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On the connectivity of the flip-graph of pseudoline arrangements

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Abstract

Pseudoline arrangements are generalizations of line arrangements, introduced in [Lev26]. They are in close relationship with zonotopal tilings [FW01], higher Bruhat orders [FZ01], and uniform extensions of certain oriented matroids [Zie93]. More generally, pseudohyperplane arrangements can represent a wide class of oriented matroids, by the famous Folkman-Lawrence representation theorem [FL93].

We can define a graph on the set of all pseudoline arrangements on a fixed number of pseudolines, by connecting two pseudoline arrangements that can be transformed into each other by local mutation. Deeper understanding of this graph could enable us to sample uniformly from the set of all pseudoline arrangements.

Not much is known about the structure of this graph. The minimal degree of this graph was established in [FK99]. [FZ01] established that this graph contains large induced polytopal subgraphs.

Felsner and Weil conjectured in [FW01], that the connectivity of this graph is equal to its minimal degree. The main result of this work is to resolve this conjecture affirmatively.

To highlight the potential of using higher Bruhat orders to understand this graph better, we will prove this graph is in general not a partial cube, using the connectivity of the cover graph of higher Bruhat orders.

1 Introduction

1.1 Pseudoline Arrangements

In this work, we will take a closer look at a geometrical object called Euclidean *pseudoline arrangement* (from now on referred to as *arrangement*). We will follow the definition in [FW01]. An arrangement is a collection of curves in the Euclidean plane, called *pseudolines*, topologically behaving like pairwise non-parallel lines. Pseudolines pairwise intersect in a crossing and removing one of them leaves the plane disconnected into two unbounded regions. We will only consider *simple* arrangements: the intersection of three pseudolines will be empty.

We will call the intersection of two pseudolines a *vertex*. A connected component of the plane we obtain after deleting all pseudolines will be called *face*. A segment on a pseudoline between two vertices is an *edge*, if there is no other crossing on the segment.

We will be working with marked arrangements, where we choose an unbounded face as a *north-face*. For each pseudoline, call the side where the north-face is to the *left* of the pseudoline and the other side to the *right* of the pseudoline. This also yields an orientation for every pseudoline.

To the right of all pseudolines there is a unique face we will call *south-face*. Moving along the unbounded regions from the north-face to the south-face in anti-clockwise direction, we meet all the pseudolines in a certain order. We will identify an arrangement A with the tuple of pseudolines $(\ell_1, \dots, \ell_n)$, where the order of lines is the previously defined order.

We will call two marked arrangements isomorphic, if there is a homeomorphism of the plane that maps one to the other, that maps the north-face to the north-face pseudoline to pseudoline and the order of the pseudolines stays the same. We will identify isomorphic arrangements. Note that some arrangements are isomorphic to a straight line arrangement. We will call these arrangements *realizable*. In the literature they are also called *stretchable*. We will say an arrangement is *realizable with slope set M* , if it is isomorphic to a line arrangement where the lines have slopes in M .

The set of all triangular faces of an arrangement A will be denoted by $\nabla(A)$. We can mutate an arrangement by choosing a triangular face and moving one of the pseudolines bounding the face over the crossing of the other two bounding pseudolines, leaving the rest of the arrangement unchanged. See Figure 1 for an example. We call this operation a flip. We can now state the definition of the flip-graph: The vertices are all arrangements on n pseudolines and two arrangements are connected by an edge, if we can obtain one from the other by a single flip. Call this graph F_n .

1.2 Minimal Degree

To gain an upper bound on the connectivity of the flip-graph F_n , we need to determine its minimal degree. It was shown in [FK99] that the minimal number

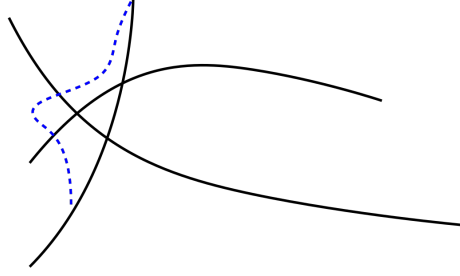


Figure 1: The black lines form a triangular face. The blue dotted line tells us, how the arrangement would look like after we flipped the triangle

of triangles in an arrangement on n pseudolines is $n - 2$. Here we reproduce this result. The corresponding result for straight line arrangements is a special case of a famous theorem of Shannon, see [Sha76].

Theorem 1. *The minimal degree of the flip-graph $\delta(F_n)$ is $n-2$.*

Proof. The core of the proof is a counting argument.

We will start by counting the number of vertices F^0 , the number of edges F^1 and the number of bounded faces F^2 in an arrangement $A \in F_n$. Since vertices are intersections of pairs of pseudolines, it follows that

$$F^0 = \binom{n}{2}.$$

Every pseudoline is subdivided by intersections into n segments. Two of these segments are unbounded. It follows that

$$F^1 = n(n - 2).$$

By flipping a triangle, the number of bounded faces is not changed. This implies F^2 is a function of the number of pseudolines. By inserting a pseudoline ℓ_i into an arrangement, such that all crossings are either on the left or on ℓ_i , we see that adding a new pseudoline adds $n - 2$ new bounded faces. This yields:

$$F^2 = \binom{n - 1}{2}.$$

We first observe that every edge e consists of three pseudolines ℓ_i, ℓ_j, ℓ_k . Let ℓ_i be the supporting pseudoline of e and the crossings (ℓ_i, ℓ_j) and (ℓ_i, ℓ_k) be the vertices of incident to e . We will assign to each side of each edge either $\oplus$ or $\ominus$. We assign to the side of an edge, that the crossing (ℓ_j, ℓ_k) is on, $\oplus$, to the other $\ominus$.

By inspecting a triangular face, we come to the conclusion that all three edges in the triangle have the $\oplus$ on the side that contains the triangular face.

By inspecting the essentially unique arrangement induced by the four pseudolines of a quadrangular face, we see that there are exactly two $\oplus$ and exactly two $\ominus$ on the side of the edges that contain the face.

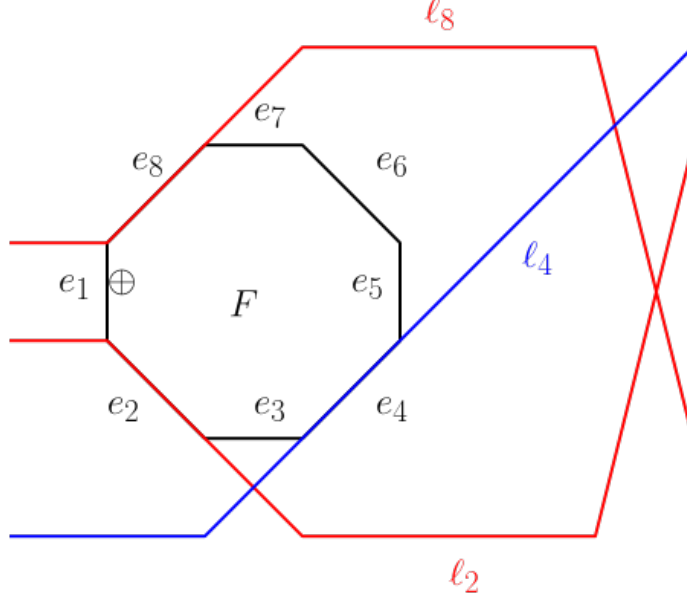


Figure 2: Sketch of a face, where the edge e_1 has $\oplus$ on the side of the face. We see that there is no alternative way to insert ℓ_4 .

For faces of higher complexity, there can also at most be two $\oplus$ on the side of the edge of the face. Assume that the face F is bounded by the edges $e_1, \dots, e_k$. Let $\ell_1, \dots, \ell_k$ be the supporting pseudolines. Assume that e_1 has an $\oplus$ on the side of F .

Looking at the sketch, we first note that ℓ_4 has to be drawn like this: it has to enter the region R , bounded by e_1, ℓ_8, ℓ_2 , since by assumption, e_4 is in this region. It also has to leave the region, so the crossings with ℓ_8 and ℓ_2 are forced. We immediately see that there can not be a $\oplus$ in F on e_3 . This generalizes to the fact that e_3 and e_{k-1} can not have a $\oplus$ on the side of F .

The cases $e_4, \dots, e_{k-2}$ follow by generalizing the fact that the edge e_6 is in the region of ℓ_5, ℓ_8 and the boundary of the face. Since the edge e_6 is in this region, ℓ_6 has to intersect ℓ_4 in the region R , but not in F . So e_5 has $\ominus$ on the side of F .

It follows, that the $\oplus$ have to be on neighboring edges in F , so there can be at most two $\oplus$ in F .

Since every edge has a $\oplus$ and a $\ominus$ side, the number of $\oplus$ is F^1 .

By double counting the number of $\oplus$ once over all edges and once over all faces, we obtain:

$$F^1 \leq 2F^2 + p_3,$$

where p_3 is the number of triangles in A .

We can substitute F^1 and F^2 to obtain:

$$n(n-2) \leq (n-1)(n-2) + p_3,$$

which implies the result

$$p_3 \geq n-2.$$

□

1.3 Poset Notation

We will heavily rely on poset notation. Let $\mathcal{P}$ be a poset. A chain C of $\mathcal{P}$ is *maximal*, if there is no other chain that contains C . The *height* of $\mathcal{P}$ is the size of the largest chain of $\mathcal{P}$. A chain is a *maximum* chain, if it has the size of the height of $\mathcal{P}$. $\mathcal{P}$ is *graded*, if all maximal chains are maximum chains.

