

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit eigenständig ohne Hilfe Dritter und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe. Alle Stellen die den benutzten Quellen und Hilfsmitteln unverändert oder sinngemäß entnommen sind, habe ich als solche kenntlich gemacht. Sofern generative KI-Tools verwendet wurden, habe ich Produktnamen, Hersteller, die jeweils verwendete Softwareversion und die jeweiligen Einsatzzwecke (z.B. sprachliche Überprüfung und Verbesserung der Texte, systematische Recherche) benannt. Ich verantworte die Auswahl, die Übernahme und sämtliche Ergebnisse des von mir verwendeten KI-generierten Outputs vollumfänglich selbst. IB / Stand: April 2024 Die Satzung zur Sicherung guter wissenschaftlicher Praxis an der TU Berlin vom 8. März 2017. https://www.static.tu.berlin/fileadmin/www/10000060/FSC/Promotion__Habilitation/Dokumente/Grundsaeetze_gute_wissenschaftliche_Praxis_2017.pdf habe ich zur Kenntnis genommen. Ich erkläre weiterhin, dass ich die Arbeit in gleicher oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt habe.

Berlin, den

.....
Unterschrift

Signotopes: Bichromatic Triangles, Pseudoconfigurations and Extensions

Yan Alves Radtke

Master's Thesis

Acknowledgements

Many ideas lead nowhere, but I am extremely grateful that Stefan Felsner took the time to discuss each one of mine with me, helping me to gain a deeper understanding of the objects I am interested in and problem solving in general. Thank you very much!

Nobody wants to read unfinished work, but I am very grateful that Larisa Janko did it anyway. Your suggestions and corrections greatly improved the readability of this thesis. I would also like to thank you for your invaluable support while I finished my write up.

Last but not least, I would like to thank my family for patiently supporting me all throughout the long years of my studies.

Contents

1	Introduction	5
2	Preliminaries	5
2.1	Pseudoline Arrangements	5
2.2	Signotopes	6
3	Triangles and Bichromatic Triangles	7
3.1	Triangles	8
3.2	Bichromatic Triangles	10
4	Degenerate Signotopes	12
4.1	Degeneracy	12
4.2	Rotations	17
4.3	Sweeps	19
5	Pseudoconfiguration of Points	20
5.1	Topological Representations of Signotopes	20
5.2	Intuition	20
5.3	Pseudoconfiguration of Points	22
6	Extensions of Signotope	27
7	Conclusion	31

1 Introduction

For a given class of geometric objects, we can often observe some simple topological properties. For example, two non-parallel lines intersect in exactly one point and each line separates the plane into two connected components. We can take these topological properties and make them the defining properties of a more general class of objects. This way we obtain pseudoline arrangements from line arrangements and pseudohyperplane arrangements from hyperplane arrangements. In a similar spirit we can generalize collections of points to pseudoconfigurations of points.

These objects can often be seen as *topological representations* of purely combinatorial structures.

A common theme is that most of these topological objects are not realizable by the original geometric objects. For example, there are way more pseudoline arrangements than line arrangements.

These considerations give rise to two questions: Firstly, which statements that hold for geometric objects also hold for their topological counterparts? And secondly, which combinatorial structures admit a topological representation?

In this work, we will consider special cases of these questions. We will look at certain triangles in pseudoline arrangements, which can easily be found in line arrangements. Proving their existence in pseudoline arrangements seems to be highly challenging. We will only prove the existence of these triangles in a special case.

We will also look at *signotopes*. We will provide a novel proof for their representation as *k-intersecting pseudoconfigurations of points*, which will generalize the relationship between pseudoline arrangements and pseudoconfigurations of points. We will use *k-intersecting pseudoconfigurations of points* to solve a conjecture about the (non)-extendability of signotopes.

2 Preliminaries

2.1 Pseudoline Arrangements

Pseudoline arrangements were introduced by Levy in 1926 [Lev26] as generalizations of line arrangements.

An *euclidean pseudoline arrangement* of n pseudolines is a collection of n curves in the euclidean plane, such that each curve extends to infinity on both ends and has no self intersection and such that every pair of curves in a pseudoline arrangement intersects exactly once in a crossing. Since we will only consider euclidean pseudoline arrangements, we will omit "euclidean" and often simply call them *arrangements*.

An arrangement is *simple*, if at most two pseudolines intersect in a point. We will call the intersection points of pseudolines *vertices*. If two vertices are consecutive on a pseudoline, we call the segment between them an *edge*. Removing the pseudolines of an arrangement from the plane divides the plane into

multiple connected components. We will call these connected components *faces*. A face is a *triangle* if it is bounded and on its boundary there are exactly three vertices.

All the arrangements we consider will be *marked*, meaning that there is a designated unbounded face, which we will call the *north pole*. After the choice of a north pole, we obtain a below-above relationship: a point is *above* a pseudoline ℓ , if it lies in the same connected component of $\mathbb{R}^2 \setminus \ell$ as the north pole. The points in the other component are *below* ℓ .

Furthermore, by the choice of a north pole we obtain a numbering on the pseudolines of an arrangement: By going counterclockwise around the plane, starting at the north pole, we meet all the pseudolines of an arrangement. We number the pseudolines in the order we meet them.

We will call two marked arrangements *isomorphic*, if they realize the same above/below relationship between vertices and pseudolines with the numbering given above.

2.2 Signotopes

The *higher Bruhat orders* were introduced by Manin and Shekhtman in [MS86]. Ziegler investigated their connection to extensions of hyperplane arrangements in [Zie93]. The elements of the higher Bruhat orders are *signotopes*. They were introduced by Felsner and Weil in [FW01]. We will give a short introduction to signotopes.

Let in the following $n \geq r \geq 2$ be natural numbers.

Definition 1. We call a function

$$\sigma : \binom{[n]}{r} \longrightarrow \{+, -\}$$

a *signotope of rank r on n elements* if for every $k+1$ subset $S = \{x_1 < x_2 < \dots < x_{r+1}\}$ of $[n]$, the sequence

$$\sigma(S \setminus x_1), \sigma(S \setminus x_2), \dots, \sigma(S \setminus x_{r+1})$$

has at most one sign change, i.e. the subsequences $+-+$ and $-+-$ are forbidden. We will call $[n]$ the elements of σ . The set of all rank r signotopes on n elements will be denoted by $S(n, k)$.

Signotopes give naturally rise to two orderings. Let $\sigma_1, \sigma_2 \in S(n, k)$ be two signotopes.

We can order $S(n, r)$ in the *inclusion order*, where $\sigma_1 \leq \sigma_2$ holds if

$$\sigma_1^{-1}(\{+\}) \subseteq \sigma_2^{-1}(\{+\}).$$

The cover relation of the *single step inclusion order* of $S(n, r)$ is $\sigma_1 \lessdot \sigma_2$ if σ_1 and σ_2 differ in a single

$$x \in \binom{[n]}{r},$$

where $\sigma_1(x) = -$ and $\sigma_2(x) = +$. Note that the single step inclusion order is also called the higher Bruhat order.

There is a bijection between marked arrangements on n pseudolines and the elements of $S(n, 3)$. We describe it briefly, the details can be found in [FW01]. Note that we can identify the elements of a signotope in $S(n, 3)$ and the pseudolines of a marked arrangement $\mathcal{A}$ by the numbering we defined in the previous section.

Given an arrangement $\mathcal{A}$, define $\sigma_{\mathcal{A}}$ the following way:

$$\sigma_{\mathcal{A}}(\{x_1 < x_2 < x_3\}) = \begin{cases} +, & x_1 \cap x_3 \text{ is above } x_2 \\ -, & \text{else.} \end{cases}.$$

Conversely, for a given signotope $\sigma \in S(n, 3)$ it is possible to construct an arrangement $\mathcal{A}$, such that $\sigma = \sigma_{\mathcal{A}}$.

Signotopes are helpful in the study of triangles due to the following relation: The pseudolines x_1, x_2, x_3 form a triangle in $\mathcal{A}$ if and only if the function we obtain by inverting the value of $\{x_1, x_2, x_3\}$ in $\sigma_{\mathcal{A}}$ is a signotope $\sigma_{\mathcal{A}'}$.

We then say that $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by flipping the triangle $\{x_1, x_2, x_3\}$. This corresponds to a local redrawing of $\mathcal{A}$, where the above/below relationship between x_1, x_2, x_3 and their intersections changes, but stays the same for every other triple of pseudolines.

Signotopes of higher rank can be represented as pseudohyperplane arrangements. We will keep this notion fuzzy, as it will not be important, but we refer the interested reader to [Bjö+93].

3 Triangles and Bichromatic Triangles

The existence of simplicial cells in pseudohyperplane arrangements is one of the central questions about oriented matroids, see [Bjö+93, Conjecture 7.3.10]. While the high dimensional case seems to be highly challenging, the rank 3 case is better understood: In projective arrangements, every pseudoline is incident to three triangles, therefore every non-trivial projective pseudoline arrangement on n pseudolines has at least n triangles. This was already proved by Levi in [Lev26].

We can therefore investigate more refined properties of triangles in pseudoline arrangements.

One of them is the so called *bichromatic triangle conjecture*. As far as we know it is first mentioned in [Bjö+93] as exercise 6.27.

We call an arrangement *bicolored*, if every pseudoline is assigned either *red* or *blue*. We will disallow the trivial all-red and all-blue colorings. We call a triangle *bichromatic*, if not all three pseudolines incident to the triangle have the same color.

Conjecture. *Every bicolored pseudoline arrangement contains a bichromatic triangle.*

We define the triangle incidence graph of an arrangement $\mathcal{A}$ as follows: the vertices are the pseudolines and triangles of $\mathcal{A}$. Add an edge between a pseudoline and a triangle, if the pseudoline supports the triangle. The bichromatic triangle conjecture is equivalent to:

Conjecture ([Bjö+93, Exercise 6.11]). *The triangle incidence graph of every pseudoline arrangement is connected.*

As stated in the preliminaries, every arrangement admits an ordering of the pseudolines. We call a pseudoline arrangement of n pseudolines *block-bicolored* if the first m pseudolines are red and the following $n - m$ pseudolines are blue.

