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HEXAGONAL CIRCLE PATTERNS AND INTEGRABLE SYSTEMS: PATTERNS WITH CONSTANT ANGLES

ALEXANDER I. BOBENKO and TIM HOFFMANN

Abstract

Hexagonal circle patterns with constant intersection angles are introduced and studied. It is shown that they are described by discrete integrable systems of Toda type. Conformally symmetric patterns are classified. Circle pattern analogs of holomorphic mappings z^c and $\log z$ are constructed as special isomonodromic solutions. Circle patterns studied in the paper include Schramm's circle patterns with the combinatorics of the square grid as a special case.

1. Introduction

In recent years the theory of circle packings and, more generally, of circle patterns has enjoyed a fast development and a growing interest among specialists in complex analysis and discrete mathematics. This interest was initiated by Thurston's rediscovery of the Koebe-Andreev theorem (see [K]) about circle packing realizations of cell complexes of a prescribed combinatorics and by his idea about approximating the Riemann mapping by circle packings (see [T], [RS]). Since then many other remarkable facts about circle patterns have been established, such as the discrete maximum principle and Schwarz's lemma (see [R]) and the discrete uniformization theorem (see [BS]). These and other results demonstrate a surprisingly close analogy to the classical theory and allow one to talk about an emergence of the "discrete analytic function theory" (see [DS]), containing the classical theory of analytic functions as a small circles limit.

Approximation problems naturally lead to infinite circle patterns, for an analytic description of which it is advantageous to stick with fixed regular combinatorics. The most popular are hexagonal packings where each circle touches exactly six neighbors. The C^∞-convergence of these packings to the Riemann mapping was established in [HS]. Another interesting and elaborated class with similar approximation properties to be mentioned here are circle patterns with the combinatorics of the square grid

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introduced by O. Schramm [S]. The square grid combinatorics of Schramm's patterns results in an analytic description that is closer to the Cauchy-Riemann equations of complex analysis than the one of the packings with hexagonal combinatorics. Various other regular combinatorics also have similar properties (see [H]).

Although computer experiments give convincing evidence for the existence of circle packing analogs of many standard holomorphic functions (see [DS]), the only circle packings that have been described explicitly are Doyle spirals (which are analogs of the exponential function) (see [BDS]) and conformally symmetric packings (which are analogs of a quotient of Airy functions) (see [BH]). Schramm's patterns are richer in explicit examples: discrete analogs of the functions $\exp(z)$, $\operatorname{erf}(z)$, Airy (see [S]) and z^c, $\log(z)$ (see [AB]) are known.

A natural question is, what property is responsible for this comparative richness of Schramm's patterns? Is it due to the packing-pattern (or hexagonal-square) combinatorics difference? Or maybe it is the integrability of Schramm's patterns which is crucial. Indeed, Schramm's square grid circle patterns in conformal setting are known to be described by an integrable system (see [BP2]), whereas for the packings it is still unknown.*

In the present paper we introduce and study *hexagonal circle patterns with constant angles*, which merge features of the two circle patterns discussed above. Our circle patterns have the combinatorics of the regular hexagonal lattice (i.e., of the packings) and the intersection properties of Schramm's patterns. Moreover, the latter are included as a special case into our class. An example of a circle pattern studied in this paper is shown in Figure 1. Each elementary hexagon of the honeycomb lattice corresponds to a circle, and each common vertex of three hexagons corresponds to an intersection point of the corresponding circles. In particular, each circle carries six intersection points with six neighboring circles, and at each point there meet three circles. To each of the three types of edges of the regular hexagonal lattice (distinguished by their directions), we associate an angle $0 \le \alpha_n < \pi$, $n = 1, 2, 3$, and require that the corresponding circles of the hexagonal circle pattern intersect at this angle. It is easy to see that α_n's are subject to the constraint $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.

We show that despite the different combinatorics the properties and description of the hexagonal circle patterns with constant angles are quite parallel to those of Schramm's circle patterns. In particular, the intersection points of the circles are described by a discrete equation of Toda type known to be integrable (see [A]). In Section 4 we present a conformal (i.e., invariant with respect to Möbius transformations) description of the hexagonal circle patterns with constant angles and show that one can vary the angles α_n, arbitrarily preserving the cross-ratios of the intersection points on circles; thus each circle pattern generates a two-parameter deformation family. Analytic reformulation of this fact provides us with a new integrable system possessing a Lax representation in (2×2)-matrices on the regular hexagonal lattice.

*It should be said that, generally, the subject of discrete integrable systems on lattices different from $\mathbb{Z}^n$ is underdeveloped at present. The list of relevant publications has been almost exhausted by [A], [KN], and [ND].