If there is an element that is smaller than every other element, we denote it with $\mathbf{0}$. If there is an element that is larger than every other element, we denote it with $\mathbf{1}$. If $\mathcal{P}$ has $\mathbf{0}$ and $\mathbf{1}$, we call $\mathcal{P}$ *bounded*.

An *ideal* is a subset $\mathcal{I}$ of $\mathcal{P}$, so that every element that is smaller than an element in $\mathcal{I}$ is contained in $\mathcal{I}$.

It is well known that the set of all ideals of $\mathcal{P}$ ordered by inclusion is a lattice. We will call this lattice the *ideal lattice*. It is well-known that a chain in the ideal lattice of $\mathcal{P}$ corresponds to a linear extension of $\mathcal{P}$.

1.4 Sweeps and Wiring Diagrams

Here we reproduce results from [FW01]. We will prove the existence of a *sweep* of an arrangement. We will relate sweeps with standardized drawings of arrangements, so called *wiring diagrams*.

Definition 1. A sweep of an arrangement $A = (\ell_1, \dots, \ell_n)$ is a sequence of pseudolines $c_0, \dots, c_r$, such that $(c_i, \ell_0, \dots, \ell_n)$ is a simple arrangement, for $i = 0, \dots, r$. The sequence has to satisfy the following properties:

- The pseudolines $c_0, \dots, c_r$ are pairwise disjoint.
- There is exactly one vertex of A that is both to the left of c_i and to the right of c_{i+1} , for $i = 1, \dots, r-1$. Alternatively, we can obtain c_{i+1} by a triangle flip that moves a crossing from the left of c_i to the right of c_i .
- For every vertex v of A , there is exactly one i , such that v is to the left of c_i and to the right of c_{i+1} .

For an example, see Figure 3.

Theorem 2. *Every arrangement has a sweep.*

For the proof we need the following definition:

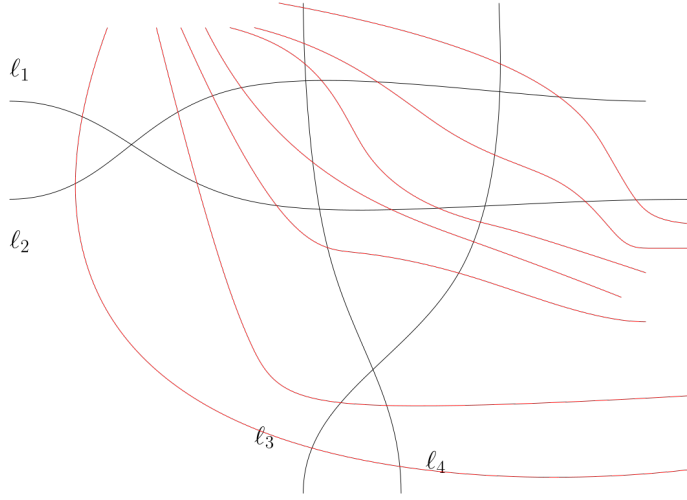


Figure 3: An arrangement with sweep

Definition 2. For an arrangement A let $\mathbf{G}(A)$ be the graph defined the following way: the vertices of $\mathbf{G}(A)$ are the vertices of A and the edges of $\mathbf{G}(A)$ are the edges of A , oriented in accordance with the north-face of A .

The following proposition will allow us to use order theory to deal with arrangements.

Proposition 1. $\mathbf{G}(\mathcal{A})$ is acyclic.

Proof. Proof by contradiction. Assume there is a cycle $C = (e_1, \dots, e_k)$ in $\mathbf{G}(\mathcal{A})$. Each cycle describes a closed curve in the plane. We can choose a cycle with minimal area inscribed in the corresponding curve, with respect to inclusion. We argue that this cycle describes a face. Firstly it needs to be a simple curve, otherwise we would be able to choose a smaller cycle. Secondly it is a face, otherwise there would be a pseudoline crossing the boundary of the cycle twice. Since all edges on this pseudoline are a linearly ordered path, this would imply the existence of a smaller cycle.

Assume that $e_1, \dots, e_k$ are the edges of the cycle. Let $\ell_{i_1}, \dots, \ell_{i_k}$ be the supporting pseudolines. Assume that the cycle is oriented counterclockwise. Then the indices of the pseudolines supporting the edges have to be a strictly decreasing sequence. Since the first edge was chosen arbitrarily, this is a contradiction. $\square$

From now on we will abuse notation and treat $\mathbf{G}(\mathcal{A})$ as the poset we obtain after taking its transitive closure.

Proof of Theorem 2. We will prove the following statement: let $\mathcal{I}$ be an ideal in $\mathbf{G}(\mathcal{A})$. Then there is a pseudoline c , such that the vertices to the left of c are exactly the elements in $\mathcal{I}$.

We prove this by induction. If $\mathcal{I}$ is empty, we draw c going from the north-face to the south-face, clockwise and crossing only unbounded faces.

For the induction step, choose a maximal element $x \in \mathcal{I}$ and use the induction hypothesis on $\mathcal{I} \setminus x$ to obtain a pseudoline c , such that the crossings to the left of c are exactly $\mathcal{I} \setminus x$.

Let x be the intersection of ℓ_i and ℓ_j . The segments between x and the intersections of ℓ_i and c and ℓ_j and c are edges, because x is maximal in $\mathcal{I}$. This implies that the segment on c between the intersections with ℓ_i and ℓ_j is empty. This implies (c, ℓ_i, ℓ_j) is a triangle. We can flip this triangle to obtain the wanted pseudoline.

We obtain a sweep by choosing a maximal chain in the ideal lattice of $\mathbf{G}(\mathcal{A})$. $\square$

A sweep is also a linear extension of $\mathbf{G}(\mathcal{A})$. Note that in the algorithm we were free to choose between multiple maximal elements, implying that an arrangement has multiple sweeps.

The *local sequence* of a pseudoline is the order in which it intersects with the other pseudolines. The local sequences of $c_0, \dots, c_r$ have an interesting interpretation: they are a maximal chain in the poset of permutations on $[n]$ ordered by inclusion of the inversion set, the so called *weak Bruhat order*.

We can see this by considering the effect moving the sweep line c over a crossing of two pseudolines ℓ_i, ℓ_j has: (ℓ_i, ℓ_j, c) form a triangle which we flip to move c over the crossing. This implies that ℓ_i, ℓ_j are neighbors in the local sequence of c . The flip induces a neighbor transposition of ℓ_i, ℓ_j in the local sequence of c .

This gives us an alternative description of the local sequence of c . Let $(\ell_{i_1}, \ell_{k_1}), \dots, (\ell_{i_m}, \ell_{k_m})$ be the crossings to the right of c . Then the local sequence of c is the permutation with the inversion set $\{(i_1, k_1), \dots, (i_m, k_m)\}$.

We want to prove the other direction: for every maximal chain

$$C = (p_0 < \dots < p_{\binom{n}{2}})$$

in the weak Bruhat order, we can construct a pseudoline arrangement $\mathcal{A}$, so that C are the local sequences of a sweep of $\mathcal{A}$.

We do this by constructing a so called *wiring diagram*. Start by drawing n vertical lines. Iterate through C . For every p_k , draw a crossing between the two elements ℓ_i, ℓ_j , that are inverted in p_k , but not in p_{k-1} . For all other elements, extend their lines by a straight segment. The lines ℓ_i, ℓ_j have to be neighbors in p_k , since C was maximal. Also, since C is a maximal chain, each pair gets inverted exactly once, so every pair of pseudolines crosses exactly once. We constructed a pseudoline arrangement with the wanted properties. An example can be seen in Figure 4.

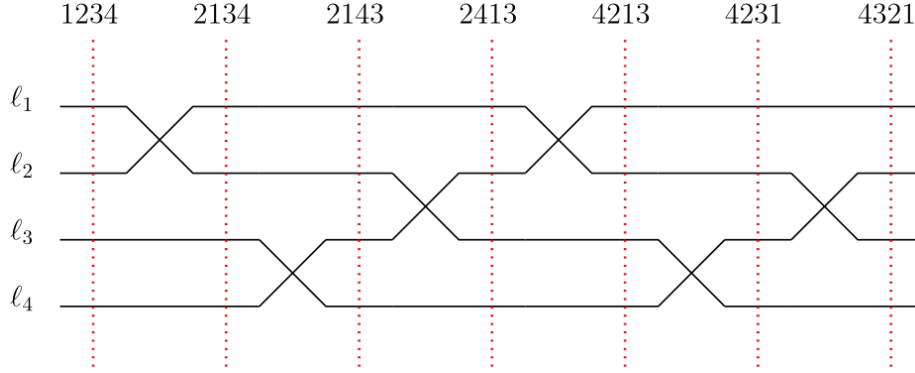


Figure 4: The wiring diagram of the arrangement and sweep in Figure 3. Note that the local sequences of the dotted lines are a maximum chain in the weak Bruhat order.

If we take a sweep $(c_0, \dots, c_r)$ of an arrangement $\mathcal{A}$ and construct the corresponding wiring diagram, the two are isomorphic. The regions in $\mathcal{A}$ and in the wiring diagram between two sweep lines c_i and $c_i + 1$ are homeomorphic. We can glue these homeomorphisms together to obtain a homeomorphism between the wiring diagram and the arrangement $\mathcal{A}$.