The main result of this section concerns a weakening of the bichromatic triangle conjecture:

Theorem 2. *Every block-bicolored pseudoline arrangement contains a bichromatic triangle.*

3.1 Triangles

As already stated, the number of triangles in arrangements is well understood. We survey some results.

We start by giving a new proof for *Shannon's Theorem* [Sha79] in the plane. Another proof using an area argument can be found in [Fel04].

We will need some additional structure of line arrangements, developed by Felsner and Ziegler in [FZ01]. We call F a face of an hyperplane arrangement $\mathcal{H}$ if it is an inclusionwise maximal collection of points, such that for every $p_1, p_2 \in F$, p_1 and p_2 are contained in the same hyperplanes of $\mathcal{H}$ and on the same side of every hyperplane that does not contain p_1 and p_2 .

Theorem 3 ([FZ01]). *Let $S = \{s_1 < s_2 < \dots < s_n\}$ be a set of real number. There is a $n - 2$ dimensional real hyperplane arrangement $\mathcal{H}_S$, such that each face of $\mathcal{H}_S$ correspond to a combinatorial type of a line arrangement realizable with slopes in S . All the hyperplanes of $\mathcal{H}_S$ contain 0.*

We briefly sketch the construction of $\mathcal{H}_S$. We can interpret $\mathbb{R}^n$ as the set of all line arrangements realizable with a given slope set the following way: the point $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ corresponds to the line arrangement consisting of the lines $s_i x + b_i = y$.

There is a bijection between triples of lines ℓ_1, ℓ_2, ℓ_3 and hyperplanes H in $\mathcal{H}_S$, such that all points of H correspond to arrangements with slopes S where the given triple is degenerate, i.e. ℓ_1, ℓ_2, ℓ_3 intersect in a common point.

Every line arrangement can be translated such that a given pair of lines intersect at 0. This way we obtain an $(n - 2)$ -dimensional hyperplane arrangement.

Theorem 4 ([Sha79]). *Every (simple or non-simple) euclidean line arrangement of n lines has at least $n - 2$ triangles, if not all lines intersect in a common point.*

Proof. Let $\mathcal{A}$ be a line arrangement realized with slopes in S . Let $\mathcal{H}_S$ be the aforementioned hyperplane arrangement.

Let ℓ_1, ℓ_2 be two lines that have an ordinary crossing in $\mathcal{A}$, i.e. there is no other line going through the intersection of ℓ_1 and ℓ_2 . Such a pair of lines exist by the Sylvester–Gallai theorem.

We will consider $\mathcal{H}_S$ as embedded into $\mathbb{R}^{n-2}$ by letting ℓ_1 and ℓ_2 intersect at 0. $\mathcal{A}$ corresponds to a point p in $\mathbb{R}^{n-2}$. We say that p *sees* a hyperplane H , if p is not contained in H and there is a line segment s connecting p with H , intersecting only H or hyperplanes that contain p .

Claim. *If p sees $H \in \mathcal{H}_S$, H corresponds to a triangle of $\mathcal{A}$.*

Proof of Claim. The segment s corresponds to a continuous deformation of $\mathcal{A}$ into an arrangement, where the triangle T corresponding to H is contracted. Since s does not enter any other hyperplane, no triangle except T is flipped or contracted. This implies that T is empty in $\mathcal{A}$. ■

Claim. *p sees at least $n - 2$ hyperplanes from $\mathcal{H}_S$.*

Proof of Claim. For every direction $v \neq 0$, the line through p with direction v does intersect hyperplanes of $\mathcal{H}_S$ that do not contain p , since eventually some line ℓ_3 will pass 0 and the hyperplane corresponding to the triple ℓ_1, ℓ_2, ℓ_3 will be intersected. The hyperplanes met along v cannot contain p , since they corresponds to a triple that is not contracted in $\mathcal{A}$.

If p would only see $n - 3$ hyperplanes, we could choose v as orthogonal to all of their normal vectors, contradicting the previous observation. ■

The two claims imply the theorem. □

Theorem 5 ([FK99]). *Every simple arrangement $\mathcal{A}$ on n pseudolines has at least $n - 2$ triangles. For every $n \geq 3$ we can find arrangements attaining this bound.*

It is noteworthy that the proof of this theorem utilizes a counting argument and does not entail information how the triangles are distributed in the arrangement.

One tool to locate triangles in an arrangement is the so called *sweeping lemma*. It solves the bichromatic triangle conjecture in the case that a single pseudoline is red.

Lemma 6 ([FW01]). *Let ℓ be a pseudoline in a marked arrangement. If there is a crossing above ℓ , then there is a crossing above ℓ that forms a triangle with ℓ . The same holds for crossings below ℓ .*

The sweeping lemma gives a tight lower bound for the number of triangles in a possibly non-simple arrangement.

Theorem 7 ([FK99]). *Every (simple or non-simple) arrangement $\mathcal{A}$ of n pseudolines has at least $\frac{2}{3}n$ triangles. For every $n \geq 6, 3 \mid n$ there are arrangements attaining this bound.*

This is the first property we meet that distinguishes pseudoline arrangements from line arrangements.

It was asked in [Alv23], whether it is possible to cover the set of all pseudoline arrangements by combinatorial abstractions of the hyperplane arrangements $\mathcal{H}_S$.

Since our proof for the number of triangles in a line arrangement followed fairly straightforward from the structure of these hyperplane arrangements, we are sceptical that there are meaningful abstractions that contain the pseudoline arrangements with fewer than $n - 2$ triangles.

The following result locates triangles in pairs of arrangements. It will be fundamental in proving Theorem 2.

Theorem 8 ([FW00]). *Let $\mathcal{A}_1, \mathcal{A}_2$ be two pseudoline arrangements with the same number of pseudolines. Assume $\sigma_{\mathcal{A}_1} < \sigma_{\mathcal{A}_2}$ in the inclusion order. Then $\sigma_{\mathcal{A}_1} < \sigma_{\mathcal{A}_2}$ in the single-step inclusion order. This implies that there is a sequence of “- to +” triangle flips that transform $\mathcal{A}_1$ into $\mathcal{A}_2$.*

3.2 Bichromatic Triangles

The bichromatic triangle conjecture holds for line arrangements. The argument is simple: let $\mathcal{A}$ be a bicolored line arrangement. After a small rotation, the vertices of $\mathcal{A}$ have pairwise different x -coordinates. Without loss of generality there is a crossing of two blue lines that lies above a red line. We now shift all the red lines in parallel and continuously upwards, while the blue lines stay the same.

The combinatorics of the arrangement has to change at some point, since there is a blue-blue crossing above a red line. Since the combinatorics of the induced blue and red sub-arrangements stay the same, the first moment the combinatorics of the arrangement changes, a bichromatic triangle was flipped. This gives us the existence of such triangle.

Since the number of line arrangements is in $2^{\theta(n \log n)}$, see [GP86], and the number of pseudoline arrangements is in $2^{\theta(n^2)}$, see [Cor+24], this argument only applies to a small class of arrangements. In [FPS22] this technique was generalized to a class of arrangements that has similar asymptotics to the set of all arrangements.

Definition 9. We call an arrangement $\mathcal{A} \in S(n, 3)$ *approaching*, if it is realizable as a collection of curves, such that every pseudoline ℓ_i is represented by a continuous function $f_i : \mathbb{R} \rightarrow \mathbb{R}$, such that $f_i - f_j$ is surjective and strictly monotonically decreasing, if $i < j$.

It is clear that we can shift curves in an approaching arrangement upwards and downwards and it stays a pseudoline arrangement. This gives us the existence of a bichromatic triangle.

It is noteworthy that this shift is not possible with an arbitrary collection of curves representing a pseudoline arrangement: the shifting could lead to curves that do not intersect anymore or that intersect more than once.

The following theorem tells us that the number of pseudoline arrangements for which the bichromatic triangle conjecture holds is not small.

Theorem 10 ([FPS22]). *The number of approaching arrangement is in $2^{\theta(n^2)}$. There is a pseudoline arrangement that is not isomorphic to an approaching arrangement.*

We now prove Theorem 2.

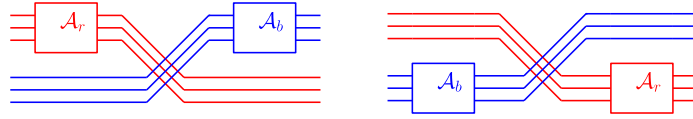


Figure 1: Both extremal arrangements with prescribed sub-arrangements

Proof. Given a block-bicolored arrangement $\mathcal{A}$, we call the arrangement induced by the red pseudolines $\mathcal{A}_r$ and the arrangement induced by the blue pseudolines $\mathcal{A}_b$. Consider the arrangements shown in Figure 1.

In the arrangement to the left, call it $\mathcal{A}_{min}$, for every triple of lines, not all of them of the same color, the pseudoline with the highest index is going below the crossing of the other two. This implies that every bichromatic triple has $-$ orientation.

A similar argument implies that every bichromatic triple in the right arrangement, call it $\mathcal{A}_{max}$, has orientation $+$.

The remaining, monochromatic, triples in $\mathcal{A}$, $\mathcal{A}_{max}$ and $\mathcal{A}_{min}$ have the same orientation.

By the observation above clearly $\mathcal{A}_{min} \leq \mathcal{A} \leq \mathcal{A}_{max}$ holds in the inclusion order.

By Theorem 8, there is a sequence (possibly of length zero!) of “ $-$ to $+$ ” triangle flips, transforming $\mathcal{A}$ into $\mathcal{A}_{max}$. Since both arrangements agree on the monochromatic triples, the sequence only consists of bichromatic triangle flips.

A similar statement holds for $\mathcal{A}_{min}$.

Clearly $\mathcal{A}_{min} \neq \mathcal{A}_{max}$, so not both $\mathcal{A} = \mathcal{A}_{min}$ and $\mathcal{A} = \mathcal{A}_{max}$ can hold, so not both flip sequences can be empty, so there exists at least one bichromatic triangle in $\mathcal{A}$. Moreover if both $\mathcal{A} \neq \mathcal{A}_{min}$ and $\mathcal{A} \neq \mathcal{A}_{max}$, $\mathcal{A}$ has two bichromatic triangles, of opposite orientation. □

Note that this proof gives us a sequence of arrangements looking like we shift a sub-arrangement to the left, similar to the proof for line arrangements.