Conformally symmetric hexagonal circle patterns are introduced in Section 5. These are defined as patterns with conformally symmetric flowers; that is, each circle with its six neighbors is invariant under a Möbius involution (Möbius 180° rotation). A similar class of circle packings has been investigated in [BH]. Let us also mention that a different subclass of hexagonal circle patterns—with the multiratio property instead of the angle condition—has been introduced and discussed in detail in [BHS]. In particular, it has been shown that this class is also described by an integrable system. Conformally symmetric circle patterns comprise the intersection set of the two known integrable classes of hexagonal circle patterns: "with constant angles" and "with multiratio property." The corresponding equations are linearizable and can be easily solved.

Further, in Section 6 we establish a rather remarkable fact. It turns out that Doyle circle packings and analogous hexagonal circle patterns with constant angles are built out of the circles with the same radii. Moreover, given such a pattern, one can arbitrarily vary the intersection angles α_n, preserving the circle radii.

Extending the intersection points of the circles by their centers, one embeds hexagonal circle patterns with constant angles into an integrable system on the dual Kagome lattice (see Sec. 7). Having included hexagonal circle patterns with constant angles into the framework of the theory of integrable systems, we get an opportunity to apply the immense machinery of the latter to study the properties of the former. This is illustrated in Section 8, where we introduce and study some isomonodromic solutions of our integrable systems on the dual Kagome lattice. The corresponding circle patterns are natural discrete versions of the analytic functions z^c and $\log z$. The results of Section 8 constitute an extension to the present, somewhat more intricate, situation of the similar construction for Schramm's circle patterns with the combinatorics of the square grid (see [BP2], [AB]).

2. Hexagonal circle patterns with constant angles

The present paper deals with *hexagonal circle patterns*, that is, circle patterns with the combinatorics of the regular hexagonal lattice (the honeycomb lattice). An example is shown in Figure 1. Each elementary hexagon of the honeycomb lattice corresponds to a circle, and each common vertex of three hexagons corresponds to an intersection point of the corresponding circles. In particular, each circle carries six intersection points with six neighboring circles, and at each intersection point exactly three circles meet.

For an analytic description of hexagonal circle patterns, we introduce some con-

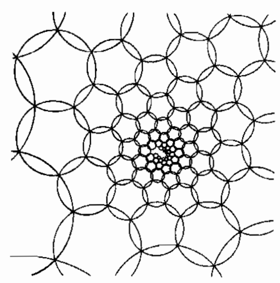

Figure 1. Doyle hexagonal isotropic circle pattern

venient lattices and variables. First, we define the *regular triangular lattice* $\mathcal{TL}$ as the cell complex whose vertices are

$$V(\mathcal{TL}) = \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}\}, \quad \text{where } \omega = \exp\left(\frac{2\pi i}{3}\right), \tag{1}$$

whose edges are the nonordered pairs

$$E(\mathcal{TL}) = \{[\mathfrak{z}_1, \mathfrak{z}_2] : \ \mathfrak{z}_1, \mathfrak{z}_2 \in V(\mathcal{TL}), \ |\mathfrak{z}_1 - \mathfrak{z}_2| = 1\}, \tag{2}$$

and whose 2-cells are all regular triangles with the vertices in $V(\mathcal{TL})$ and the edges in $E(\mathcal{TL})$. We use triples $(k, \ell, m) \in \mathbb{Z}^3$ as coordinates of the vertices. On the regular triangular lattice, two such triples are equivalent and should be identified if they differ by the vector (n, n, n) with $n \in \mathbb{Z}$.

The vertices of the regular triangular lattice correspond to centers and intersection points of hexagonal circle patterns. Associating one of the centers with the point $k = \ell = m = 0$, we obtain the *regular hexagonal sublattice* $\mathcal{HL}$ with all 2-cells being the regular hexagons with the vertices in

$$V(\mathcal{HL}) = \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m \not\equiv 0 \ (\mathrm{mod}\ 3)\} \tag{3}$$

and the edges in

$$E(\mathcal{HL}) = \{[\mathfrak{z}_1, \mathfrak{z}_2] : \ \mathfrak{z}_1, \mathfrak{z}_2 \in V(\mathcal{HL}), \ |\mathfrak{z}_1 - \mathfrak{z}_2| = 1\}. \tag{4}$$

The cells and the vertices correspond to circles and to the intersection points of the hexagonal circle patterns, respectively. Natural labelling of the faces

$$F(\mathcal{HL}) = \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m \equiv 0 \ (\mathrm{mod}\ 3)\}$$

yields

$$V(\mathcal{HL}) \cup F(\mathcal{HL}) = V(\mathcal{TL}).$$

Definition 2.1
We say that a map $w : V(\mathcal{HL}) \mapsto \hat{\mathbb{C}}$ defines a *hexagonal circle pattern* if the following condition is satisfied.