We want to characterize all maximal chains in the weak Bruhat order that give rise to the same pseudoline arrangement. For this, let

$$C = \{p_0 < \dots < p_{\binom{n}{2}}\}$$

be a maximal chain in the weak Bruhat order. We define the flip sequence of C as $A_0, \dots, A_{\binom{n}{2}-1}$, such that A_i is the unique inversion in which p_i and p_{i+1} differ, for $i = 0, \dots, \binom{n}{2} - 1$. If the flip sequence of C, C' differ by exactly one neighbor transposition, we call C, C' *directly swap-equivalent*. Assume the flip sequence of C has the subsequence A_t, A_{t+1} and the flip sequence of C' has the subsequence A_{t+1}, A_t . It follows, that A_t, A_{t+1} are disjoint, otherwise C and C' can not both be maximal chains in the weak Bruhat order, since there is only one way we can move an entry in a permutation two times by neighbor transpositions in a row and still obtain comparable permutations. In the corresponding wiring diagrams and the corresponding sweeps, the region between c_{t-1} and c_{t+1} are homeomorphic. This can be seen in Figure 4: we could have drawn the crossing between ℓ_3 and ℓ_4 before the crossing of ℓ_1 and ℓ_2 , but the drawing between the first and the third dotted line would be homeomorphic. This implies that directly swap equivalent chains yield the same pseudoline arrangement.

We call two chains C, C' swap-equivalent, if there is a sequence

$$C = C_0, C_1, \dots, C_k = C',$$

such that C_i and C_{i+1} are directly swap-equivalent, for $i = 0, \dots, k-1$. Swap-equivalent maximal chains describe the same pseudoline arrangement.

We will later prove that any two topological sortings can be transformed into each other by a sequence of neighbor transitions, such that each intermittent sorting is a topological sorting.

Altogether, this implies the main result of this section:

Theorem 3. *Pseudoline arrangements are in correspondence with swap-equivalent classes of maximal chains of the weak Bruhat order.*

1.5 Signotopes and Higher Bruhat Orders

In this section we reproduce results from [FW01]. The results will allow us to deal with certain paths in F_n , which will aid us in proving that F_n is not a partial cube.

We already saw the connection between arrangements and maximal chains of the weak Bruhat orders. We can define a hierarchy of objects, called *signotopes*, where the relation between arrangements and permutations is a special case.

We define the following operation on sets: for a finite subset P of the natural numbers, let $P^{[i]}$ be the set P without its i -th largest element. For a finite set P , we call $\{P^{[i]} \mid i = 1, \dots, |P|\}$ the *package* of P . A set of sets S is a package, if there is a P , such that S is the package of P .

Definition 3. Let α be a function that maps r -subsets of $[n]$ to $\{+, -\}$. We call α an *r -signotope on $[n]$* , if for every $(r+1)$ -subset P of $[n]$, the sequence

$$\alpha(P^{[1]}), \alpha(P^{[2]}), \dots, \alpha(P^{[r+1]})$$

has at most one change in sign.

In the following we will need to build new signotopes from old ones. We introduce the following constructions:

Lemma 1. *Let α be an r -signotope on $[n]$.*

- *The complement of α is the r -signotope on $[n]$, where every sign is reversed.*
- *The contraction of α by n is the $(r-1)$ -signotope on $[n-1]$ β , such that $\beta(P) = \alpha(P \cup \{n\})$, for every $(r-1)$ -subset P of $[n-1]$.*
- *The deletion of α by n is the r -signotope on $[n-1]$, such that $\beta(P) = \alpha(P)$, for every r -subset P of $[n-1]$.*
- *The weak boundary of α is the $r+1$ -signotope on $[n]$ $\partial^* \alpha$, such that $\partial^* \alpha(P) = \alpha(P^{[r+1]})$, for all $(r+1)$ -subsets of $[n]$.*

Proof. The fact that the first three constructions yield signotopes is straightforward.

It remains to prove that $\partial^* \alpha$ is an $(r+1)$ -signotope. Let Q be an $(r+2)$ -subset of $[n]$. Let $P = Q^{\lfloor r+2 \rfloor}$. Let $1 \leq i < j < k \leq r+2$. We have to do a case distinction:

- For $h < r+2$, it holds that $Q^{\lfloor h \rfloor \lfloor r+1 \rfloor} = P^{\lfloor h \rfloor}$. So for $k \leq r+1$ the monotonicity follows from the monotonicity of α .
- It holds that $Q^{\lfloor r+2 \rfloor \lfloor r+1 \rfloor} = Q^{\lfloor r+1 \rfloor \lfloor r+1 \rfloor} = P^{\lfloor r+1 \rfloor}$. For $k = r+2$ and $j < r+1$, by the preceding case, the monotonicity holds for the indices $i, j, k-1$. Since $\partial^* \alpha(Q^{\lfloor k \rfloor}) = \partial^* \alpha(Q^{\lfloor k-1 \rfloor})$, the monotonicity follows for i, j, k .
- If $k = r+2$ and $j = r+1$, $Q^{\lfloor k \rfloor \lfloor r+1 \rfloor} = Q^{\lfloor j \rfloor \lfloor r+1 \rfloor}$ holds. The monotonicity follows for i, j, k , since $i < j < k$.

□

We can define two posets on the set of all r -signotopes on $[n]$. The *higher Bruhat order* was introduced by Manin and Shekhtman in [MS86].

Definition 4. For $n \geq r \geq 1$ the *signotope order* $S(n, r)$ is a poset on the set of all r -signotopes on $[n]$. In $S(n, r)$, a signotope α is smaller than a signotope β , if all r -subsets that α maps to $+$ are also mapped to $+$ by β . We say that $S(n, r)$ is *ordered by inclusion*.

Definition 5. For $n-1 \geq k \geq 0$ the *higher Bruhat order* $B(n, k)$ is a poset on the set of all $(k+1)$ -signotopes on $[n]$. In $B(n, k)$ the signotope α is covered by the signotope β , if α and β only differ at exactly one $(k+1)$ -subset of $[n]$, where α is $-$ and β is $+$. We say that $B(n, k)$ is *ordered by single-step inclusion*.

These two orders coincide in some cases:

- $S(n, 1) = B(n, 0)$ is isomorphic to the Boolean lattice [FW01].
- $S(n, 2) = B(n, 1)$ is isomorphic to the weak Bruhat order [FW01].
- $S(n, 3) = B(n, 2)$ is isomorphic to the transitive closure of a certain orientation of F_n [FW00].

In general these orders do not coincide [Zie93].

For our interest, the higher Bruhat order will be more helpful, since the cover graph of $B(n, k)$ can be interpreted as a flip-graph.

A very important tool to work with signotopes will be a graph that has similarities with $\mathbf{G}(A)$. In fact, both give rise to the same poset, in the case $r = 3$.

Definition 6. Let α be an r -signotope on $[n]$. Then we define the directed graph $G_\alpha = (V, E)$, where V is the set of all $(r-1)$ -subsets of $[n]$. Let P be an r -subset of $[n]$. For $1 \leq i < j \leq r$, let

$$(P^{\lfloor i \rfloor}, P^{\lfloor j \rfloor})$$

be an edge, if $\alpha(P) = +$. Otherwise, let

$$(P^{\lfloor j \rfloor}, P^{\lfloor i \rfloor})$$

be an edge.

The following lemma emphasizes that G_α is analogous to $\mathbf{G}(A)$.

Lemma 2. *Let α be an r -signotope on $[n]$. Then G_α is acyclic.*

Proof. We prove the statement by induction. For $r = 2$, G_α is a complete graph on n vertices. The monotonicity condition on α implies that there are no cyclic oriented triangles in G_α . It is well known that this implies that G_α is acyclic.

For $n = r$, the graph is the lexicographic or reverse-lexicographic order of the $(n - 1)$ -subsets of $[n]$.

Let $n > r > 2$. Let β be the signotope we obtain after the deletion of $\{n\}$ in α . In a pseudoline arrangement, this would correspond to the deletion of the n -th line. Let γ be the signotope we obtain after the contraction of $\{n\}$ in α . In a pseudoline arrangement, this would correspond to the local permutation sequence of the n -th pseudoline.

For this, let

$$X^s = \left\{ A \in \binom{[n-1]}{r-1} : \alpha(A \cup \{n\}) = s \right\},$$

for $s \in \{+, -\}$.

Let

$$Y = \left\{ A \in \binom{[n]}{r-1} \mid n \in A \right\}.$$

By induction $G_\alpha[X^+ \cup X^-]$ is acyclic, since it is isomorphic to G_β . Since $G_\alpha[Y]$ is isomorphic to G_γ , it is also acyclic.

By definition, the edges between X^+ and Y are oriented from X^+ to Y . The edges between X^- and Y are oriented from Y to X^- .

We need to show that the edges between X^+ and X^- are oriented from X^- to X^+ . For this, let $A \in X^-$ and (B, A) an edge in G_β . We want to show that $B \in X^+$. Since there is an edge between them, it follows that $P = A \cup B$ is an r -subset of $[n - 1]$. There is i, j , such that

$$P^{\lfloor j \rfloor} = A, P^{\lfloor i \rfloor} = B.$$

We want to use monotonicity, so we extend P to an $(r + 1)$ -subset of $[n]$, by looking at $Q = P \cup \{n\}$. The r -subsets we are interested in are

$$B \cup n = Q^{\lfloor i \rfloor}, A \cup n = Q^{\lfloor j \rfloor}, P = Q^{\lfloor r+1 \rfloor}.$$

We have to consider the possible lexicographic orders of these three sets.