4 Degenerate Signotopes

4.1 Degeneracy

We will now introduce degeneracy into signotopes. In the topological picture, this means we are talking about pseudohyperplane arrangements in $\mathbb{R}^d$, where more than d pseudohyperplanes are allowed to intersect in a common point. The previous theory of signotopes does not include this.

The *package* of a $k + 1$ element set S is the tuple of all k subsets of S , in ascending lexicographic order. We will use the notation $P(S)$ for the package of S . For example,

$$P(\{1, 2, 3, 4\}) = (\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}).$$

Let $\prec$ be an order on $\{+, -, 0\}$ defined the following way:

$$- \prec 0 \prec +.$$

We also define

$$- \cdot - = +, \quad - \cdot + = -, \quad - \cdot 0 = 0.$$

Definition 11. We call a function

$$\sigma : \binom{[n]}{r} \longrightarrow \{+, -, 0\}$$

a *degenerate signotope of rank r on n elements* if it obeys the following rules for the package P of any $(r + 1)$ -subset of $[n]$:

1. σ is either (weakly) monotonically ascending or descending on P with regard to $\prec$, and
2. if two sets in P are mapped to 0 by σ , every set in P is mapped to 0 by σ .

By $DS(n, r)$ we will denote the set of all degenerate signotopes of rank r on n elements. We will often slightly abuse notation and allow the elements of signotopes to be arbitrary subsets of $\mathbb{N}$. It is clear how this can be translated to signotopes on $[n]$.

Back to the topological picture, a tuple is assigned zero, if all of its $d + 1$ elements meet in a single point. Note that all signotopes are also degenerate signotopes. We will call degenerate signotopes where no tuple is assigned 0 *uniform*.

Let in the following $\sigma \in DS(n, r)$ be a degenerate signotope.

Definition 12 (Deletion, Contraction and Extension). For an $i \in [n]$, let the restriction of σ to

$$\binom{[n] \setminus i}{r}$$

be called the *deletion of i* of σ .

For an $i \in [n]$ let

$$\alpha : \binom{[n] \setminus i}{r-1} \longrightarrow \{+, -, 0\}, \quad x \mapsto \sigma(x + i)$$

be the *contraction to i* of σ .

We call σ^* an *extension* of σ , if σ is a deletion of σ^*

Fact. *The deletions of σ are in $DS(n-1, r)$ and the contractions of σ are in $DS(n-1, r-1)$.*

Definition 13. Let G_σ be a graph with vertices

$$V(G_\sigma) := \binom{[n]}{r-1}.$$

Let $v_1, v_2 \in V(G_\sigma)$ be two sets, such that $|v_1 \cup v_2| = r$. Without loss of generality assume that v_1 is lexicographically smaller than v_2 .

G_σ has the following edges:

- If $\sigma(v_1 \cup v_2)$ is $-$, (v_1, v_2) is a (directed) edge.
- If $\sigma(v_1 \cup v_2)$ is $+$, (v_2, v_1) is a (directed) edge.
- If $\sigma(v_1 \cup v_2)$ is 0 , $\{v_1, v_2\}$ is an (undirected) edge.

Note that if S is a r -subset of $[n]$ and $\sigma(S) \neq 0$, $P(S)$ is a directed path in G_σ .

We will call two vertices v_1, v_2 *equivalent*, in symbols $v_1 \sim_\sigma v_2$, if they are connected by a path of undirected edges.

Define a relation $\leq_\sigma$ on $\mathcal{C}_\sigma = V/\sim_\sigma$ the following way: for $c_1, c_2 \in \mathcal{C}_\sigma$ $c_1 \leq_\sigma c_2$ holds if there exist $v_1 \in c_1, v_2 \in c_2$ and a path $v_1 = w_1, \dots, w_k = v_2$ in G_σ such that no edge has orientation (w_{i+1}, w_i) , for $i = 1, \dots, k-1$. We will often refer to $\mathcal{C}_\sigma$ as the *vertices* of the signotope σ .

The following notation allows us to think about the elements of $\mathcal{C}_\sigma$ as a collection of pseudohyperplanes that intersect in a single point, i.e. vertices of pseudohyperplane arrangements.

Definition 14. For a given degenerate signotope σ and $c \in \mathcal{C}_\sigma$, let

$$U_c := \bigcup_{v \in c} v.$$

In the proof of the next lemma, we will need the following structure.

Definition 15. The *degeneracy hypergraph* $\mathcal{H}_\sigma = ([n], D_\sigma)$ is the r -uniform hypergraph with hyperedges

$$D_\sigma = \{x \subseteq [n] \mid |x| = r, \sigma(x) = 0\}.$$

The following lemma gives us the main tool to deal with degeneracies. The result seems very intuitive in the topological view of signotopes.

Lemma 16 (Equivalence Class Lemma). *Let $c \in \mathcal{C}_\sigma$. Then it holds*

- (i) *c is the set of all $(r-1)$ -subsets of U_c .*
- (ii) *Every edge incident to two elements of c is undirected, i.e. σ is 0 on every r -subset of U_c .*

Proof. If c consists of only one $(r-1)$ -set, the statement is trivially true.

The following technical claim will imply the two statements of the lemma.

Claim. *Let $U \subseteq [n], |U| \geq r$ such that $\mathcal{H}_\sigma[U]$ is a complete r -uniform hypergraph. Assume there is an hyperedge E in $\mathcal{H}_\sigma$, such that $E \setminus U$ is a singleton $\{u\}$. Then $\mathcal{H}_\sigma[U+u]$ is a complete r -uniform hypergraph.*

Proof of Claim. The second property of degenerate signotopes translates to: if a $(r+1)$ -subset of $[n]$ contains at least two hyperedges in $\mathcal{H}_\sigma$, it induces a complete r -uniform hypergraph.

Let U, u, E be as in the statement. Let S be a r -subset of $U+u$ that contains u . We show that S is an hyperedge of $\mathcal{H}_\sigma$.

We proceed by induction on $i := r - |S \cap E|$. For $i = 0$ the statement holds by assumption on E . We prove the statement for $i \geq 1$, assuming it holds for every edge S' with $r - |S' \cap E| < i$. Let v be an element from $E \setminus S$ and let w be an element from $S \setminus E$. Note that v is unequal to u . The package $P(S+v)$ contains at least two hyperedges of $\mathcal{H}_\sigma$: $S+v-u$ by the assumption on U and $S+v-w$ by the induction hypothesis, since $|(S-w+v) \cap E| + 1 = |S \cap E|$, implying that $S \in D_\sigma$.

It follows that $\mathcal{H}_\sigma[S+u]$ is the complete k -uniform hypergraph. ■

It is sufficient to prove that $\mathcal{H}_\sigma[U_c]$ is the complete r -uniform hypergraph on U_c .

Since c is non-trivial, there is an undirected edge $\{v_0, v_1\}$, with $v_1, v_2 \in c$. We traverse the component connected by undirected edges of G_σ that contains v_1 starting in v_1 , for example by breadth-first-search.

During this process we will maintain a set S , which contains all the elements we have seen, i.e. S is the union of all the vertices we have seen so far.

We start with $S = \{v_1 \cup v_2\}$. Note that this implies that $\mathcal{H}_\sigma[S]$ is a complete r -uniform hypergraph. If we now traverse the edge $\{v_{k-1}, v_k\}$, such that v_k contains an element that is not already in S , it has exactly one such element u , by the definition of the edges of G_σ . Then we apply the claim proven above to S and $v_{k-1} \cup v_k$ to see that $\mathcal{H}_\sigma[S+u]$ is a complete hypergraph.

This implies (ii).

(i) follows by the fact that we can easily construct a path between

$$v_1, v_2 \in \binom{[U_c]}{r-1}$$

by repeatedly replacing elements in $v_1 \setminus v_2$ by elements in $v_2 \setminus v_1$. $\square$

The following will allow us to lift properties from uniform to degenerate signotopes.

Lemma 17 (Ordered Perturbation Lemma). *Let $X \sim_\sigma Y$. Then there exists $\sigma' \in DS(n, r)$, where $X \leq_{\sigma'} Y$, such that σ' is a perturbation of σ , i.e. if $\sigma(x) \neq \sigma'(x)$, then $\sigma(x) = 0$.*

Proof.

Claim. *Let $X, Y \in \binom{[n]}{r-1}$. There is a (uniform) signotope σ of rank r with n elements, such that $X \leq_\sigma Y$.*

Proof of Claim. For for a n -permutation $z = (z_1, \dots, z_n)$ and $s \in \{+, -\}^n$, let

$$f_{s,z} : \binom{[n]}{r} \longrightarrow \{+, -\}$$

be defined the following way:

$$f_{s,z}(\{u_1 < \dots < u_r\}) = s_i(-)^{(j+1)},$$

if z_i is the first element in z that is in $\{u_1 < \dots < u_r\}$ and $u_j = z_i$. It is straightforward to check that $f_{s,z}$ is a signotope.

Let $\{x_1 < \dots < x_m\} = X \setminus Y$ and $\{y_1 < \dots < y_m\} = Y \setminus X$. For $k = 1, \dots, m$, let

$$d_k := \{y_1, \dots, y_k, x_{k+1}, \dots, x_m\} \cup (X \cap Y) \in \binom{[n]}{r-1}$$

and

$$e_k := \{y_1, \dots, y_k, x_k, \dots, x_m\} \cup (X \cap Y) \in \binom{[n]}{r}.$$

Also let $d_0 = X$.

Note that d_{k-1} and d_k are elements of $P(e_k)$.

We now choose $z = (x_1, x_2, \dots, x_m, \dots)$ a permutation of $[n]$. We still need to choose $s = (s_1, \dots, s_n)$. Note that the smallest element in e_k that is also in X is x_k , so $f_{s,z}(e_k)$ only depends on s_k .

This allows us to choose s in such a way that for every $k = 1, \dots, m$, $P(e_k)$ is oriented from d_{k-1} to d_k , in the graph of the signotope $f_{s,z}$.