- Let

$$\mathfrak{z}_k = \mathfrak{z}' + \varepsilon^k \in V(\mathcal{HL}), \quad k = 1, 2, \ldots, 6, \text{ where } \varepsilon = \exp\left(\frac{\pi i}{3}\right),$$

be the vertices of any elementary hexagon in $\mathcal{HL}$ with the center $\mathfrak{z}' \in F(\mathcal{HL})$. Then the points $w(\mathfrak{z}_1), w(\mathfrak{z}_2), \ldots, w(\mathfrak{z}_6) \in \hat{\mathbb{C}}$ lie on a circle, and their circular order is just the listed one. We denote the circle through the points $w(\mathfrak{z}_1), w(\mathfrak{z}_2), \ldots, w(\mathfrak{z}_6)$ by $C(\mathfrak{z}')$, thus putting it into a correspondence with the center $\mathfrak{z}'$ of the elementary hexagon above.

As a consequence of this condition, we see that if two elementary hexagons of $\mathcal{HL}$ with the centers in $\mathfrak{z}', \mathfrak{z}'' \in F(\mathcal{HL})$ have a common edge $[\mathfrak{z}_1, \mathfrak{z}_2] \in E(\mathcal{HL})$, then the circles $C(\mathfrak{z}')$ and $C(\mathfrak{z}'')$ intersect at the points $w(\mathfrak{z}_1)$ and $w(\mathfrak{z}_2)$. Similarly, if three elementary hexagons of $\mathcal{HL}$ with the centers in $\mathfrak{z}', \mathfrak{z}'', \mathfrak{z}''' \in F(\mathcal{HL})$ meet in one point $\mathfrak{z}_0 \in V(\mathcal{HL})$, then the circles $C(\mathfrak{z}')$, $C(\mathfrak{z}'')$, and $C(\mathfrak{z}''')$ also have a common intersection point $w(\mathfrak{z}_0)$.

Remark. We also consider circle patterns defined not on the whole of $\mathcal{HL}$ but rather on some connected subgraph of the regular hexagonal lattice.

To each pair of intersecting circles, we associate the corresponding edge $\mathfrak{e} \in E(\mathcal{HL})$ and denote by $\phi(\mathfrak{e})$ the intersection angle of the circles, $0 \le \phi < 2\pi$. The edges of $E(\mathcal{HL})$ can be decomposed into three classes:

$$\begin{aligned} E_1^H &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm 1\}, \\ E_2^H &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega\}, \\ E_3^H &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{HL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega^2\}. \end{aligned} \tag{5}$$

These three sets correspond to three possible directions of the edges of the regular hexagonal lattice.

We study in this paper a subclass of hexagonal circle patterns intersecting at given angles defined globally on the whole lattice.

Definition 2.2
We say that a map $w : V(\mathcal{HL}) \mapsto \hat{\mathbb{C}}$ defines a *hexagonal circle pattern with constant angles* if, in addition to the condition of Definition 2.1, the intersection angles of circles are constant within their class E_n^H, that is, if

$$\phi(\mathfrak{e}) = \alpha_n, \quad \forall \mathfrak{e} \in E_n^H, \ n = 1, 2, 3.$$

This angle condition implies, in particular,

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi. \tag{6}$$

We call the circle pattern *isotropic* if all its intersection angles are equal, that is, if $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$.

The existence of hexagonal circle patterns with constant angles can be easily demonstrated via solving a suitable Cauchy problem. For example, one can start with the initial circles $C(n(1-\omega))$, $C(n(\omega-\omega^2))$, $C(n(\omega^2-1))$, $n \in \mathbb{N}$.

Remark. In the special case of $\alpha_1 = \alpha_2 = \pi/2$, $\alpha_3 = 0$, one obtains two pairs of touching circles intersecting orthogonally at each vertex. The hexagonal circle pattern becomes in this case a circle pattern of Schramm [S] with the combinatorics of the square grid. The generalization $\alpha_1 + \alpha_2 = \pi$, $\alpha_3 = 0$ was introduced in [BP2].

Figure 2 shows a nearly Schramm pattern (α_3 is small).