- if $i < j$, then $(B, A) \in G_\beta$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = +$. Since $A \in X^-$, $\alpha(A \cup \{n\}) = \alpha(Q^{\lfloor j \rfloor}) = -$. Since $i < j$, $\alpha(Q^{\lfloor i \rfloor})$ has to be minus.

- if $j < i$, then $(B, A) \in G_\beta$ implies $\beta(P) = \alpha(Q^{\lfloor r+1 \rfloor}) = -$. Again, $\alpha(A \cup \{n\}) = \alpha(Q^{\lfloor j \rfloor})$ is minus. Since $j < i$, this implies that $\alpha(Q^{\lfloor i \rfloor})$ is minus.

□

We will again abuse notation and identify G_α with the poset induced by the transitive closure of G_α .

The next lemma is a generalization of the fact that local sequences are permutations.

Lemma 3. *Let $2 \leq r \leq n$ and α be an r -signotope on $[n]$. Let $\mathcal{I}$ be an ideal in G_α . Then the function*

$$\beta : \binom{n}{r-1} \longrightarrow \{+, -\}, \beta(x) = \begin{cases} +, & x \in \mathcal{I} \\ -, & \text{else} \end{cases},$$

is an $(r-1)$ -signotope.

Proof. Let P be an r -subset of $[n]$. In G_α it either holds that

$$P^{\lfloor 1 \rfloor} < \dots < P^{\lfloor r \rfloor} \text{ or } P^{\lfloor 1 \rfloor} > \dots > P^{\lfloor r \rfloor}.$$

Since $\mathcal{I}$ is an ideal, $\mathcal{I}$ can only contain an initial or a final segment of $P^{\lfloor 1 \rfloor}, \dots, P^{\lfloor r \rfloor}$. This implies that there can not be more than one sign change in $\beta(P^{\lfloor 1 \rfloor}), \dots, \beta(P^{\lfloor r \rfloor})$. So β is a $(r-1)$ -signotope. □

Corollary 1. *We can conclude from this that an r -signotope α with a topological sorting of G_α describes a maximum chain in $S(n, r-1)$.*

We will introduce new notation: Let $1 < r \leq n$ and

$$C = \beta_0 < \dots < \beta_{\binom{n}{r-1}}$$

be a maximum chain in $S(n, r-1)$. Note that this implies that C is a maximal chain in $B(n, r)$. For $t = 1, \dots, \binom{n}{r-1}$, let A_t be the unique $(r-1)$ -set where β_{t-1} and β_t differ. $(A_1, \dots, A_{\binom{n}{r-1}})$ is called the *flip-sequence* of C .

The other way around, assuming a maximum chain C was constructed like in the preceding lemma, we can look whether any r -package is traversed lexicographically or reverse-lexicographically and deduce the value of the signotope on this r -package. By being *traversed lexicographically*, we mean the elements in a package are sorted in lexicographically descending order, in the flip-sequence of the maximum chain.

This leads to the following:

Lemma 4. *Let $1 < r \leq n$ and $C = \beta_0 < \dots < \beta_{\binom{n}{r-1}}$ be a maximum chain in $S(n, r-1)$. Then there is an r -signotope α on $[n]$ so that the flip sequence $(A_1, \dots, A_{\binom{n}{r-1}})$ of C is a topological sorting of G_α .*

Proof. For a given package $P \subseteq \binom{[n]}{r}$, $\beta_{0|P}, \dots, \beta_{(\binom{n}{r-1})|P}$ is a chain in $S(r-1, r-1)$, with duplicates. We already pointed out that the corresponding A_t are lexicographically or reverse-lexicographically ordered.

For an r -subset P of $[n]$, define $\alpha(P)$ to be plus if the package of P traversed reverse-lexicographically, else let $\alpha(P)$ be minus.

Let Q be an $(r+1)$ -subset of $[n]$. Let $1 \leq i < j < k \leq r+1$. We assume that the packages of $Q^{[i]}$ and $Q^{[k]}$ are traversed lexicographically in C . We want to show that the package of $Q^{[j]}$ is also traversed lexicographically.

We consider two elements from the package of $Q^{[j]}$, namely

$$Q^{[i]} \cap Q^{[j]} \text{ and } Q^{[j]} \cap Q^{[k]}.$$

We also consider the element the packages of $Q^{[i]}$ and $Q^{[k]}$ share:

$$Q^{[i]} \cap Q^{[k]}.$$

In the lexicographic order

$$Q^{[i]} \cap Q^{[j]} < Q^{[i]} \cap Q^{[k]} < Q^{[j]} \cap Q^{[k]}$$

holds. The packages of $Q^{[i]}$ and $Q^{[k]}$ are traversed lexicographically, implying that $Q^{[i]} \cap Q^{[j]}$ was traversed before $Q^{[j]} \cap Q^{[k]}$. This implies that the package of $Q^{[j]}$ was traversed lexicographically. The reverse-lexicographic case follows similarly. $\square$

The following lemma proves that the cover graph of $B(n, k)$ is connected. We recall that we defined the weak boundary of α , as the $(r+1)$ -signotope on $[n]$ $\partial^* \alpha$, such that $\partial^* \alpha(P) = \alpha(P)^{[r+1]}$, for every $r+1$ -subset of $[n]$.

Lemma 5. *Every element of $S(n, r)$ is contained in a chain of length $\binom{n}{r} + 1$.*

Proof. Let α be an r -signotope on $[n]$. We consider the $(r+1)$ -signotope $\partial^* \alpha$. We want to show, that $\alpha^{-1}(-)$ is an ideal in $G_{\partial^* \alpha}$. For this, let $(P^{[i]}, P^{[j]})$ be an edge in $G_{\partial^* \alpha}$ and $\alpha(P^{[j]})$ be minus. We have to consider two cases.

- if $i < j$, then $\partial^* \alpha(P)$ is plus. This implies that $\alpha(P^{[r+1]})$ is plus. Since $P^{[i]} < P^{[j]} < P^{[r+1]}$, this implies that $P^{[i]}$ is minus.
- if $j < i$, it follows that $\partial^* \alpha(P)$ is minus. It follows that $\alpha(P^{[r+1]})$ is minus. Since $P^{[j]} < P^{[i]} < P^{[r+1]}$, this implies that $P^{[i]}$ is minus.

Now we choose a maximal chain in the ideal lattice that contains $\alpha^{-1}(-)$. The complement of the signotopes we obtain after applying Lemma 3 is the chain we were looking for. $\square$

The following will be important to show the connectivity of a graph on the maximum chains of $S(n, 3)$ later on. We will call two maximal chains C, C' in $S(n, 3)$ that represent the same $(r+1)$ -signotope *swap-equivalent*.

Lemma 6. *For $r \geq 3$ the following two assertions hold:*

- *Two maximum chains C, C' in $S(n, r)$, whose flip-sequence differs by exactly one neighbor transposition, represent the same $(r + 1)$ -signotope.*
- *If C, C' represent the same $(r + 1)$ -signotope, there is a sequence of maximum chains $C = C_1, C_2, \dots, C_k = C'$, such that the flip sequences of C_i and C_{i+1} differ by exactly one neighbor transposition, for $i = 1, \dots, k - 1$.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2$ be the flip sequences of C, C' . Let α be the signotope represented by C .

By assumption, C, C' differ only in that there is a t , such that A_t, A_{t+1} is a subsequence of $\mathbf{A}_1$ and A_{t+1}, A_t is a subsequence of $\mathbf{A}_2$. We claim that there is no edge between A_t and A_{t+1} .

If there were such an edge, A_t and A_{t+1} would be in a common package. Since a package has size $r \geq 3$, we can not change it from being in lexicographic to reverse-lexicographic order with a single neighbor transposition. So there can not be an edge between A_t and A_{t+1} . This implies the first claim of the lemma.

For the second claim, we delete the initial segment that coincides in the two flip sequences. Of all the elements that are still in the truncated flip sequences, choose one that is maximal with regards to G_α . We can move this element to the front by a sequence of neighbor transpositions and then repeat the procedure. $\square$

1.6 Signotopes and Arrangements

Still following [FW01], we prove a bijection between $S(n, 3)$ and the set of all arrangements on n pseudolines. More than that, the following result implies that the cover graph of $S(n, 3)$ is F_n .

For this, let A be an arrangement on n lines. We define a function f that assigns each triple $\{i < j < k\}$ a plus, if the crossing of ℓ_i and ℓ_k is to the left of ℓ_j , otherwise $f(i, j, k)$ is minus.

Proposition 2. The triple orientation function f is a 3-signotope on $[n]$.

Proof. Every set of four lines induces an arrangement on four lines. After considering all eight marked arrangements on four lines, we conclude that the monotonicity property holds. $\square$

Proposition 3. Every 3-signotope on $[n]$ is the triple orientation function of an arrangement.

Proof. For a given 3-signotope α on $[n]$, we construct an arrangement whose triangle function is α . For this, choose a maximum chain in $S(n, 2)$ we obtain from corollary 1. The wiring diagram we obtain from this maximum chain by Theorem 3 is a pseudoline arrangement $\mathcal{A} = (\ell_1, \dots, \ell_n)$. We claim that α is a triple orientation function of A .

To see this, consider $i < j < k$. If $\alpha(i, j, k)$ is plus, we traverse the package of $\{i, j, k\}$ in lexicographic descending order. This implies that the crossing of ℓ_i and ℓ_k is left to ℓ_j . The case $\alpha(i, j, k) = -$ follows identically. $\square$

This leads us to the following conclusion:

Theorem 4. *The triple orientation functions of pseudoline arrangements on n pseudolines are exactly the 3-signotopes on $[n]$.*

Since $S(n, 3) = B(n, 2)$, $S(n, 3)$ is ordered by single-step inclusion. This implies that the cover graph of $S(n, 3)$ is F_n .