Overall there is a path $X = d_0, \dots, d_m = Y$ in the graph of $f_{s,z}$. $\blacksquare$

To prove the lemma, we take the equivalence class $c \in \mathcal{C}_\sigma$ that contains X and Y . Then choose a signotope σ^* on U_c where $X \leq_{\sigma^*} Y$ to obtain the perturbation σ' :

$$\sigma'(x) = \begin{cases} \sigma^*(x), & \text{if } x \subseteq U_c \\ \sigma(x), & \text{else.} \end{cases}$$

It is straightforward to see that the perturbation is again a signotope with the wanted properties. $\square$

The following statement was proved for uniform signotopes in [FW01]. We could prove it in a similar fashion by a deletion-contraction argument, but we will see that Lemma 17 gives a pleasant proof.

Let $\mathbf{P}_\sigma := (\mathcal{C}_\sigma, \leq_\sigma)$.

Lemma 18. $\mathbf{P}_\sigma$ is a partial order.

Proof. Transitivity and reflexivity are clear by the definition of $\leq_\sigma$. We still need to prove acyclicity.

Towards a contradiction, assume that there is a directed cycle in $\leq_\sigma$. Then there is a cycle in G_σ , that does not go against any oriented edges. Choose a shortest such cycle $C = v_1, v_2, \dots, v_1$.

If $v_j, v_{j+1}, \dots, v_{j+k}$ is a segment of the cycle where all edges are unoriented, we can perturb σ (i.e. change 0 to + or -) to σ' with the help of Lemma 17, such that $v_j \leq_{\sigma'} v_{j+k}$.

We can do this for every unoriented edge in C . We then can change every remaining 0 to +, to obtain an uniform signotope whose graph contains a cycle, contradicting the fact that the statement holds for uniform signotope. $\square$

The following is a generalization of Proposition 3.1 in [BFS23a] to degenerate signotopes.

Lemma 19. Let $\sigma \in DS(n, r)$ and $\mathcal{I}$ be an ideal of $\mathbf{P}_\sigma$ and $\mathcal{T} \subseteq \max(\mathcal{I})$. We define a function

$$\sigma^* : \binom{[n] + 0}{r} \longrightarrow \{+, -, 0\}$$

by setting¹

$$\sigma^*(x) = \begin{cases} \sigma(x), & 0 \notin x \\ +, & [x \setminus 0] \in \mathcal{I} \setminus \mathcal{T} \\ 0, & [x \setminus 0] \in \mathcal{T} \\ -, & \text{else.} \end{cases}$$

Then $\sigma^*(x)$ is an extension of σ .

Proof. We need to prove that σ^* is a degenerate signotope. Then, by definition, it is clear that σ^* is an extension of σ .

It is sufficient to consider the package of $\mathcal{U}$, a $(r+1)$ -subset of $[n] + 0$ that contains 0. In $P(\mathcal{U})$, only the last set S does not contain 0. In the following we rely on the fact that if $x < y$ and $y \in \mathcal{I}$, $x \in \mathcal{I}$ holds.

If $\sigma(S)$ is plus, $P(S)$ is descendingly ordered by $\mathbf{P}_\sigma$ and all sets in $P(S)$ belong to different equivalence classes. It follows that the sets in $P(S)$ that are in $\mathcal{I} \setminus \mathcal{T}$ form a terminal segment of $P(S)$. The sets in $P(S)$ that are not in $\mathcal{I}$ form an initial segment of $P(S)$. Since $P(S)$ is a descending chain in $\mathbf{P}_\sigma$, there is at most one set in $P(S)$ that is also in $\mathcal{T}$. This implies that the properties of degenerate signotopes are respected.

¹Note that in this definition $[x]$ denotes the equivalence class of x under $\sim_\sigma$.

The case $\sigma(S) = -$ follows analogously.

If $\sigma(S) = 0$, all the sets in $P(S)$ are equivalent. It follows that σ^* is constant on $P(\mathcal{U}) \setminus S$. Since S is the last element in $P(\mathcal{U})$, this respects the properties of degenerate signotopes.

So σ^* is a degenerate signotope. $\square$

Geometrically speaking, the next lemma tells us that if two vertices lie on many common pseudohyperplanes, they lie on a line.

Lemma 20. *Let $c_1 \neq c_2 \in \mathcal{C}_\sigma$. If $|U_{c_1} \cap U_{c_2}| = r - 2$, c_1 and c_2 are comparable in $\mathbf{P}_\sigma$.*

Proof. Let $I = U_{c_1} \cap U_{c_2}$ and let $x \in U_{c_1} \setminus U_{c_2}$ and $y \in U_{c_2} \setminus U_{c_1}$. By Lemma 16, $I + x \in c_1$ and $I + y \in c_2$.

It cannot be that $\sigma(I + x + y)$ is zero, otherwise $I + x$ and $I + y$ would be equivalent contradicting the assumption that $c_1 \neq c_2$.

Since there is a directed edge between $I + x$ and $I + y$, c_1 and c_2 are comparable in $\mathbf{P}_\sigma$. $\square$

4.2 Rotations

We will now repeat some facts about rotations of signotopes, introduced by Bergold, Felsner and Scheucher in [BFS23a]. Note that full proofs can be found in [BFS23b]. We will slightly alter the definitions and generalize the notions to degenerate signotopes.

For a given n , let s_1 be the shift function

$$s : [n] \longrightarrow [n], \quad s(n) = 1, \quad s(n-1) = n, \quad \dots, \quad s(1) = 2.$$

For $2 \leq i \leq n$, let

$$s_i := s_{i-1} \circ s_1.$$

For convenience, let s_0 be the identity.

Definition 21. Given a degenerate signotope $\sigma \in DS(n, r)$, let $\text{rot}(\sigma, i)$ be the following function:

$$\text{rot}(\sigma, i) : \binom{[n]}{r} \longrightarrow \{+, -, 0\}, \quad x \mapsto \begin{cases} \sigma(x), & |x \cap [i]| \text{ even} \\ -\sigma(x), & \text{else.} \end{cases}$$

Note that $\text{rot}(\sigma, 0) = \sigma$.

The rank three case gives intuition for rotations. A signotope $\sigma \in DS(n, 3)$ can be represented as a marked pseudoline arrangement. The *rotation* of σ can be seen as the pseudoline arrangement we obtain by moving the northpole over an edge to the unbounded face to the left of the previously marked face, but keeping the current numbering of pseudolines. After renaming the pseudolines according to our rule, we would obtain another signotope. We consider an example in Figure 2.

The following is a straightforward generalization of Lemma 2.1 in [BFS23a].

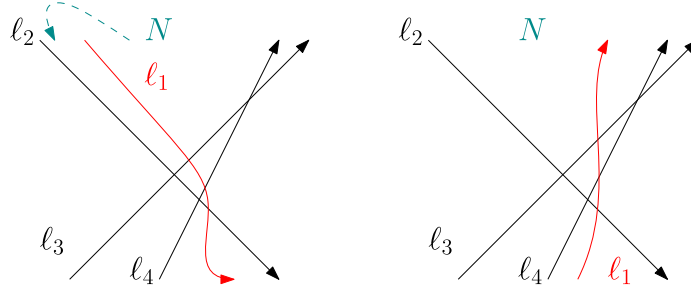


Figure 2: Example for a rotation. Let the signotope given by the pseudoline arrangement on the left be σ . Note that the plus tuples of σ are exactly $(1, 2, 3)$ and $(1, 2, 4)$. It follows that the only plus tuple of $\text{rot}(\sigma, 1)$ is $(1, 3, 4)$. After renumbering the pseudolines according to the rule stated in the preliminaries, we would obtain the signotope σ' with the only plus tuple $(2, 3, 4)$. This is the same function as $\text{rot}(\sigma, 1) \circ s_1$.

Lemma 22. $\text{rot}(\sigma, i) \circ s_i$ is a degenerate signotope.

Each rotation of σ will give rise to an order of $\mathcal{C}_\sigma$ in a natural way.

Definition 23. For $c_1, c_2 \in \mathcal{C}_\sigma$ and $0 \leq i \leq n$, let

$$c_1 \leq_{\text{rot}(\sigma, i)} c_2$$

if and only if $s_i^{-1}(c_1) \leq_{\sigma'} s_i^{-1}(c_2)$ in the order given by the signotope $\sigma' := \text{rot}(\sigma, i) \circ s_i$.

Define the graph $G_{\text{rot}(\sigma, i)}$ similarly as a relabelling of $G_{\text{rot}(\sigma', i)}$.

The next statement is a simple generalization of Proposition 3.3 and the following remark in [BFS23b]. It again follows by assuming the uniform case and applying some perturbations.

Lemma 24 (Rotation Order Lemma). *For two elements $c_1, c_2 \in \mathcal{C}_\sigma, c_1 \neq c_2$ with $c_1 \leq_{\text{rot}(\sigma, i)} c_2$ it holds that $c_1 \geq_{\text{rot}(\sigma, i+1)} c_2$ if and only if $i+1 \in U_{c_1} \cap U_{c_2}$.*

Proof. We first prove the "if" part.

We perturb σ with the help of Lemma 17, until we obtain a uniform signotope σ' with $v_j \in c_j, i+1 \in v_j, j = 1, 2$, where $v_1 \leq_{\text{rot}(\sigma', i)} v_2$. Since the statement holds for uniform signotopes, it follows that $v_2 \leq_{\text{rot}(\sigma', i+1)} v_1$. The path in $G_{\text{rot}(\sigma', i+1)}$ connecting v_2 to v_1 can be embedded into a path in $G_{\text{rot}(\sigma, i+1)}$, where some directed edges become undirected. But this implies $c_2 \leq_{\text{rot}(\sigma, i+1)} c_1$.

The "only if" part follows similarly: towards a contradiction, assume there are c_1, c_2 where $i+1 \notin U_{c_1} \cap U_{c_2}$, with

$$c_1 \leq_{\text{rot}(\sigma, i)} c_2 \quad \text{and} \quad c_2 \leq_{\text{rot}(\sigma, i+1)} c_1.$$

Let p_1 be the path in $G_{\text{rot}(\sigma, i)}$ going from $v_1 \in c_1$ to $v_2 \in c_2$ and let p_2 be the path in $G_{\text{rot}(\sigma, i+1)}$ from $v'_2 \in c_2$ to $v'_1 \in c_1$.