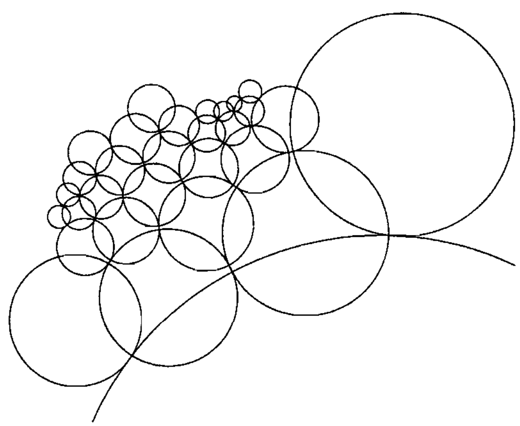

Figure 2. A nearly Schramm pattern

3. Point and radii descriptions

In this paper three different analytic descriptions are used to investigate hexagonal circle patterns with constant angles. Obviously, these circle patterns can be characterized through the *radii* of the circles. On the other hand, they can be described through the coordinates of some natural *points*, such as the intersection points or the centers of circles. Finally, note that the class of circle patterns with constant angles is invariant

with respect to arbitrary fractional-linear transformations of the Riemann sphere $\hat{\mathbb{C}}$ (Möbius transformations). Factorizing with respect to this group, we naturally come to a *conformal* description of the hexagonal circle patterns with constant angles in Section 4.

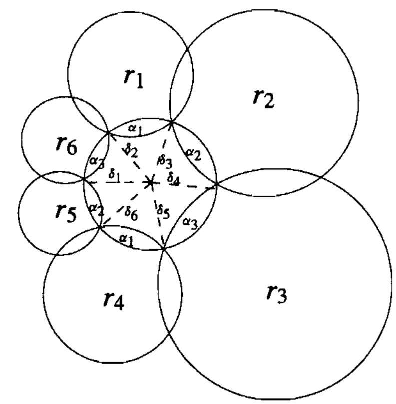

Figure 3. Circle flower

The basic unit of a hexagonal circle pattern is the *flower*, illustrated in Figure 3 and consisting of a center circle surrounded by six petals. The radius r of the central circle and the radii r_n, $n = 1, \ldots, 6$, of the petals satisfy

$$\arg \prod_{n=1}^{6} (r + e^{i\alpha_n} r_n) = \pi, \tag{7}$$

where α_n is the angle between the circles with radii r and r_n. Specifying this for the hexagonal circle patterns with constant angles and using (6), one obtains the following theorem.

THEOREM 3.1
The mapping $r : F(\mathscr{HL}) \to \mathbb{R}_+$ is the radius function of a hexagonal circle pattern with constant angles $\alpha_1, \alpha_2, \alpha_3$ if and only if it satisfies

$$\arg \prod_{n=1}^{3} (1 + e^{i\alpha_n} R_n)(e^{-i\alpha_n} + R_{n+3}) = 0, \quad R_n = \frac{r_n}{r}, \tag{8}$$

for every flower.

Conjugating (8) and dividing it by the product $R_1 \cdots R_6$, one observes that for every hexagonal circle pattern with constant angles there exists a *dual* one.

Definition 3.2
Let $r : F(\mathcal{HL}) \to \mathbb{R}_+$ be the radius function of a hexagonal circle pattern CP with constant angles. The hexagonal circle pattern CP^* with the same constant angles and the radii function $r^* : F(\mathcal{HL}) \to \mathbb{R}_+$ given by

$$r^* = \frac{1}{r} \tag{9}$$

is called *dual** to CP.

For deriving the point description of the hexagonal circle patterns with constant angles, let us consider the intersection points jointly with the centers of the circles. Fix some point $P_\infty \in \hat{\mathbb{C}}$. The reflections of this point in the circles of the pattern are called *conformal centers* of the circles. In the particular case of $P_\infty = \infty$, the conformal centers become the centers of the corresponding circles. We call an extension of a circle pattern by conformal centers a *center extension.*

Möbius transformations play a crucial role for the considerations in this paper. Recall that the *cross-ratio* of four points

$$q(z_1, z_2, z_3, z_4) := \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_2)(z_1 - z_4)} \tag{10}$$

is invariant with respect to these transformations. We start with a simple lemma.

LEMMA 3.3
Let z_2, z_4 be the intersection points, and let z_1, z_3 be the conformal centers of two circles intersecting with the angle α, as in Figure 4. The cross-ratio of these points is

$$q(z_1, z_2, z_3, z_4) = e^{-2i\alpha}. \tag{11}$$

The claim is obvious for the Euclidean centers. These can be mapped to conformal centers by an appropriate Möbius transformation that preserves the cross-ratio.