1.7 Zonotopal Subgraphs

Here we reproduce the main result from [FZ01]. It is remarkable that the object we will construct has different interpretations. We could interpret it as a *fiber polytope* or as an *adjoint* of the cyclic oriented matroid.

Our construction will remain completely in the language of hyperplane arrangements. We call $H = H_0, \dots, H_k$ a *hyperplane arrangement* in $\mathbb{R}^n$, if H_i is a hyperplane in $\mathbb{R}^n$ that contains the origin, for $i = 1, \dots, k$. We say that H is *essential*, if the normal vectors of $H_0, \dots, H_k$ span $\mathbb{R}^n$. The *face lattice* $\mathcal{L}(H)$ of a hyperplane arrangement H are the faces of H ordered by reverse inclusion.

The dual object of a hyperplane arrangement is a *zonotope*. A zonotope is the Minkowski sum of finitely many segments in $\mathbb{R}^n$.

For a given hyperplane arrangement H , we can find a zonotope Z , such that $\mathcal{L}(Z) = \mathcal{L}(H)^{\text{op}}$, see [Bjö+93], Proposition 2.2.2.

Theorem 5. *Let $M = \{s_1 < s_2 < \dots < s_n\}$ be a subset of $\mathbb{R}$. Let $\mathcal{A}_M$ be the set of all pseudoline arrangements realizable with the slope set M . Then $\mathcal{A}_M$ induces a subgraph of F_n isomorphic to the graph of an $(n - 2)$ -zonotope.*

Proof. Let $\mathcal{A}$ be an arbitrary line arrangement with slope set M . Then $\mathcal{A}$ is completely determined by the y -intercepts $(b_1, \dots, b_n)$. Because of this, we will identify line arrangements with their y -intercepts. We will construct a hyperplane arrangement where each region corresponds to the set of all realizations with slope set M of a pseudoline arrangement in $\mathcal{A}_M$. We will identify each region with the corresponding pseudoline arrangement

Let ℓ_i, ℓ_j, ℓ_k be lines in $\mathcal{A}$ with $i < j < k$. We can determine the orientation of the line-triple $\{\ell_i, \ell_j, \ell_k\}$ by comparing the x -coordinate of the intersection of ℓ_i and ℓ_j and the x -coordinate of the intersection of ℓ_j and ℓ_k . A simple calculation shows us that this is the same as determining the sign of

$$\frac{b_j - b_i}{s_i - s_j} - \frac{b_k - b_j}{s_j - s_k}.$$

Since this term is linear in $(b_1, \dots, b_n)$, there is a hyperplane $H_{i,j,k}$ in $\mathbb{R}^n$ separating the line arrangements where $\{\ell_i, \ell_j, \ell_k\}$ is upwards-pointing from the

line arrangements where $\{\ell_i, \ell_j, \ell_k\}$ is downwards-pointing. This hyperplane goes through the origin.

Considering $H = \{H_{i,j,k} \mid \{i < j < k\} \subset [n]\}$, we get a hyperplane arrangement in $\mathbb{R}^n$. By construction, it is clear that each region of H corresponds to a pseudoline arrangement in $\mathcal{A}_M$. If two regions share the same boundary hyperplane, the corresponding pseudolines only differ by a triangle flip. If two regions R_1, R_2 differ only by the orientation of $\{i < j < k\}$ they also share a boundary hyperplane: Taking $v_1 \in R_1, v_2 \in R_2$, the line segment that connects v_1 and v_2 only crosses the hyperplane $H_{i,j,k}$. So $H_{i,j,k}$ is a boundary hyperplane of R_1 and R_2 . This implies that the 1-skeleton of the dual of the face lattice is a subgraph of F_n induced by $\mathcal{A}_M$.

This hyperplane arrangement contains a line: in $\mathcal{A}$, fix ℓ_1 and shift all the other lines parallel to ℓ_1 . This shift can be done in both directions, without changing the combinatorial type of $\mathcal{A}$. The shift corresponds to a line in $\mathbb{R}^n$ that does not cross a hyperplane in H . If we fix ℓ_1 and ℓ_2 this is no longer possible: for every $\ell_i, i \neq 1, 2$ the crossing of ℓ_1 and ℓ_2 is on one side of ℓ_i . Moving ℓ_i over this crossing changes the combinatorial type of the line arrangement, implying we crossed $H_{1,2,i}$.

Since all stretchable pseudoline arrangements have a realization with the first two lines crossing in $(0, 0)$, the restriction of H to the subspace where the first two coordinates are zero is an essential hyperplane arrangement in $\mathbb{R}^{n-2}$ with the same face lattice as H . By the already mentioned duality between zonotopes and essential hyperplane arrangements the claim follows. $\square$

2 Partial Cube

We will come to the proof that F_n is not a partial cube. Recall the definition of a partial cube:

Definition 7. A graph $G = (V, E)$ is a partial cube if there is an m such that there exists an isometric embedding of G into the hypercube Q_m , i.e. a function $f : V \rightarrow Q_m$, such that for any two vertices $v_1, v_2 \in V$,

$$d(v_1, v_2) = |f(v_1) \Delta f(v_2)|$$

holds.

In [FZ01] it was shown, that F_n is the 1-skeleton of a zonotope, for $n \leq 5$. They also showed, that the 1-skeletons of the zonotopes constructed in the previous chapter is not isomorphic to F_n , for $n \geq 6$. We prove a stronger claim:

Theorem 6. *The flip-graph of arrangements on n pseudolines F_n is not a partial cube, for $n \geq 6$.*

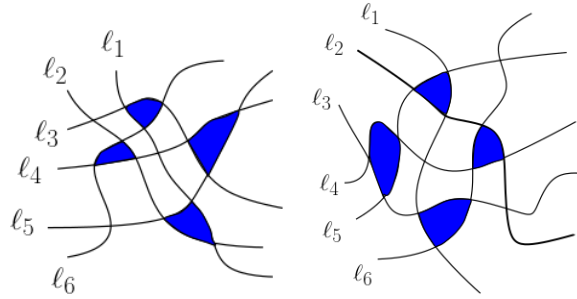


Figure 5: A pair of pseudoline arrangements whose triangular faces all have the same orientation.

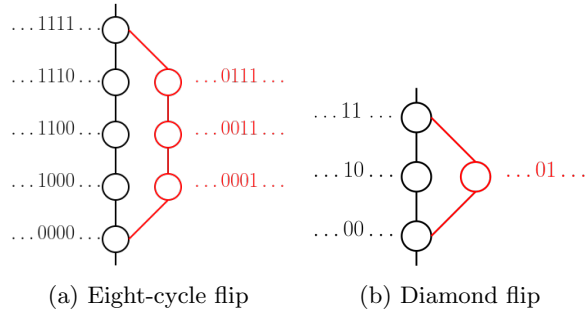


Figure 6: A part of a maximum chain in black with diamond and eight-cycle flip in red. If the labelling of the black part is prescribed for an isometric embedding, the isometric embedding has to also have the labelling of the red part. The same holds for the signotope embedding.

Since zonotopal graphs are partial cubes, see [Bjö+93, Proposition 4.2.3], Theorem 6 implies the claim in [FZ01]. Our proof will show that if there were to be an isometric embedding into the hypercube, the resulting labeling would be the corresponding signotopes. In [FW00], it was shown that there are pairs of pseudoline arrangements, whose distance in the flip-graph is larger than the bitwise distance of their signotopes, see Figure 5. This implies that the signotope mapping is not an isometric embedding into the hypercube. Note that we can expand the example from Felsner and Weil by adding pseudolines, such that all intersections are either on or to the left of this pseudoline, to obtain a similar example for arbitrary n larger than five.

To prove the main theorem, we will take a maximal chain in $S(n, 3)$ and modify it according to Figure 6. We will prove in the following, that these operations can reach all maximum chains and that they preserve the embedding in such a way, that we can conclude that it would have to be the signotope embedding.

Lemma 7. *Let $\mathcal{P}$ be a bounded, graded poset whose cover graph is a partial*

cube. Then there exists a monotone function from $\mathcal{P}$ to Q_{k-1} , where k is the height of $\mathcal{P}$. We can choose such a function that is an isometric embedding of the cover graph into Q_{k-1} .

Proof. Let f be an isometric embedding of the cover graph of $\mathcal{P}$ into Q_m . Since the diameter of Q_m is m , $m \geq k - 1$ follows. We want to show that we can choose $m = k - 1$.

For that, we can apply a hypercube isomorphism to ensure that

$$f(\mathbf{0}) = (0, \dots, 0) \text{ and } f(\mathbf{1}) = (1, \dots, 1, 0 \dots, 0),$$

with $k - 1$ non-zero entries. This is possible, because the distance between $\mathbf{0}$ and $\mathbf{1}$ in the cover graph is the height of $\mathcal{P}$ minus one, since $\mathcal{P}$ is graded.