If there is $c_3 \in \mathcal{C}_\sigma$ with $v_3, v_4 \in c_3$ and $v_3 \in p_1$ and $v_4 \in p_2$, we can exchange c_1 or c_2 with c_3 , depending which of both does not contain $i + 1$. The statement then still holds.

So, we can assume p_1 and p_2 to not have vertices of the same equivalence class of $\sim_\sigma$, except for c_1 and c_2 . Then we can apply Lemma 17 to make p_1 and p_2 into degeneracy-free paths. This leads to a contradiction, since the statement of the theorem holds for uniform signotopes. $\square$

4.3 Sweeps

Signotopes can be represented pseudohyperplane arrangements. As such, they admit *topological sweeps*. For an introduction to topological sweeps consult chapter 10 in [Bjö+93].

We will consider a less general notion of sweeps, introduced in [FW01].

Definition 25. Let $\sigma \in DS(n, r)$ be a degenerate signotope. We call a linear extension of $\mathbf{P}_\sigma$ a sweep of σ . We call a linear extension of $(\mathcal{C}_\sigma, \leq_{\text{rot}(\sigma, i)})$ a sweep of $\text{rot}(\sigma, i)$.

Example. Consider the signotope σ as described in Figure 2. A possible sweep of $\text{rot}(\sigma, 1)$ is $(2, 3), (2, 4), (2, 1), (4, 1), (3, 1), (3, 4)$. We obtain the corresponding sweep of the signotope $\text{rot}(\sigma, 1) \circ s_1$ by applying s_1^{-1} to every vertex in the previous sequence: $(1, 2), (1, 3), (1, 4), (3, 4), (2, 4), (2, 3)$.

Example. Since we will need it later, we will consider the sequence of sweeps of $\text{rot}(\sigma, i), i = 0, \dots, n$, for a fixed $\sigma \in DS(n, n)$.

Assume that $\sigma([n]) = -$, the other case follows analogously. Then the sweep of $\sigma = \text{rot}(\sigma, 0)$ is

$$[n] \setminus n, [n] \setminus (n-1), \dots, [n] \setminus 1.$$

In more generality, the sweep of $\text{rot}(\sigma, i) \circ s_i$ is

$$[n] \setminus n, [n] \setminus (n-1), \dots, [n] \setminus 1$$

for even i and

$$[n] \setminus 1, [n] \setminus 2, \dots, [n] \setminus n$$

for odd i . We now apply s_i to obtain that

$$[n] \setminus s_i(n), [n] \setminus s_i(n-1), \dots, [n] \setminus s_i(1)$$

is a sweep of $\text{rot}(\sigma, i)$ for even i . Then $\text{rot}(\sigma, i+1)$ has the sweep

$$[n] \setminus s_{i+1}(1), [n] \setminus s_{i+1}(2), \dots, [n] \setminus s_{i+1}(n) = [n] \setminus s_i(2), [n] \setminus s_i(3), \dots, [n] \setminus s_i(1),$$

which we can obtain from the previous sequence by inverting the order of the first $n-1$ subsets.

A similar result holds for odd i , by inverting the order of the last $n-1$ subsets of the sequence.

We see that between $\text{rot}(\sigma, i)$ and $\text{rot}(\sigma, i+1)$ only the order of the sets containing $i+1$ changes.

5 Pseudoconfiguration of Points

Goodman in [Goo80] was the first to consider (1-intersecting) pseudoconfigurations of points as duals to projective pseudoline arrangements.

Later Agarwal and Sharir introduced in [AS05] pseudoconfigurations of points as duals of marked euclidean arrangements, i.e. rank 3 signotopes.

In [Miy21] Miyata generalized the original definition to higher degrees. He also proved that every k -intersecting pseudoconfiguration of points represents a signotope. Experts raised concerns about the validity of his construction of pseudoconfigurations representing a given signotopes. We provide a novel signotope-first proof that highlights the connection between pseudoconfiguration of points and extensions of signotopes.

5.1 Topological Representations of Signotopes

Signotopes are a special class of oriented matroids. Thus, they admit a large number of topological representations: they can be represented as extensions of the cyclic hyperplane arrangement [Zie93]. By oriented matroid duality, this implies that they can be viewed as liftings of the cyclic hyperplane arrangements, see [FZ01]. These liftings were called C_d -arrangements in [FW01]. It was also pointed out in [FZ01] that signotopes can be seen as zonotopal tilings of cyclic zonotopes.

For signotopes with high rank and co-rank, all these representations are high-dimensional. Pseudoconfigurations of points will offer a plane representation for all signotopes.

5.2 Intuition

We will give some intuition about the objects considered in our main results. Let $P = \{p_1 = (x_1, y_1), \dots, p_n = (x_n, y_n)\}$ be a collection of points in the plane, such that the x coordinates are pairwise different. For a given k , let P' be the following collection of points in $\mathbb{R}^{k+2}$:

$$P' := \{p'_i = (1, x_i, x_i^2, \dots, x_i^k, y_i) \mid (x_i, y_i) \in P\}.$$

Let H be a hyperplane through at least $k + 1$ points $\{p'_{i_1}, \dots, p'_{i_m}\}$ with normal vector ν . Note that $0 \in H$. It can be shown that $\nu_{k+2} \neq 0$. Since a hyperplane does not change by scaling its normal vector, we assume $\nu_{k+2} = -1$. We can associate an univariate polynomial with ν :

$$q(x) = \sum_{i=0}^k \nu_i x^i.$$

Now the following holds:

$$q(x_i) - y_i = \langle \nu, p'_i \rangle.$$

This shows that H corresponds to a polynomial that goes through the points $p_{i_1}, \dots, p_{i_m}$. Call this set q^0 . There are points in P that lie below the curve of

q . Call these q^+ . Analogously define q^- . This gives us a partition of P into three parts.

H partitions P' into three subsets:

$$\begin{aligned} H^+ &= \{p'_i \mid \langle p'_i, \nu \rangle > 0, i = 1, \dots, n\} \\ H^0 &= \{p'_i \mid \langle p'_i, \nu \rangle = 0, i = 1, \dots, n\} \\ H^- &= \{p'_i \mid \langle p'_i, \nu \rangle < 0, i = 1, \dots, n\}. \end{aligned}$$

The partitions induced by q and H are identically in a natural way. So in a certain sense the combinatorial properties of the hyperplanes spanned by P' are identical to the polynomial curves of degree k going through points in P .

This leads us to the following observation: certain high-dimensional point sets and the hyperplanes spanned by them can be embedded into a two dimensional point set and polynomial curves through these points.

We want to characterize these point sets: We lifted n points from the k -th order moment curve $M_k(t) = (1, t, t^2, \dots, t^k) \in \mathbb{R}^{k+1}$ into $\mathbb{R}^{k+2}$ by adding a new coordinate. This is a special case of the following theorem:

Theorem 26 ([FZ01]). *Liftings of the cyclic point configuration $C_{n,k}$ correspond to the signotopes in $S(n, k)$.*

Note that in [FZ01], liftings are purely topological/combinatorial objects. Until now we only considered the geometric notion of lifting.

Grasping for intuition, we will look at vertical lines in a collection of points and the polynomial curves PP spanned by them. Let σ be the corresponding signotope, which exists by Theorem 26. We will only state some intuitive facts, which we will later prove in more generality.

Let $H = \{x = x_0\}$ be such a line. Every point (x_0, y) on this line corresponds to the point

$$(1, x_0, x_0^2, \dots, x_0^k, y) \in \mathbb{R}^{k+2}.$$

If we start at $y = -\infty$ and continuously increase y , we intersect the polynomials in PP in a certain order. In the high dimensional collection of points, we intersect the corresponding hyperplanes in the same order. This is a sweep of the corresponding signotope!

We can also observe what happens if we continuously change the parameter x_0 : if x_0 is smaller than the x -value of p_1 , the sweep is an ordinary sweep.

If we now increase x_0 , two things can happen: Our line H could contain an intersection of curves, but not the point p_1 . This corresponds to a sweep through the corresponding hyperplane arrangement, where the sweep point intersects multiple hyperplanes $H_1, \dots, H_m$ at once. It is notable that we can "stop" the sweep and find an extension contained in $H_1, \dots, H_m$.

The other option is that H crosses p_1 . Then the corresponding sweep hits all the hyperplanes that contain p_1 at the same time (in the point p_1). In a certain sense this corresponds to a sweep parallel to p_1 .

If we increase x_0 further, we obtain a sweep by an element that comes *after* p_1 . This means that H then corresponds to a sweep of $\text{rot}(\sigma, 1)$.

5.3 Pseudoconfiguration of Points

It is natural to ask whether signotopes admit a representation as a topological version of the objects considered in the previous section.

Definition 27 ([Miy21]). Let P be a collection of n points in the plane, with pairwise different x -coordinates. Let L be a collection of x -monotone² curves. $PP = (P, L)$ is a k -intersecting pseudoconfiguration of points if:

1. On every $\ell \in L$, there are at least $k + 1$ points of P .
2. For every $k + 1$ points of P , there is a unique $\ell \in L$ that goes through them.
3. Any two curves in L intersect at most k times. All intersections between curves are crossings and not touchings.

If $(p_1, \dots, p_n)$ are the elements of P ordered by x -coordinate, we will often identify p_i with i . We will also often identify curves in L with the collection of points in P that lie on them. We will refer to P as the *points* of PP and similarly to L as the *curves* of PP . We will call the set of all k -intersecting pseudoconfigurations of points on n points $PP(n, k)$.

Given a k -intersecting pseudoconfiguration of n points $PP = (P, L)$, we construct a signotope that represents the combinatorial properties of (P, L) . For this we introduce the notation $\ell(p_{i_1}, \dots, p_{i_{k+1}})$ for the unique curve through $p_{i_1}, \dots, p_{i_{k+1}}$.

We will prove next that the following map gives rise to a signotope. We say that a point is *below* a x -monotone curve, if the point is below the intersection of the curve with the vertical line through the point.