Thus a circle pattern provides a solution $z : V(\mathcal{TL}) \to \hat{\mathbb{C}}$ to (11) with the corresponding angles α. This solution is defined at the vertices of the lattice $\mathcal{TL}$ with quadrilateral sites composed of the pairs of points, as in Lemma 3.3. For a hexagonal circle pattern with constant angles, one can derive from (11) equations for the intersection points $z : V(\mathcal{HL}) \to \hat{\mathbb{C}}$ and conformal centers $z : V(\mathcal{TL} \setminus \mathcal{HL}) \to \hat{\mathbb{C}}$.

*Note that in the theory of circle packings there is a notion of a dual packing which is completely different from the present definition.

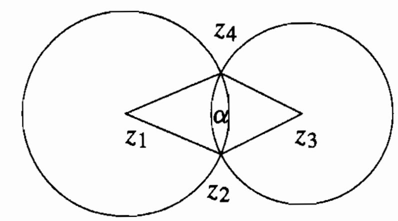

Figure 4. Cross-ratio of an elementary quadrilateral

THEOREM 3.4
Let $z, z_1, z_2, z_3, z_4, z_5, z_6$ be conformal centers of a flower of a hexagonal circle pattern with constant angles $\alpha_1, \alpha_2, \alpha_3$, where α_n, $n = 1, 2, 3$, are the angles of pairs of circles corresponding to z, z_n and z, z_{n+3}. Define $\delta_1, \delta_2, \delta_3$ through

$$2\alpha_n = \delta_{n+2} - \delta_{n+1} \pmod{2\pi}, \quad n \in \{1, 2, 3\} \pmod 3. \tag{12}$$

Then z_n satisfy a discrete equation of Toda type on the hexagonal lattice

$$\sum_{n=1}^{3} A_n \Big(\frac{1}{z - z_n} + \frac{1}{z - z_{n+3}} \Big) = 0, \tag{13}$$

where

$$A_n = e^{i\delta_{n+2}} - e^{i\delta_{n+1}}, \quad n \in \{1, 2, 3\} \pmod 3.$$

Let w_1, w_2, w_3 be the intersection points neighboring the point w of a hexagonal circle pattern, and let α_n be the angle between the circles intersecting at w, w_n. Then the following identity holds:

$$\sum_{n=1}^{3} A_n \Big(\frac{1}{w - w_n} \Big) = 0. \tag{14}$$

In the special case of Schramm's patterns $\alpha_3 = 0$, $\alpha_1 + \alpha_2 = \pi$, flowers contain only four petals, and we arrive at the following.

THEOREM 3.5
The intersection points $w, w_r, w_u, w_l, w_d \in \mathbb{C}$ and the conformal centers $z, z_r, z_u, z_l, z_d \in \mathbb{C}$ of the neighboring circles of a Schramm pattern (labelled as in Fig. 5) satisfy the discrete equation of Toda type on a $\mathbb{Z}^2$-lattice:

$$\frac{1}{w - w_r} + \frac{1}{w - w_l} = \frac{1}{w - w_u} + \frac{1}{w - w_d}, \tag{15}$$

$$\frac{1}{z - z_r} + \frac{1}{z - z_l} = \frac{1}{z - z_u} + \frac{1}{z - z_d}. \tag{16}$$

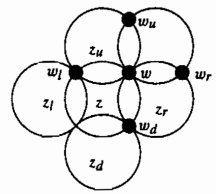

Figure 5. Intersection points and conformal centers of a Schramm circle pattern

The proofs of these two theorems are presented in Appendix B (in the case of complex $\alpha \in \mathbb{C}$).

It should be noticed that equations (16) and (13) both appeared in the theory of integrable equations in a totally different context (see [Su], [A]). The geometric interpretation in the present paper is new.

The sublattices of the centers and of the intersection points are dependent, and one can be essentially uniquely reconstructed from the other. The corresponding formulas, which are natural to generalize for complex cross-ratios (10), hold for discrete equations of Toda type (13), (14) and (15), (16). We present these relations, which are of independent interest in the theory of discrete integrable systems, in Appendix B.

In Section 7 we show that the hexagonal circle patterns with constant angles are described by an integrable system on a regular lattice closely related to the lattices introduced in Section 2.

4. Conformal description

Let us now turn our attention to a conformal description of the hexagonal circle patterns. This description is used in the construction of conformally symmetric circle patterns in Section 5. We derive equations for the cross-ratios of the points of the hexagonal lattice which allow us to reconstruct the lattice up to Möbius transformations.

First, we investigate the relations of cross-ratios inside one hexagon of the hexagonal lattice shown in Figure 6.