Let

$$\mathbf{0} = c_0 < c_1 < \dots < c_{k-2} < c_{k-1} = \mathbf{1}$$

be a maximal chain in $\mathcal{P}$. Since the chain is maximal and $\mathcal{P}$ is graded, it follows that

$$d(c_i, c_j) = |i - j|,$$

where $d(\cdot, \cdot)$ is the distance in the cover graph of $\mathcal{P}$. Since $f(\mathbf{0}) = \mathbf{0}$, this implies that we obtain $f(c_{i+1})$ by changing a zero in $f(c_i)$ to a one. This implies that

$$f(c_0) < \dots < f(c_{k-1})$$

is also a maximal chain in Q_m . By the choice of $f(\mathbf{1})$, it follows that we can choose $m = k - 1$. □

It was observed in [Zie93] that the higher Bruhat orders are graded. We recall that $S(n, 3)$ is isomorphic to the higher Bruhat order $B(n, 2)$ and the cover graph of $S(n, 3)$ is F_n . $S(n, 3)$ has a $\mathbf{0}$ and a $\mathbf{1}$, the all minus and the all plus signotope.

In the following we will assume that f is a monotone function of F_n into $Q_{\binom{n}{3}}$ according to Lemma 7. Let d be the distance in F_n .

Denote by $\mathbf{S}_n$ the mapping of arrangements to their signotope, interpreted as a mapping from $S(n, 3)$ to $Q_{\binom{n}{3}}$. Recall from the introduction that maximal chains in $S(n, 3)$ can be interpreted as a linear order of triples, the so called *flip-sequence*.

Definition 8. Let C_1, C_2 be maximal chains in $S(n, 3)$. We call C_1 and C_2 connected by a *diamond flip* if their flip-sequences differ in exactly one neighbor transposition.

We call C_1 and C_2 connected by an *eight-cycle flip* if the flip-sequences of C_1 and C_2 only differ in the order of one package.

We can interpret an eight-cycle flip as "walking around an F_4 the other way". In the signotope we obtain from Lemma 4, this is a change at exactly one 4-subset of $[n]$.

Two chains that are connected by a diamond flip are swap-equivalent and represent the same element of $S(n, 4)$, see Lemma 6.

Lemma 8. *Let C_1, C_2 be maximal chains in $S(n, 3)$. Then*

$$f|_{C_1} = \mathbf{S}_n|_{C_1} \implies f|_{C_2} = \mathbf{S}_n|_{C_2},$$

if C_1 and C_2 are connected by a diamond or an eight-cycle flip.

Proof. Let M be either a diamond or an eight-cycle. Let v_0 and v_1 be the minimal and the maximal node of M . Note that f has to map M into the interval

$$\mathcal{I} = [f(v_0), f(v_1)] \cong Q_{\lfloor \frac{|M|}{2} \rfloor}$$

in $Q_{\binom{n}{3}}$.

Let $v_2 \in C_1, v_3 \in C_2$ be two antipodal points in M . By the choice of M , it holds that

$$d(v_2, v_3) = \frac{|M|}{2}.$$

Thus $f(v_3)$ is the relative complement in $\mathcal{I}$ of $f(v_2)$. We see that $\mathbf{S}_n$ does the same, by considering $\mathbf{S}_4$ and the signotopes involved in a diamond flip. $\square$

An essential part of the proof of theorem 6 is the connectedness of the following graph:

Definition 9 (Chain graph). Let $CG(n) = (V, E)$ be the graph where V is the set of all maximal chains of $S(n, 3)$ and two vertices are connected if and only if they differ by a diamond or 8-cycle flip.

We will need to proof a proposition first. This will allows us to translate the connectivity of $B(n, 3)$ to the connectivity of $CG(n)$

Proposition 4. Let S_1, S_2 be two signotopes adjacent in the cover graph of $B(n, 3)$. Then there are two maximal chains C_1, C_2 of $S(n, 3)$, representing S_1 and S_2 , that are connected by an eight-cycle flip.

Proof. Let G_{S_1} and G_{S_2} be the graphs defined like in definition 6. Since S_1 and S_2 are adjacent, they only differ in whether a fixed package P is lexicographically or reverse-lexicographically ordered. Without loss of generality, let S_1 be the signotope where P is lexicographically ordered in G_{S_1} . It follows that the upset $\text{up}(P)$ and downset $\text{down}(P)$ of P are the same in G_{S_1} and G_{S_2} .

We will use the following:

Claim. In G_{S_1} the sets $\text{up}(P)$, $\text{down}(P)$ and P are a partition of the vertex set $\binom{[n]}{3}$.

If the claim were to be false, there would be $p_1, p_2 \in P$ and $x \in \binom{[n]}{3}$, such that $p_1 < x < p_2$ in G_{S_1} . For a given x choose such a p_2 minimal and such a p_1 maximal in P . Then there is a directed path from p_2 to x and a directed path from x to p_1 that does not contain other elements in P . These paths have the

same orientation in G_{S_2} . In G_{S_2} there is a directed path from p_1 to p_2 , implying the existence of a directed cycle. This is a contradiction to Lemma 2.

We now choose the topological sortings s_1, s_2 of G_{S_1} and G_{S_2} the following way: the initial segment of s_1 and s_2 is a topological sorting of $\text{up}(P)$. The following segment in s_1 is P lexicographically ordered. In s_2 it is reverse-lexicographically ordered. Finally we append a topological sorting of $\text{down}(P)$ to s_1 and s_2 .

We constructed the topological sortings we can interpret as maximum chains in $S(n, 3)$, by corollary 1. By construction it is clear that they only differ in an eight-cycle flip. $\square$

Lemma 9. *$CG(n)$ is connected.*

Proof. Let C, C' be two maximal chains in $S(n, 3)$. They represent two signotopes S, S' in $B(n, 3)$, by Lemma 4. Since the cover graph of $B(n, 3)$ is connected, see Lemma 5, we obtain a sequence of signotopes $S = S_0, \dots, S_k = S'$, so that they form a path in the cover graph of $B(n, 3)$. By Proposition 4, we find chains $C_{1,i}, C_{2,i}, i = 1, \dots, k$, representing S_i , such that $C_{2,i}$ and $C_{1,i+1}$ are separated by a single eight-cycle flip. By Lemma 6, we obtain a sequence of maximal chains connecting $C_{1,i}$ and $C_{2,i}$ by diamond flips. We showed the connectivity of $CG(n)$ $\square$

Now we can prove the main theorem of this section.

Proof of Theorem 6. Choose a maximal chain C of $S(n, 3)$. After taking a permutation, we can assume $f|_C = \mathbf{S}_n|_C$. For an arbitrary $x \in S(n, 3)$, there is a maximum chain C' of $S(n, 3)$ containing x .

Since $CG(n)$ is connected, there is a sequence of diamond and eight-cycle flips, connecting C to C' . It follows that $f(x) = \mathbf{S}_n(x)$. This is a contradiction, since $\mathbf{S}_n$ is not an isometric embedding into the hypercube, for $n \geq 6$. In this case, F_n is not a partial cube $\square$

3 Shellable Arrangements

We will introduce a new class of arrangements, the so called shellable arrangements. The arrangements in this class have simple combinatorial descriptions. Shellable arrangements will be important to work around the fact that the signotope mapping is not an isometric embedding into a hypercube, allowing us to guarantee the disjointness of certain paths later on. They also have remarkable realization properties. We suspect that they generalize to so called *lexicographic extensions* in general higher Bruhat orders, since they share the properties we will prove now, see [SZ93] and [Zie93].

Let $\mathcal{A}$ be an arrangement and ℓ a pseudoline in the arrangement. We say ℓ is *going over* (*going under*) $\mathcal{A}$, if all crossings of $\mathcal{A}$ are either on or to the right (to the left) of ℓ . We call ℓ in $\mathcal{A}$ *extreme*, if ℓ goes over or under $\mathcal{A}$.

Definition 10 (Shellable arrangements). An ordering $\ell_{i_1}, \ell_{i_2}, \dots, \ell_{i_n}$ of the pseudolines of an arrangement is a *shelling sequence*, if for every $k \in \{1, \dots, n\}$, the line ℓ_{i_k} is extreme in the subarrangement induced by $\ell_{i_k}, \dots, \ell_{i_n}$. If an arrangement has a shelling sequence, we call it *shellable*. The set of all shellable arrangements on n pseudolines is $\mathcal{S}_n$.

We note that all arrangements on 4 or less pseudolines are shellable. There is an essentially unique non-shellable arrangement on 5 lines, compare [FZ01, Figure 2].

Proposition 5. Shellable arrangements are realizable with any given slope set $s_1 < s_2 < \dots < s_n$.

Proof. We prove the proposition by induction. For $n = 3$ there are only two arrangements. Both are shellable and both are realizable with any given slope set. Let $\mathcal{A}$ be a shellable arrangement on $n > 3$ pseudolines. Let ℓ_i be an extreme line in $\mathcal{A}$. The arrangement $\mathcal{A}' = \mathcal{A} \setminus \ell_i$ is by definition also a shellable arrangement. By induction $\mathcal{A}'$ has a realization with slope set $s_1 < \dots < s_{i-1} < s_{i+1} < \dots < s_n$. We reinsert ℓ_i with a y -intercept large enough if ℓ_i goes over $\mathcal{A}$. If ℓ_i goes under $\mathcal{A}$, we choose a small enough y -intercept. $\square$

We call a path in F_n *monotone*, if every triple changes orientation at most once along the path.

Lemma 10. For $S \in \mathcal{S}_n$ and $\mathcal{A} \in F_n$, there is a monotone path between S and $\mathcal{A}$.

Proof. We prove this by induction on n . For $n = 3$ the statement is clearly true, since there are only two arrangements separated by a single triangle flip.

For the case $n \geq 4$, let $\mathcal{A} \in V(F_n)$ and $S \in \mathcal{S}_n$. We can assume without loss of generality that ℓ_1 is extreme in S . Otherwise we can reorient S and $\mathcal{A}$ simultaneously.