Definition 28. In the following, always assume that $i_j < i_{j'}$, for $j < j' \in [n]$. For $PP \in PP(n, k)$, define σ_{PP} to be the following function:

$$\sigma_{PP} : \binom{[n]}{k+2} \longrightarrow \{+, -, 0\}$$

$$\sigma_{PP}(\{p_{i_1}, \dots, p_{i_{k+2}}\}) := \begin{cases} -, & p_{i_1} \text{ below } \ell(p_{i_2}, \dots, p_{i_{k+2}}) \\ 0, & p_{i_1} \text{ on } \ell(p_{i_2}, \dots, p_{i_{k+2}}) \\ +, & p_{i_1} \text{ above } \ell(p_{i_2}, \dots, p_{i_{k+2}}). \end{cases}$$

We will now start our investigation of pseudoconfiguration of points guided by the idea that vertical lines should correspond to sweeps.

Since a function

$$f : \binom{[n]}{n} \longrightarrow \{+, -, 0\}$$

is always a signotope, the next lemma is sensible.

²Here x -monotone curves are exactly the graphs of functions that map $\mathbb{R}$ to $\mathbb{R}$.

Lemma 29. *Let $PP \in PP(k+2, k)$. Let H_i be a vertical line that does not go through any intersections or points of PP , such that exactly i points of PP are to the left of H_i . The order in which H_i intersects the curves of PP from bottom to top is a sweep of $\text{rot}(\sigma_{PP}, i)$.*

Proof. Let $S = \{p_1, \dots, p_{k+2}\}$ be the points of PP ordered by x -coordinate.

Note that two curves of PP have k points of PP in common. This implies that there are no intersections of curves that are not on a point of PP .

We first prove the claim by induction over k and for $i = 0$. This implies that the point p_1 is to the right of H_0 . By inspection the claim holds for $k = 1$. Assume now $k > 1$ and that the statement holds for all smaller values of k . Assume $\sigma_{PP}(S) = -$, the other cases follow similarly. We delete the curve $\ell(S \setminus p_1)$. Then we delete p_1 and locally redraw the remaining curves such that they do not intersect in p_1 . These curves and the points $p_2, \dots, p_{k+2}$ are a $(k-1)$ -intersecting pseudoconfiguration PP' .

We now show: $\sigma_{PP'}(S \setminus p_1) = +$. Since p_1 is below $\ell(S \setminus p_1)$ and $\ell(S \setminus p_2)$ goes through p_1 , $\ell(S \setminus p_2)$ has to go below p_2 , otherwise there would be an additional crossing.

By induction, the curves in PP' are lexicographically descending along H_0 . After reintroducing p_1 and the crossings in p_1 , they are lexicographically ascending, since $\ell(S \setminus p_1)$ goes above p_1 and $S \setminus p_1$ is the lexicographically maximal set in $P(S)$.

Since we have just proven the claim for $i = 0$, we now prove the claim for $i > 0$ by induction. Note that the unique curve of PP not containing p_i , $\ell(S \setminus p_i)$, goes above or below the point p_i . This implies that it is above or below all the other curves between p_{i-1} and p_{i+1} . We obtain the order of intersections along H_i by considering the order of intersections along H_{i-1} and reversing the order of all the curves that contain p_i .

Since the order of intersections along H_{i-1} was a sweep of $\text{rot}(\sigma_{PP}, i-1)$, the element corresponding to p_i is the first element of $\text{rot}(\sigma_{PP}, i-1)$. The sweep of $\text{rot}(\sigma_{PP}, i)$ is exactly the order of intersections along H_i , compare with section 4.3.

□

Theorem 30. *For a given $PP \in PP(n, k)$, σ_{PP} is in $DS(n, k+2)$.*

Proof. We now prove that σ_{PP} respects the properties of signotopes. Since it is sufficient to prove monotonicity for a single package, we can assume that PP contains exactly $k+3$ points $S = \{p_1, \dots, p_{k+3}\}$. Assume that $\sigma_{PP}(S \setminus p_1) = -$, the other cases follows similarly. By Lemma 29, the curves that do not contain p_1 are lexicographically ascending when intersected with a vertical line through p_1 . Since $S \setminus p_1$ is lexicographically maximal in the package of S and the curves that are below p_1 form a lexicographic downset in the package, monotonicity holds. See also Figure 3. □

The preceding theorem and lemma imply the following.

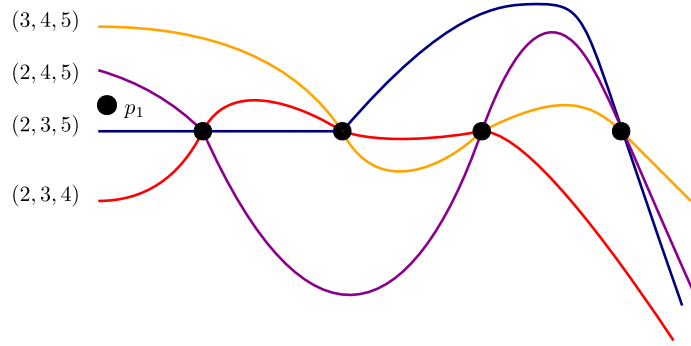


Figure 3: The situation in the proof of Theorem 30. The value of σ_{PP} will be plus on $(1,2,3,4)$ and $(1,2,3,5)$, since the curves $(2,3,4)$ and $(2,3,5)$ are below the point p_1 . The value of σ_{PP} will be minus on $(1,2,4,5)$ and $(1,3,4,5)$ since the curves $(2,4,5)$ and $(3,4,5)$ go above p_1 . In the proof we assumed that $\sigma_{PP}(2,3,4,5) = -$.

Corollary 31. *Let $PP \in PP(n, k)$. Let H_i be a vertical line that does not go through any intersections or points of PP , such that exactly i points of PP are to the left of H_i . The order in which H_i intersects the curves of PP from bottom to top is a sweep of $\text{rot}(\sigma_{PP}, i)$.*

Proof. We consider a sub-pseudoconfiguration PP' of PP consisting of $k+2$ points $p_{i_1}, \dots, p_{i_{k+2}}$ and the curves in L spanned by $p_{i_1}, \dots, p_{i_{k+2}}$. Let $S = \{i_1, \dots, i_{k+2}\}$. There are $m \leq i$ points of PP' to the left of H_i . Let σ' be the signotope represented by PP' and σ the signotope represented by PP .

We are interested in the order of $S \setminus i_1, \dots, S \setminus i_{k+2}$ in $\text{rot}(\sigma, i)$. By Lemma 29, the order in which H_i intersects PP' is a sweep of $\text{rot}(\sigma', m)$. Since $|S \cap [i]| = m$, it holds that $\text{rot}(\sigma, i)(S) = \text{rot}(\sigma', m)(S)$, by the definition of rotations. Since the elements of S appear in the same order in $s_i(1), s_i(2), \dots, s_i(n)$ and $s_m(1), s_m(2), \dots, s_m(n)$, the package of S has the same order in the sweep of $\text{rot}(\sigma, i)$ and $\text{rot}(\sigma', m)$. □

We will now come to a novel construction of a pseudoconfiguration representing a given signotope.

Definition 32. Let $\sigma \in DS(n, r)$. We call σ^* a *completion* of σ , if it fullfills the following conditions:

1. We obtain σ by repeatedly deleting elements in σ^* . Especially the vertices of σ correspond to some vertices of σ^* (in symbols, by abuse of notation: $\mathcal{C}_\sigma \subset \mathcal{C}_{\sigma^*}$).
2. In every rotation of σ^* , every two vertices of σ are comparable, i.e. the relation $\leq_{\text{rot}(\sigma^*, i)}$ induces a chain on $\mathcal{C}_\sigma$ for every i that is between 0 and the number of elements of σ^* .

Lemma 33. *Every signotope admits a completion.*

Proof. Assume that σ^* contains σ as a deletion, but it is not a completion of σ . Then there is an i and $c_1, c_2 \in \mathcal{C}_\sigma$, such that c_1 and c_2 are incomparable in $\text{rot}(\sigma^*, i)$. By Lemma 19, there is an extension of $\text{rot}(\sigma^*, i) \circ s_i$ with an element going through c_1 and c_2 .

By Lemma 20, if c_1 and c_2 share $r - 2$ elements, c_1 and c_2 are comparable in every rotation. This implies that the process stops at some point, proving the existence of a completion. $\square$

Note that the subposet of $\mathbf{P}_{\sigma^*}$ induced by $\mathcal{C}_\sigma$ is an extension of $\mathbf{P}_\sigma$, i.e. for $c_1, c_2 \in \mathcal{C}_\sigma$ with $c_1 \leq_\sigma c_2$ it holds that $c_1 \leq_{\sigma^*} c_2$. The same holds for rotations. Also note that in general the completion is highly degenerate.

Theorem 34. *Let $\sigma \in DS(n, r)$. There exists a pseudoconfiguration of points $PP \in PP(n, r - 2)$, such that $\sigma = \sigma_{PP}$*

A similar technique as in the following proof was used in [Goo80] to construct a pseudoconfiguration of points dual to a given pseudoline arrangement.

Proof. Let σ^* be a completion of σ . Assume that σ^* consists of m elements.

We will look at sweeps of $\text{rot}(\sigma^*, i)$, for $i = 0, \dots, m$. For a fixed i , the elements of $\mathcal{C}_\sigma$ appear in the same order in every sweep of $\text{rot}(\sigma^*, i)$, since they form a chain. We will denote this order by π_i for $i = 0, \dots, m$.

We want to describe the difference between π_i and π_{i+1} . By Lemma 24, the order of two elements $c_1, c_2 \in \mathcal{C}_\sigma$ changes between π_i and π_{i+1} if and only if the element $i + 1$ is contained in both c_1, c_2 , since in no rotation of σ^* pairs of elements of $\mathcal{C}_\sigma$ are incomparable. This implies that all vertices of σ that contain $i + 1$ appear contiguously in π_i and π_{i+1} .

For every $c \in \mathcal{C}_\sigma$, denote by $\pi_i(c)$ the position of c in π_i . Consider the following collection of points:

$$\{c_i := (i, \pi_i(c)) \mid i = 0, \dots, m \text{ and } c \in \mathcal{C}_\sigma\}.$$

Now interpolate for every c linearly between the points $c_i, i = 0, \dots, m$. Extend these polygonal curves by horizontal rays at c_0 and c_m . Call the resulting curves $\ell_c, c \in \mathcal{C}_\sigma$. Note that ℓ_c is x -monotone.