LEMMA 4.1

Given a map $w : V(\mathscr{HL}) \to \hat{\mathbb{C}}$, let $\mathfrak{z}_1, \dots, \mathfrak{z}_6$ be six points of a hexagon cyclically ordered (see Fig. 6). To each edge $[\mathfrak{z}_i, \mathfrak{z}_{i+1}]$ of the hexagon, let us assign a cross-ratio T_i of successive points $w_i = w(\mathfrak{z}_i)$:

$$T_i := q(w_i, w_{i+1}, w_{i+2}, w_{i-1}) \ (i \bmod 6). \tag{17}$$

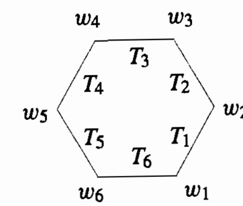

Figure 6. Cross-ratios in a hexagon

Then the equations

$$\frac{T_1}{T_4} = \frac{T_3}{T_6} = \frac{T_5}{T_2} = \frac{T_1 + T_3 - 1 - T_1T_2T_3}{1 - T_2} \tag{18}$$

hold.

Proof

Let m_i be the Möbius transformation that maps w_{i-1}, w_i, and w_{i+1} to 0, 1, and ∞, respectively. Then $M_i := m_{i+1}^{-1} m_i$ maps w_{i-1}, w_i, and w_{i+1} to w_i, w_{i+1}, and w_{i+2}, respectively, and has the form

$$M_i = \begin{pmatrix} 1 & -1 \\ 1 & -T_i \end{pmatrix}.$$

Now $(M_6M_5M_4)^{-1} = \rho M_3M_2M_1$ for some ρ gives the desired identities. □

Since every edge in $E(\mathscr{HL})$ belongs to two hexagons, it carries two cross-ratios in general. We now investigate the relation between them and show that in the case of a hexagonal circle pattern with constant angles the two cross-ratios coincide. This way, Lemma 4.1 furnishes a map $T : E(\mathscr{HL}) \to \mathbb{C}$.

Together with the cross-ratios of successive points in each hexagon (the T_i), we need the cross-ratios of a point and its three neighbors.

Definition 4.2

Let $\mathfrak{z}_1$, $\mathfrak{z}_2$, and $\mathfrak{z}_3$ be the neighbors of $\mathfrak{z} \in V(\mathscr{HL})$ counterclockwise ordered, and let $[\mathfrak{z}_i, \mathfrak{z}] \in E_i^H$, $i = 1, 2, 3$. Any map $w : V(\mathscr{HL}) \to \hat{\mathbb{C}}$ furnishes three cross-ratios in each point $\mathfrak{z} \in V(\mathscr{HL})$:

$$S_{\mathfrak{z}}^{(i)} := q(w_i, w_{i+1}, w, w_{i+2}), \quad i = 1, 2, 3 \ (\bmod 3). \tag{19}$$

They are linked by the modular transformation

$$S_{\mathfrak{z}}^{(i+1)} = 1 - \frac{1}{S_{\mathfrak{z}}^{(i)}} \ (i \bmod 3). \tag{20}$$

In Appendix A it is shown how this observation implies a Lax representation on the hexagonal lattice for the system (18).

Figure 2 shows a nearly Schramm pattern. One can obtain Schramm's description by taking combinations of T_i's and $S^{(i)}$'s which stay finite in the limit $\alpha_3 \to 0$.

5. Conformally symmetric circle patterns

The basic notion of conformal symmetry introduced in [BH] for circle packings can be easily generalized to circle patterns: every elementary flower is invariant under the Möbius equivalent of a 180° rotation.

Definition 5.1

(1) An elementary flower of a hexagonal circle pattern with petals C_i is called *conformally symmetric* if there exists a Möbius involution sending C_i to C_{i+3} (i mod 3).

(2) A hexagonal circle pattern is called *conformally symmetric* if all of its elementary flowers are.

For investigation of conformally symmetric patterns, we need the notion of the multiratio of six points:

$$m(z_1, z_2, z_3, z_4, z_5, z_6) := \frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_3 - z_4}{z_4 - z_5} \cdot \frac{z_5 - z_6}{z_6 - z_1}. \tag{28}$$

Hexagonal circle patterns with multiratio -1 are discussed in [BHS] (see [KS] for a further geometric interpretation of this quantity). It turns out that in the case of conformally symmetric hexagonal circle patterns the two known integrable classes "with constant angles" and "with multiratio -1" coincide.

PROPOSITION 5.2

(1) *An elementary flower of a hexagonal circle pattern is conformally symmetric if and only if the opposite intersection angles are equal and the six intersection points with the central circle have multiratio -1.*

(2) *A hexagonal circle pattern is conformally symmetric if and only if it has constant intersection angles and for all circles the six intersection points have multiratio -1.*

Proof

First, let us show that six points z_i have multiratio -1 if and only if there is a Möbius involution sending z_i to z_{i+3}.