We can treat ℓ_1 as a sweeping line in $\mathcal{A} \setminus \ell_1$. This implies that there is a sequence of flips moving ℓ_1 in going over or going under $\mathcal{A} \setminus \ell_1$, see Theorem 2. Each flip moves a crossing across ℓ_1 , but by the definition of a sweep, no crossing is moved across ℓ_1 more than once. Only triangles containing ℓ_1 were flipped.

By induction, there is a monotone path from $\mathcal{A}' \setminus \ell_1$ to $S \setminus \ell_1$. Since ℓ_1 is extreme in $\mathcal{A}'$, every triangle in $\mathcal{A}' \setminus \ell_1$ is also a triangle in $\mathcal{A}'$. This implies that we can lift the monotone path from $\mathcal{A}' \setminus \ell_1$ to $S \setminus \ell_1$ to a monotone path from $\mathcal{A}'$ to S . The flipped triangles in this path do not contain ℓ_1 . After combining the path from $\mathcal{A}$ to $\mathcal{A}'$ and the path from $\mathcal{A}'$ to S we constructed a monotone path. $\square$

We want to point out, that Lemma 10 is equivalent to the statement, that the reorientation of the order $S(n, 3)$ by a shellable arrangement is bounded and graded. It was shown in [FW00], that this property does not hold for arbitrary arrangements.

For a result in the next section we will need a lower bound on the number of shellable arrangements.

Proposition 6. There are at least $(n - 1)!$ shellable arrangements on n pseudolines.

Proof. Let $P = (p_1, \dots, p_{n-1})$ be a permutation of $[n - 1]$. We extend P to a shelling sequence $(p_1, \dots, p_{n-1}, n)$. If all lines are shelled from below, the local permutation sequence of n is $(p_1, \dots, p_{n-1})$. $\square$

There are $n!$ shelling sequences. In each shelling step, we can shell the line either from above or below. So the upper bound of $2^n n!$ is straightforward. This implies

$$|\mathcal{S}_n| \in 2^{\Theta(n \log n)}.$$

Computer experiments yielded that the sequence $(|\mathcal{S}_n|)_{n \in \mathbb{N}}$ has no entry in the OEIS.

4 Connectivity

One of the main components of the proof of our main theorem will be the connectivity of the zonotope we constructed in Theorem 5. We recall that the zonotope is $(n - 2)$ -dimensional. We obtain the $(n - 2)$ -connectivity of this zonotope by the following theorem.

Theorem 7 (Balinski's theorem [Bal61]). *The graph of an n -polytope is n -connected.*

Before we start the proof of our main theorem, we give a weaker result that is already implicit in [SZ93].

Proposition 7. The induced subgraph of all realizable arrangements in F_n is $(n - 2)$ -connected.

Proof. For $n = 3$ this is trivially true. There is a finite number of $(n - 2)$ -zonotopes, obtained from our construction in Theorem 5, that cover all realizable arrangements. All these zonotopes share the shellable arrangements, by Proposition 5. After deleting $n - 3$ vertices, the zonotopes are connected and there still is at least one shellable arrangement left, by Proposition 6. This proves the connectivity. $\square$

This already proves the $(n - 2)$ -connectivity of F_n for $n \leq 8$, since all arrangements on less than nine lines are realizable, see [GP80].

We will now start to prove the main theorem of this work.

Definition 11. We call a subset T of triangles of an arrangement $\mathcal{A}$ a *good triangle set* of $\mathcal{A}$, if for every choice of an orientation for these triangles, there is a shellable arrangement that has this orientation on T .

Lemma 11. *Every Euclidean arrangement on n pseudolines has a good triangle set of size at least $\frac{n}{3}$.*

Proof. We will prove the following:

Claim. There is a sequence $t_1, \dots, t_k$, $k \geq \frac{n}{3}$ of triangles of $\mathcal{A}$, such that for each $i = 1, \dots, k$, there is a line $\ell_i \in t_i$, such that ℓ_i is not contained in a triangle with an index lower than i .

For t_1 , we choose an arbitrary triangle in $\mathcal{A}$. Choose an arbitrary pseudoline in t_1 as ℓ_1 . Assume that we already have chosen $t_1, \dots, t_j$ and $\ell_1, \dots, \ell_j$. If there is no pseudoline that is not contained in

$$t_1 \cup \dots \cup t_j,$$

j has to be at least $\frac{n}{3}$.

If this is not the case, let ℓ_{j+1} be a pseudoline that is not contained in this set. Since after reorientation we can treat ℓ_{j+1} as a sweep line, there has to be a triangle incident to ℓ_{j+1} . Let this triangle be t_{j+1} . Repeat until the first case occurs.

This procedure will yield $t_1, \dots, t_k$ and $\ell_1, \dots, \ell_k$ as wanted.

Claim. $t_1, \dots, t_k$ is a good triangle set of $\mathcal{A}$.

We can extend $\ell_1, \dots, \ell_k$ to a sequence of all lines of $\mathcal{A}$ by inserting the pseudolines in $\mathcal{A} \setminus \ell_1, \dots, \ell_k$ in an arbitrary order in front of $\ell_1, \dots, \ell_k$ to obtain $p_1, \dots, p_n$.

We insert the pseudolines in order of $p_1, \dots, p_n$, by inserting p_{i+1} going over the arrangement $\{p_1, \dots, p_i\}$, if p_{i+1} is not in $\ell_1, \dots, \ell_k$.

The two pseudolines in t_j that are not ℓ_j appear earlier in $p_1, \dots, p_n$ than ℓ_j . This implies that we can freely choose the orientation of t_j by inserting ℓ_j going over or going under $\{p_1, \dots, p_{n-k+j}\}$. The sequence $p_1, \dots, p_n$ is the reverse shelling sequence of the resulting arrangement, so it is shellable. $\square$

We introduce the following notation:

Definition 12. Let $\mathcal{A} \in F_n$ and D be a collection of triangles of $\mathcal{A}$. If it exists, $\mathcal{A}[D]$ is the arrangement we obtain after flipping all the triangles D in $\mathcal{A}$.

Definition 13. For an arrangement $\mathcal{A}$ and $D \subseteq \nabla(\mathcal{A})$, we define the graph $G = G(\mathcal{A}, D)$ such that the vertices of G are the triangles D and two vertices are adjacent if and only if they do not share two pseudolines.

The following is a well-known theorem from graph theory.

Theorem 8 ([Dir52]). *Let $G = (V, E)$ be a graph with*

$$\delta(G) \geq \frac{|V|}{2}.$$

Then G contains a Hamiltonian cycle.

Lemma 12. *Let $\mathcal{A}$ and D be as in the previous definition. Then the following observations about $G = G(\mathcal{A}, D)$ hold:*

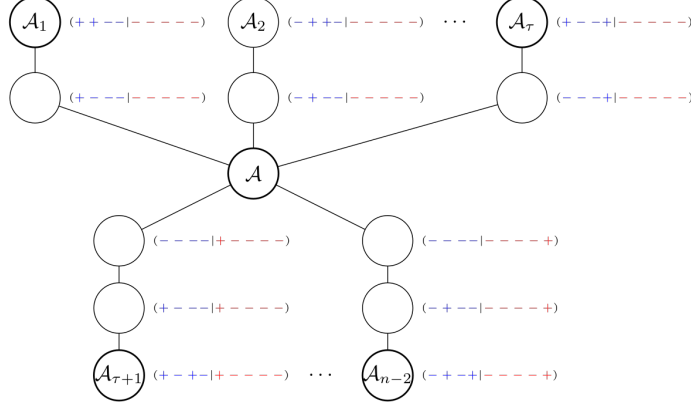


Figure 7: The star-shaped tree from Lemma 13. The blue part of the signature represent the orientation of the good triangles. The red part are the elements in $D \setminus T$,

1. The minimal degree of G satisfies $\delta(G) \geq |D| - 4$.
2. Let $C = \{t_1, \dots, t_i\}$ be a clique in G . Then $\mathcal{A}[C]$ exists.
3. If $|D| \geq \frac{n}{3}$ and n is large enough, there exists a Hamiltonian cycle in G .

Proof. For statement 1, let ℓ_1, ℓ_2 be two pseudolines in $\mathcal{A}$. After inspecting the essentially unique arrangement on four pseudolines, we note that two triangles incident to ℓ_1 and ℓ_2 intersect only at the intersection of ℓ_1 and ℓ_2 .

This implies that at most two triangles can be incident to a pair of pseudolines. It follows that there are at most 3 other triangles in D that share two pseudolines with any given triangle in D .

For statement 2, after looking at a triangular face F in an arrangement drawn into the plane, we note that the only cells that are affected by a triangle flip of F are the cells that share two pseudolines with F .

Statement 1 and the theorem of Dirac imply statement 3. $\square$

In the following, let $\mathcal{A}|_T$ be the orientation of the triples T in $\mathcal{A}$.

Definition 14. We call a tree $\mathcal{T}$ *star-shaped with center* $v \in V(\mathcal{T})$ if all vertices except v have degree less than three. The degree of $\mathcal{T}$ is the degree of v .

For this, we will need the following lemma:

Lemma 13. Let n be large enough and let $\mathcal{A} \in F_n$ and T be a good triangle set of $\mathcal{A}$, $\tau := |T| \geq \frac{n}{3}$. There is a subgraph $\mathcal{T}$ of F_n , such that all of the following hold:

- $\mathcal{T}$ is a star-shaped tree with center $\mathcal{A}$.