By the remark above, all vertices whose curves have intersections with x -coordinates in the interval $[i, i + 1]$ share the element $i + 1$ and intersect in the same point. If the element $i + 1$ corresponds to the element j of σ , call this point p_j . Since σ is contained in σ^* as a deletion, every element of σ is rotated exactly once. This makes the following definition sensible:

$$P := \{p_j \mid j \in [n]\}.$$

Let in the following k be equal to $r - 2$.

Claim. $PP = (P, \{\ell_c \mid c \in \mathcal{C}_\sigma\})$ is a k -intersecting pseudoconfiguration of points.

Proof of Claim. Two different vertices can share at most k elements, by Lemma 16. This implies that the curves ℓ_c pairwise intersect at most k times.

Note that every $c \in \mathcal{C}_\sigma$ contains at least $k + 1$ elements, implying that ℓ_c goes through at least $k + 1$ points of PP .

Let $I \subseteq [n]$ be an index set of size $k + 1$. By Lemma 16, there is a unique vertex c in $\mathcal{C}_\sigma$ that contains the elements of I . Note that the curve ℓ_c goes through the points $\{p_j \mid j \in I\}$.

This shows that PP is a k -intersecting pseudoconfiguration of points. ■

Claim. $\sigma_{PP} = \sigma$.

Proof of Claim. By construction, the order in which the vertical line through 0 intersects the curves of PP is a sweep of σ . By Corollary 31, the vertical line is also sweep of σ_{PP} .

It was shown in [FW01] that signotopes can be uniquely reconstructed from their sweeps, so the claim follows. ■

The two claims imply the theorem. See Figure 4 for an example of this construction. □

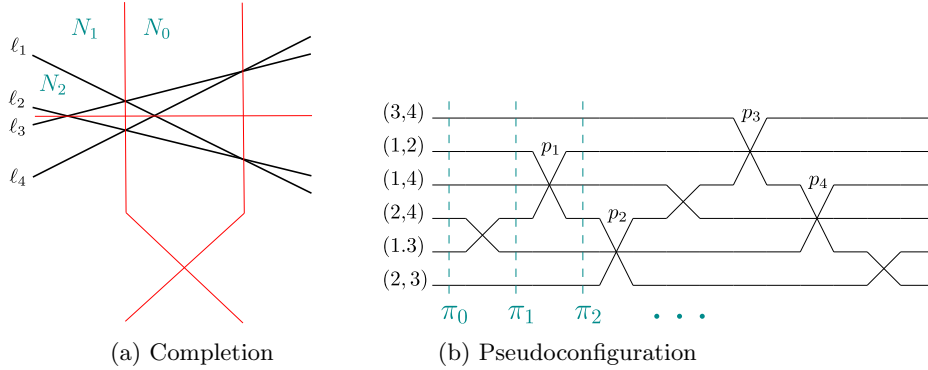


Figure 4: We construct a pseudoconfiguration that corresponds to the signotope given by the black pseudoline arrangement, where the northpole is at position N_0 . We add the red pseudolines to obtain a completion. Afterwards we consider the sweeps of the completion and its rotations, indicated by the different choices of northpoles. We construct a pseudoconfiguration from them like indicated in the right picture.

Since we established the connection between signotopes in $S(n, r)$ and $(r-2)$ -intersecting pseudoconfigurations of points in $PP(n, r-2)$, we will simply call them pseudoconfigurations, if their parameters can be inferred from the context.

6 Extensions of Signotope

Extending a given d -dimensional hyperplane arrangement with a hyperplane that contains d prescribed points is simple. In the case of pseudohyperplane arrangements, this is no longer true: Richter-Gebert described a pseudohyperplane arrangement $\mathcal{M}$ in [Ric93] with two points, such that there is no extension of $\mathcal{M}$ by a pseudohyperplane containing these two points.

In this section we will consider the extensions of signotopes. Despite the fact that signotopes can be considered as pseudohyperplane arrangements, an extension of a signotope is again a signotope. This means that extensions of signotopes are a subset³ of the extensions of the underlying pseudohyperplane arrangements.

Let in the following $\sigma \in S(n, r)$ be a signotope. Let x be an element of σ . We say that x goes through the vertex $c_1 \in \mathcal{C}_\sigma$, if $\sigma(c_1 + x) = 0$.

We call a signotope *2-extendable*, if for every choice of two disjoint vertices c_1, c_2 there is an extension of σ , such that the additional element goes through c_1 and c_2 .

The 2-extendability of rank three signotopes is long known as *Levi's Extension Lemma*.

Theorem 35 ([Lev26]). *Let $\mathcal{A}$ be a pseudoline arrangement, with c_1 and c_2 two points that are not on the same pseudoline. There exists a pseudoline p , such that $\mathcal{A} \cup p$ is a pseudoline arrangement and c_1 and c_2 lie on p .*

Bergold, Felsner and Scheucher generalized this result to all signotopes of odd rank.

Theorem 36 ([BFS23a]). *Signotopes of odd rank are 2-extendable.*

By computer search, they found examples of signotopes that are not 2-extendable with ranks 4, 6, 8, 10 and 12. For the rank 4 case, see Figure 5. They conjectured the following.

Theorem 37. *For every even $r \in \mathbb{N}, r \geq 4$, there is a rank r signotope, that is not 2-extendable.*

To resolve this conjecture we need a criterion for the existence of extensions through vertices of $\sigma \in S(n, k)$

Lemma 38. *If there is an extension of σ through c_1 and c_2 , where $c_1, c_2 \in \mathcal{C}_\sigma$ does not have an extension through the two vertices represented by them, there is a pseudoconfiguration representing σ , where ℓ_{c_1} and ℓ_{c_2} intersect.*

Proof. Let σ' be the extension of σ by the element x in question. Let $PP = (P, L)$ be a pseudoconfiguration representing σ' by Theorem 34. Note that in PP the curves corresponding to c_1 and c_2 intersect in the point corresponding to x . Delete first the point x from P and then all the curves that contain less than $k - 1$ of the remaining points in P from L . This is a pseudoconfiguration representing σ , where ℓ_{c_1} and ℓ_{c_2} intersect. $\square$

³As it will turn out, a proper subset.

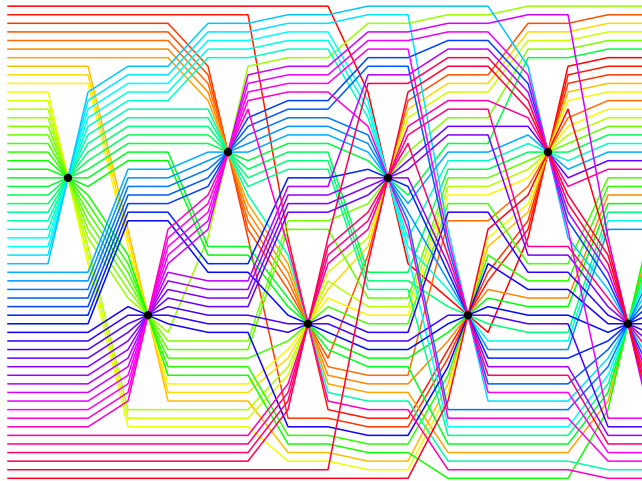


Figure 5: A signotopograph of rank 4 that is not 2-extendable. For a vertex that is an intersection of elements of odd index and a vertex that is an intersection of elements of even index, there is no extension through both of them.⁴

Note that by Corollary 31, the converse holds: if two curves of a pseudoconfiguration intersect, there is an extension of the underlying signotopograph through the vertices represented by the two intersecting curves.

We can extend a given collection of points P and polynomial curves L of degree k to a pseudoconfiguration in $PP(n, k)$, by adding points to P such that there are $k + 1$ many on each polynomial curve and then adding all the polynomial curves spanned by $k + 1$ points in P to L .

In the following we will work with pseudoconfiguration that are realizable as a collection of points and curves. By the considerations in Section 5.2, they correspond to hyperplane arrangements. Thus they are 2-extendable as a pseudohyperplane arrangement.

Proof of Theorem 37. We start with the case $r = 4$. Consider the configuration C of parabolas described in Figure 6 (the red "lines" are parabolas, with very small leading coefficients of opposite sign).

For every crossing c_i , let ℓ_{i_1} and ℓ_{i_2} be the two parabolas that intersect in the crossing. Place two points very closely next to the crossing in the following manner: one of the points is above ℓ_1 and below ℓ_2 , the other one vice versa, compare Figure 7. Call these points p_{i_1} and p_{i_2} . Call the set of all these points P_F .

We now construct a 2-intersecting pseudoconfiguration PP by the procedure

⁴Reproduced with friendly permission from Helena Bergold. Taken from https://helenabergold.github.io/supp/3d_signotopes/nonextendable_pcp48.pdf. Thank you, Helena!

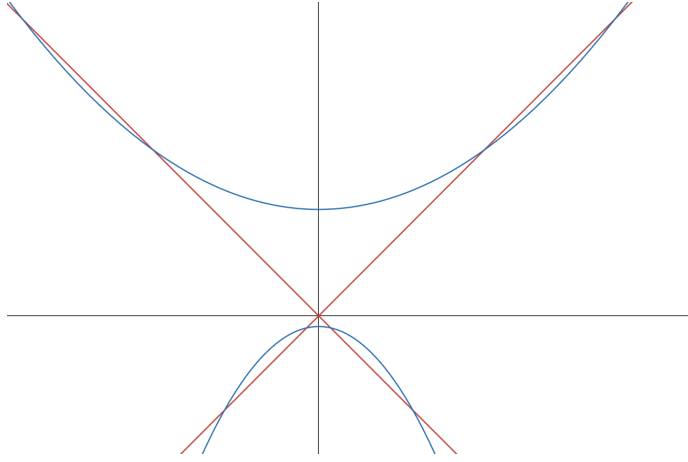


Figure 6: The configuration C of parabolas for the $r = 4$ case. There is no vertical segment connecting the two blue parabolas, that does not intersect a red parabola. Note that the red parabolas have a second intersection at a very large x -value.