If there is such a Möbius transformation, it is clear that $q(z_1, z_2, z_3, z_4) = q(z_4, z_5, z_6, z_1)$ and

$$m(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{-q(z_1, z_2, z_3, z_4)}{q(z_4, z_5, z_6, z_1)}, \tag{29}$$

which implies

$$m(z_1, z_2, z_3, z_4, z_5, z_6) = -1. \tag{30}$$

Conversely, let M be the Möbius transformation sending z_1, z_2, and z_3 to z_4, z_5, and z_6, respectively. Then for $z_* := M(z_4)$ equation (30) implies $q(z_4, z_5, z_6, z_1) = q(z_1, z_2, z_3, z_4) = q(M(z_1), M(z_2), M(z_3), M(z_4)) = q(z_4, z_5, z_6, z_*)$, and thus $\mathfrak{z}_* = \mathfrak{z}_1$. The same computation yields $\mathfrak{z}_2 = M(\mathfrak{z}_5)$ and $\mathfrak{z}_3 = M(\mathfrak{z}_6)$.

Now the first statement of the theorem is proven since the intersection points and angles determine the petals completely. For the proof of the second statement, the only thing left to show is that all flowers being conformally symmetric implies that the three intersection angles per flower sum up to π. So let us look at a flower around the circle C with petals C_i. Let the angle between C and C_i be α_i (and we know that $\alpha_i = \alpha_{i+3}$). Then the angle $\beta_1 = \pi - \alpha_1 - \alpha_2$ is the angle between C_1 and C_2, and $\beta_3 = \pi - \alpha_3 - \alpha_1$ is the one between C_3 and C_4. Since the flowers around C_2 and C_3 are conformally symmetric too, we now have two ways to compute the angle β_2 between C_2 and C_3. Namely,

$$\pi - \alpha_2 - \alpha_3 = \beta_2 = \pi - (\pi - \alpha_3 - \alpha_1) - (\pi - \alpha_1 - \alpha_2),$$

which implies $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. □

Using (29), we see that in the case of multiratio -1 the opposite cross-ratios T defined in Section 4 must be equal: $T_i = T_{(i \bmod 3)}$. Thus on $E(\mathscr{H}\mathscr{L})$ the T's must be constant in the direction perpendicular to the edge they are associated with. For the three cross-ratios in a hexagon, we get from (18),

$$T_1 + T_2 + T_3 - T_1 T_2 T_3 = 2. \tag{31}$$

Let us rewrite this equation by using the labelling shown in Figure 8. We denote by a_k, b_ℓ, and c_m the cross-ratios (24) associated with the edges of the families E_1^H, E_2^H, and E_3^H, respectively. Note that for the labels in Figure 8,

$$k + \ell + m = 1$$

holds. Written in terms of a_k, b_ℓ, c_m, equation (31) is linear:

$$a_k + b_\ell + c_m = 1 \tag{32}$$

and can be solved explicitly.

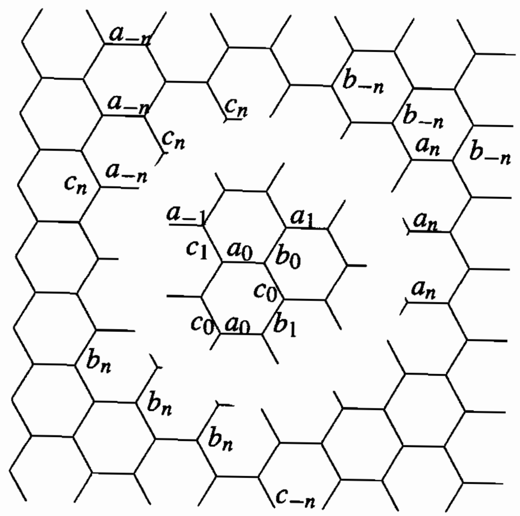

Figure 8. Cross-ratios for conformally symmetric circle patterns

LEMMA 5.3
The general solution to (32) on $E(\mathscr{H}\mathscr{L})$ is given by

$$\begin{aligned} a_k &= a_0 + k\Delta, \\ b_\ell &= b_0 + \ell\Delta, \\ c_m &= c_0 + m\Delta, \end{aligned} \tag{33}$$

with $a_0, b_0, c_0 \in \mathbb{C}$ and $\Delta = 1 - a_0 - b_0 - c_0$.