- $\mathcal{T}$ has degree $n - 2$.
- Let $\mathcal{A}_1, \dots, \mathcal{A}_{n-2}$ be the leaves of $\mathcal{T}$. Then $\mathcal{A}_i|_T \neq \mathcal{A}_j|_T$, if $i \neq j$.

Proof. Theorem 1 states, that $|\nabla(\mathcal{A})| \geq n - 2$. Choose D such that $\nabla(\mathcal{A}) \supseteq D \supseteq T$ and $|D| = n - 2$.

We choose the neighborhood of $\mathcal{A}$ in $\mathcal{T}$ as $\{\mathcal{A}[t] \mid t \in D\}$. We will add paths to the neighbors of $\mathcal{A}$ to fulfill the third condition in the statement of the lemma. We need to treat the cases $t \in T$ and $t \in D \setminus T$ differently.

Case. $t \in T$

For n large enough, it follows from Lemma 12 that $G(\mathcal{A}, T)$ has a Hamiltonian cycle C . We can choose an orientation of this cycle to get $C = ((t_1, t_2), \dots, (t_\tau, t_1))$. Again by Lemma 12, for $(t_i, t_{i+1}) \in C$, with indices modulo τ , $\mathcal{A}[t_i]$ and $\mathcal{A}[t_{i+1}]$ exist. We add the vertex $\mathcal{A}[t_i, t_{i+1}]$ and the edge $\{\mathcal{A}[t_i], \mathcal{A}[t_{i+1}]\}$ to $\mathcal{T}$. By the choice of (t_i, t_{i+1}) it is clear that we did not close a cycle and that the third condition holds.

Case. $t \in D \setminus T$

For a given $t \in D \setminus T$, we find a lower bound for $|\{\mathcal{A}[t_i, t_j] \mid t_i, t_j \in T\}|$. By Lemma 12 it is enough to find a lower bound for the number of triangles in $G(\mathcal{A}, T \cup \{t\})$ containing t . Let $e = \{v, t\}$ be an edge incident to t . By Lemma 12 it holds that

$$|N(v) \cap N(t)| \geq \frac{n}{3} - 8.$$

Since there are at least $\frac{n}{3} - 4$ edges incident to t , there is a $c > 0$ such that

$$|\{\mathcal{A}[t, t_i, t_j] \mid t_i, t_j \in T\}| \geq cn^2$$

holds, for n large enough.

We assign to each $t \in D \setminus T$ a pair of triangles $\{t_i, t_j\} \subseteq T$, such that $f(\mathcal{A}, \{t_i, t_j\})$ exists. By the lower bound we just obtained, we can choose them pairwise distinct and also distinct from any edge in C , if n is large enough. For each $t \in D \setminus T$ add the vertices $\mathcal{A}[t, t_i]$ and $\mathcal{A}[t, t_j]$ and the edges $\{\mathcal{A}[t], \mathcal{A}[t, t_i]\}$ and $\{\mathcal{A}[t, t_i], \mathcal{A}[t, t_j]\}$ to $\mathcal{T}$.

The paths we added to $\mathcal{T}$ are pairwise disjoint. This implies we do not close a cycle. The claim on $\mathcal{A}_i|_T$ follows directly by the choice of (t_i, t_j) . $\square$

Theorem 9. *The flip-graph of Euclidean, marked arrangements on n pseudolines, F_n , is $(n - 2)$ -connected, if n is sufficiently large.*

Proof. The idea of the proof can be seen in Figure 8. Let n be large enough to apply Lemma 13. Let $\mathcal{A} \in F_n$. Choose a set of slopes $s_1 < \dots < s_n$ and let Z be the corresponding zonotope, according to Theorem 5. Choose a good triangle

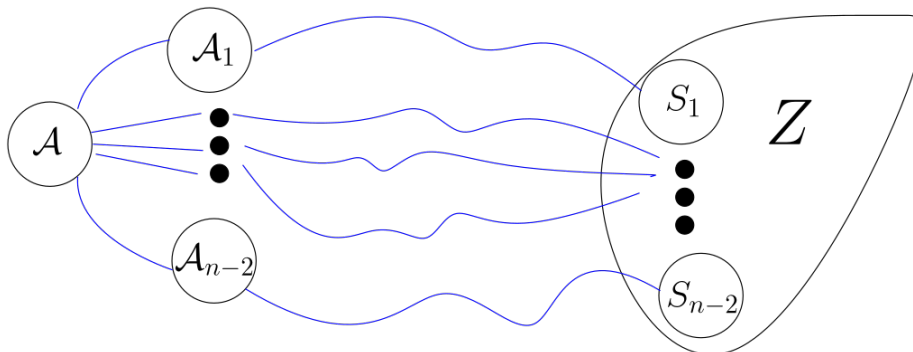


Figure 8: The idea behind the proof of the connectivity. Note that all the blue paths are pairwise disjoint by construction

set of $\mathcal{A}$ of size at least $\frac{n}{3}$ and let $\mathcal{T}$ be the star-shaped tree we obtain from Lemma 13. Let $\mathcal{A}_1, \dots, \mathcal{A}_{n-2}$ be the leaves of $\mathcal{T}$.

By the definition of a good triangle set, there is a set $S_1, \dots, S_{n-2}$ of shellable arrangements, such that $\mathcal{A}_i|_T = S_i|_T$. By Lemma 10 and Proposition 5, we obtain $n - 2$ disjoint paths from $\mathcal{A}$ to Z , since $\mathcal{A}_i|_T \neq \mathcal{A}_j|_T$, for $i \neq j$. By Balinski's theorem 7, Z is $(n - 2)$ -connected.

After deleting the vertices $X = \{x_1, \dots, x_{n-3}\} \not\ni \mathcal{A}$ and the incident edges in F_n , there is a path from $\mathcal{A}$ to $Z \setminus X$ and $Z \setminus X$ is connected. This implies that there is a path from $\mathcal{A}$ to every vertex in $Z \setminus X$. Since $\mathcal{A}$ was chosen arbitrarily, the $(n - 2)$ -connectivity follows. $\square$

5 Conclusion

The connectivity of the flip-graph of Euclidean pseudoline arrangements equals its minimal degree, if the arrangements are sufficiently large. Another graph parameter that would be highly interesting to determine is the *edge expansion*. If F_n has a high edge expansion, Markov chains on F_n are rapidly mixing, allowing us to efficiently sample uniformly from all arrangements. It is possible to obtain a canonical path system from our construction of the disjoint paths. This canonical path system has a high path congestion: there are only $2^{\Theta(n \log n)}$ shellable arrangements, while there are $2^{\Theta(n^2)}$ arrangements in general [FV11].

Determining whether F_n is polytopal for $n \geq 6$ is a possible research direction. There seem to be difficulties in generalizing the result of the special cases $n \leq 5$, especially the fact that F_n is not a partial cube in general.

It seems plausible that there is a generalization of our result to higher Bruhat orders. The *lexicographic extensions* behave similarly to shellable arrangements: they have good realization and reorientation properties, see [SZ93], [Zie93]. Also,

the construction in [FZ01] yields zonotopal subgraphs for every higher Bruhat order.

Another direction for further research could be to generalize the construction for zonotopes. This could allow us to cover F_n with well-behaved graphs. The most intuitive approach would be to lift the restriction on the realizability on the adjoints of the cyclic oriented matroid considered in [FZ01]. There is reason for pessimism: it is known that the dual of the non-Desargues matroid does not have an adjoint, [AKW90] possibly indicating that its vertex in F_n can not be covered by the tope graph of an adjoint of the cyclic oriented matroid.

Shellable arrangements seem to have a particularly simple combinatorial structure. It seems possible that there are questions that can be solved more easily on shellable arrangements than on the larger set of all pseudoline arrangements.

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6 Appendix

Due to regulations of the Technical University of Berlin, I have to include a short german summary.

In dieser Arbeit haben wir die Struktur des sogenannten Flip-Graphen auf euklidischen Pseudogeradenarrangements untersucht. Wir werden diesen Graphen im Folgenden F_n nennen.

Wir haben zuerst die bekannten Resultate zu diesem Flip-Graphen wiederholt. Dazu gehören das Resultat in [FK99], dass F_n einen Minimalgrad von $n - 2$ hat. Dann haben wir die Theorie der Signotope betrachtet, die in [FW01] eingeführt wurde. Wir kamen dann zu dem Resultat, dass F_n 1-Skelette von $(n - 2)$ -Zonotopen als induzierte Subgraphen enthält, siehe auch [FZ01].

Wir bewiesen dann mithilfe von Arrangements, die in [FW00] entdeckt wurden, dass keine isometrische Einbettung von F_n in einen Hyperwürfel existierten. Wir nutzten dafür Eigenschaften der höheren Bruhat Ordnungen.

Wir führten dann eine neue Klasse von Arrangements ein, die sogenannten *schälbaren Arrangements*. Wir konnten zeigen, dass diese Arrangements uns erlauben, die Kombinatorik gewisser Pfade in F_n zu kontrollieren.

Der Abschluss dieser Arbeit bildet der Beweis, der zeigt, dass der Minimalgrad von F_n der Konnektivität von F_n entspricht, wenn n groß genug ist. Das Resultat, dass F_n $n - 2$ zusammenhängend ist, für $n \leq 8$, gab es in einer impliziten Form in [SZ93]. Dies war eine Vermutung, die in [FW01] geäußert wurde. Wir nutzten dazu die schälbaren Arrangements, die Zonotope aus [FZ01] und das Resultat zu dem Minimalgrad von F_n aus [FK99].