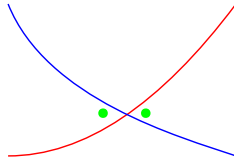


Figure 7: We place points around every crossing in C the following way.

described above, where the points are P_F as defined in the previous paragraph and L_C consists of the parabolas in C . Note that we add the points that lie on the parabolas far to the left. Let σ_{PP} be the corresponding signotope.

If we now consider any pseudoconfiguration $PP' = (P, L)$ representing σ_{PP} , the order of the x -coordinate of the points in P is fixed. Note that the red curves in C correspond to some curves in PP' . Call them r_1 and r_2 . Similarly we obtain b_1 and b_2 . Recall that all curves in a pseudopointconfiguration are x -monotone.

Note that the above-below relationship between curves in L and points in P is implied by the signotope σ_{PP} . This implies that between the points of the pseudoconfiguration PP' corresponding to p_{i_1} and p_{i_2} the crossing c_i is forced. It is also forced which curve comes from above and which comes from below.

This implies that the curves r_1, r_2, b_1, b_2 contain all the crossings of C , in the same x -order. Since the red-red and all four red-blue pairs are crossing saturated, we can not introduce further crossings without violating the properties of a 2-intersection pseudoconfiguration of points, since the blue-blue pair never sees each other along a horizontal segment. This implies that there can not be

any blue-blue crossing.

Since PP' was an arbitrary pseudoconfiguration representing $\sigma_{PP'}$, there is no extension through the vertices corresponding to b_1 and b_2 , by Lemma 38.

To obtain the general case for $r = 2n$, assume that all the crossings of C have x -coordinates in $[-1, 1]$. Let $p(x) = cx(x-1)\dots(x-(n-2))$, with large enough c , be a polynomial such that every element in $[-1, 1]$ is hit $n-1$ times by p . We plug p into every parabola in the configuration C .

We then obtain a collection of 4 polynomials of degree $r-2$, which behave like $n-1$ copies of C glued together. For an example, see Figure 8. Note that the red-red pair and all red-blue pairs are again crossing saturated.

We can again add the points that force the crossings and construct a pseudoconfiguration out of the collection of points and curves.

An analogous argument as above proves that the underlying signotope of rank r does not have an extension through the two vertices represented by the blue curves. $\square$

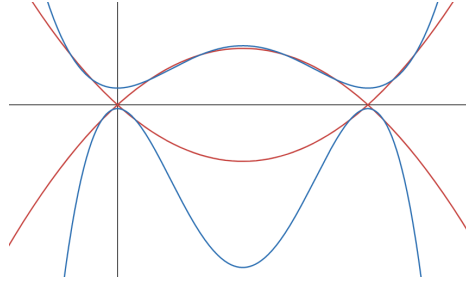


Figure 8: The configuration C after plugging in $5x(x-1)$ into every parabola in C . We omitted two crossings of the red-red pair. One should be to the right, the other to the left of all other crossings.

7 Conclusion

In this work, we found a solution to the bichromatic triangle conjecture for so called block-bicolored arrangements. The general case still remains open. A possible way to attack this problem would be to generalize Theorem 8 for reorientations of $S(n, 3)$. It was shown in [FW00] that not all reorientations of $S(n, 3)$ have good properties.

It might be possible to generalize Theorem 2 to signotopes of higher rank, giving rise to the following.

Conjecture. *Every block-bicolored signotope contains a bichromatic simplex.*

Note that Theorem 8 does not hold for higher ranked signotopes, see Theorem 4.5 in [Zie93].

We also provided a new proof for the representation of signotopes as k -intersecting pseudoconfigurations of points, introduced in [Miy21] as a generalization of the representation of pseudoline arrangements as pseudoconfigurations of points, as in [Goo80] and [AS05]. Pseudoconfigurations of points could provide a valuable tool to solve problems regarding signotopes.

We also proved that for every even rank, there are non 2-extendable signotopes. The notion of 2-extendability may be too strong, suggesting a line of research that studies the extendability of signotopes as pseudohyperplane arrangements. Towards this goal, it could be helpful to investigate different orientations of the pseudohyperplane arrangements representing signotopes.

Pseudoconfigurations in itself could be interesting objects of study. Corollary 31 implies a strong extendability property of pseudoconfigurations constructed by Theorem 34. We conjecture that every pseudoconfiguration has this property.

Conjecture. *Let $PP \in PP(n, k)$ be a pseudoconfiguration. After adding a point to any face of PP , we can add curves to PP , such that the resulting object is in $PP(n + 1, k)$.*

Sources

- [Lev26] F. Levi. “Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade”. In: *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse* 78 (1926), pp. 256–267.
- [Sha79] R.W. Shannon. “Simplicial cells in arrangements of hyperplanes”. In: *Geometriae Dedicata* 8 (1979), pp. 179–187.
- [Goo80] J. E. Goodman. “Proof of a conjecture of Burr, Grünbaum, and Sloane”. In: *Discrete Mathematics* 32.1 (1980), pp. 27–35.
- [GP86] J.E. Goodman and R. Pollack. “Upper bounds for configurations and polytopes in $\mathbb{R}^d$ ”. In: *Discrete & Computational Geometry* 1 (1986), pp. 219–227.

- [MS86] Y.I. Manin and V.V. Shekhtman. “Higher Bruhat orders, related to the symmetric group”. In: *Functional Analysis and Its Applications* 20.2 (1986), pp. 148–150.
- [Bjö+93] A. Björner et al. *Oriented Matroids*. Vol. 46. Encyclopedia of Mathematics. Cambridge: Cambridge University Press, 1993.
- [Ric93] J. Richter-Gebert. “Oriented matroids with few mutations”. In: *Discrete & Computational Geometry* 10 (1993), pp. 251–269.
- [Zie93] G. Ziegler. “Higher Bruhat orders and cyclic hyperplane arrangements”. In: *Topology* 32.2 (1993), pp. 259–279.
- [FK99] S. Felsner and K. Kriegel. “Triangles in Euclidean Arrangements”. In: *Discrete & Computational Geometry* 22 (1999), pp. 429–438.
- [FW00] S. Felsner and H. Weil. “A theorem on higher Bruhat orders”. In: *Discrete & Computational Geometry* 23.1–2 (2000), pp. 121–127.
- [FW01] S. Felsner and H. Weil. “Sweeps, arrangements and signotopes”. In: *Discrete Applied Mathematics* 109.1–2 (2001), pp. 67–94.
- [FZ01] S. Felsner and G. Ziegler. “Zonotopes associated with higher Bruhat orders”. In: *Discrete Mathematics* 322.1–3 (2001), pp. 301–312.
- [Fel04] S. Felsner. *Geometric Graphs and Arrangements*. Vieweg+Teubner Verlag, 2004.
- [AS05] P. Agarwal and M. Sharir. “Pseudo-Line Arrangements: Duality, Algorithms, and Applications”. In: *SIAM Journal on Computing* 34.3 (2005), pp. 526–552.
- [Miy21] H. Miyata. “On Combinatorial Properties of Points and Polynomial Curves”. Preprint. arXiv:1703.04963. 2021.
- [FPS22] S. Felsner, A. Pilz, and P. Schnider. “Arrangements of Approaching Pseudo-Lines”. In: *Discrete & Computational Geometry* 67 (2022), pp. 380–402.
- [Alv23] Y. Alves Radtke. “On the connectivity of the flip-graph of pseudoline arrangements”. Bachelor’s Thesis. Technische Universität Berlin. 2023.
- [BFS23a] H. Bergold, S. Felsner, and M. Scheucher. “An extension theorem for signotopes”. In: *Proceedings of 39th International Symposium on Computational Geometry (SoCG 2023)* (2023).
- [BFS23b] H. Bergold, S. Felsner, and M. Scheucher. “An extension theorem for signotopes”. Preprint. arXiv:2303.04079. 2023.
- [Cor+24] F. Cortés Kühnast et al. “An Improved Lower Bound on the Number of Pseudoline Arrangements”. In: *Proceedings of 40th International Symposium on Computational Geometry (SoCG 2024)* (2024).

Zusammenfassung

Due to regulations of the TU Berlin, I have to include a short summary of this work in german.

Wir betrachten Pseudogeraden Arrangements, in denen die Geraden entweder rot oder blau gefärbt sind.

Unser Resultat hängt mit der Vermutung zusammen, dass jedes Pseudogeradenarrangement ein Dreieck besitzt, welches aus sowohl roten als auch blauen Pseudogeraden besteht. Soweit wir wissen, wurde sie zuerst in [Bjö+93] aufgestellt.

Wir wiederholen zuerst einige Fakten über Pseudogeraden Arrangements, Signotope, Dreiecke und bichromatische Dreiecke.

Wir geben dann ein kurzes Argument für die Existenz eines bichromatischen Dreiecks, wenn das Arrangement block-gefärbt ist, also die ersten k Pseudogeraden rot und die folgenden $n - k$ blau sind. Das Argument basiert auf dem Fakt, dass zwei Ordnungen auf der Menge aller Pseudogeraden Arrangements übereinstimmen, was in [FW00] gezeigt wurde.

Wir führen dann Signotope mit Degenerationen ein, also Punkten, in denen sich mehr als $(r - 1)$ Pseudohyperebenen treffen, wenn wir ein Rang r Signotop betrachten.

Mithilfe dieser Betrachtungen konstruieren wir k -schneidende Pseudopunktkonfigurationen, die Pseudopunktkonfigurationen verallgemeinern, die in [Goo80] und [AS05] eingeführt wurden. Die Konstruktion basiert darauf, dass wir durch die Vertices eines gegebenen Signotopes viele weitere Elemente durchlegen.

Dann nutzen wir Pseudopunktkonfigurationen, um Signotope beliebigen geraden Ranges zu konstruieren, die nicht zwei-erweiterbar sind. Das bedeutet, dass es ein Signotop mit einem Paar von Punkten gibt, sodass wir keine Pseudohyperebenen durch dieses Paar von Punkten legen können, sodass das erweiterte Arrangement immer noch ein Signotop darstellt. Die Existenz solcher Signotope wurde in [BFS23a] vermutet. Die Konstruktion basiert auf einer Sammlung von Parabeln, von denen zwei einen Schnitt im Komplexen haben.