hashimoto Surfaces and a Doubly Discrete Smoke-Ring Flow." In *Discrete Differential Geometry*, edited by A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, 95–115. Springer, 2008.
- [26] Hoffmann, T. and A. O. Sageman-Furnas. "A 2x2 Lax representation, associated family, and Bäcklund transformation for circular K-nets." (2015): arXiv.org.
- [27] Kamberov, G., F. Pedit, and U. Pinkall. "Bonnet pairs and isothermic surfaces." *Duke Mathematical Journal* 92, no. 3 (1998): 637–44.
- [28] Konopelchenko, B. G., and W. K. Schief. "Trapezoidal discrete surfaces: geometry and integrability." *Journal of Geometry and Physics* 31, no. 2 (1999): 75–95.
- [29] Liu, Y., H. Pottmann, J. Wallner, Y.-L. Yang, and W. Wang. "Geometric modeling with conical meshes and developable surfaces." *ACM Transactions on Graphics* 25, no. 3 (2006): 681–89.
- [30] Morvan, J.-M. *Generalized Curvatures*. Geometry and Computing, Vol. 2. Berlin/Heidelberg: Springer, 2008.

- [31] Nimmo, J. J. C. and W. K. Schief. "Superposition principles associated with the Moutard transformation: an integrable discretization of a (2+1)-dimensional sine-Gordon system." *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 453, no. 1957 (1997): 255–79.
- [32] Nutbourne, A. W. and R. R. Martin. *Differential Geometry Applied to Curve and Surface Design*, Vol. 1. Chichester: Horwood, 1988.
- [33] Oprea, J. *The Mathematics of Soap Films: Explorations With Maple*. Student Mathematical Library, Vol. 10. American Mathematical Society, 2000.
- [34] Pedit, F., and H. Wu. "Discretizing constant curvature surfaces via loop group factorizations: the discrete sine- and sinh-Gordon equations." *Journal of Geometry and Physics* 17, no. 3 (1995): 245–60.
- [35] Pinkall, U. "Designing Cylinders with Constant Negative Curvature." In *Discrete Differential Geometry*, edited by A. I. Bobenko, P. Schröder, J. M. Sullivan, and G. M. Ziegler, 57–66. Germany: Springer, 2008.
- [36] Pinkall, U., and I. Sterling. "On the classification of constant mean curvature tori." *Annals of Mathematics* (1989): 407–451.
- [37] Pinkall, U., B. Springborn, and S. Weißmann. "A new doubly discrete analogue of smoke ring flow and the real time simulation of fluid flow." *Journal of Physics A: Mathematical and Theoretical* 40, no. 42 (2007): 12563.
- [38] Sauer, R. "Parallelogrammmitter als Modelle pseudosphärischer Flächen." *Mathematische Zeitschrift* 52, no. 1 (1950): 611–22.
- [39] Sauer, R. *Differenzgeometrie*. Berlin: Springer, 1970.
- [40] Schief, W. K. "On a maximum principle for minimal surfaces and their integrable discrete counterparts." *Journal of Geometry and Physics* 56, no. 9 (2006): 1484–95.
- [41] Sym, A. "Soliton Surfaces and Their Applications (Soliton Geometry from Spectral Problems)." In *Lecture Notes in Physics*, edited by R. Martini, 154–231. Berlin/Heidelberg: Springer, 1985.
- [42] Tabachnikov, S., and E. Tsukerman. "On the discrete bicycle transformation." *Publicaciones Matemáticas del Uruguay (Proceedings of the Montevideo Dynamical Systems Congress 2012)* 14, (2013): 201–20.
- [43] Wunderlich, W. *Zur Differenzgeometrie der Flächen konstanter negativer Krümmung*. Springer, 1951.