Proof
Obviously, (33) solves (32). On the other hand, it is easy to show that the cross-ratios on three neighboring edges determine all other cross-ratios recursively. Therefore (33) is the only solution to (32). □

THEOREM 5.4
Conformally symmetric circle hexagonal circle patterns are described as follows: Given a_n, b_n, c_n by (33), choose $S^{(1)} \in \mathbb{C}$ (or angles α_1 and α_2); then there is a conformally symmetric circle pattern with intersection angles α_1, α_2, and α_3 given by formula (27).

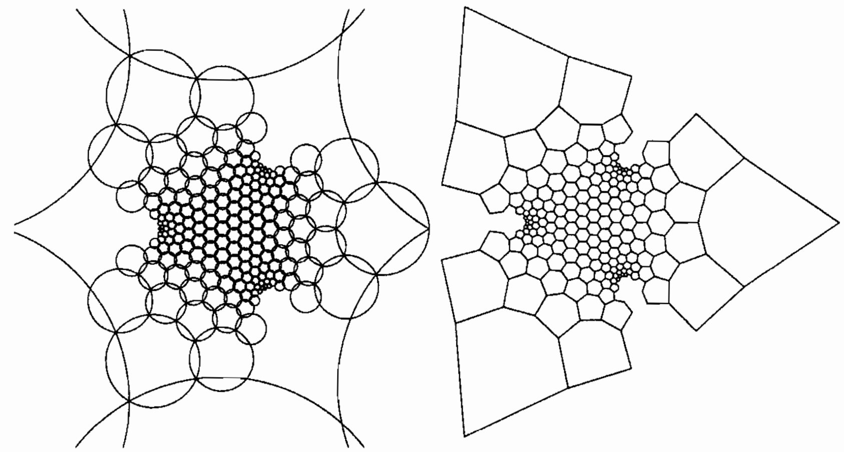

Figure 9. A conformally symmetric circle pattern and $V(\mathscr{H}\mathscr{L})$ under the quotient of two Airy functions $f(z) = (\mathrm{Bi}(z) + \sqrt{3}\mathrm{Ai}(z))/(\mathrm{Bi}(z) - \sqrt{3}\mathrm{Ai}(z))$

The cross-ratio of four points can be viewed as a discretization of the Schwarzian derivative. In this sense conformally symmetric patterns correspond to maps with a linear Schwarzian. The latter are quotients of two Airy functions (see [BH]). Figure 9 shows that a symmetric solution of (32) is a good approximation of its smooth counterpart. Figure 2 also shows a conformally symmetric pattern with $S^{(1)} = 10i$ and $a_0 = b_0 = 1/3 - 0.29$, and $c_0 = 1 - 2a_0$.

6. Doyle circle patterns

Doyle circle packings are described through their radii function. The elementary flower of a hexagonal circle packing is a central circle with six touching petals (which, in turn, touch each other cyclically). Let the radius of the central circle be R, and let $R_1, \dots, R_6$ be the radii of the petals. Doyle spirals are described through the constraint

$$R_k R_{k+3} = R^2, \qquad R_k R_{k+2} R_{k+4} = R^3. \tag{34}$$

There are two degrees of freedom for the whole packing—for example, R_1/R and R_2/R, which are constant for all flowers. The next lemma claims that the same circles that form a Doyle packing build up a circle pattern with constant angles.

LEMMA 6.1
The radii function of a Doyle packing (i.e., a solution to (34)) solves the pattern radii equation (8) for any choice of the angles $\alpha_1, \alpha_2, \alpha_3 = \pi - \alpha_1 - \alpha_2$.

Proof
Insert identities (34) into (8). □

Definition 6.2
A hexagonal circle pattern with constant angles whose radii function obeys the constraint (34) is called a Doyle pattern.

Figure 1 shows a Doyle pattern. The following lemma and theorem show how the Doyle patterns fit into our conformal description.

LEMMA 6.3
Doyle patterns are conformally symmetric.

Proof
We have to show that for a Doyle pattern the multiratio of each hexagon is -1. One can assume that the circumferencing circle has radius 1 and center zero and that $w_1 = 1$. Then the other points are given by

$$w_j = w_{j-1}\frac{1 + R_j e^{i\alpha_{(j \bmod 3)}}}{1 + R_j e^{-i\alpha_{(j \bmod 3)}}},$$

where R_j are the radii of the petals. Inserting this into the definition of the multiratio implies the claim. □

THEOREM 6.4
Doyle patterns and their Möbius transforms can be characterized in the following way: the corresponding solution to (32) is constant; that is,* $a_0 + b_0 + c_0 = 1$ *and* $a_0, b_0, c_0 > 0$.

Proof
Any Doyle pattern gives rise to a constant solution of (32) since all elementary flowers of a Doyle pattern are similar.

On the other hand, one easily sees that all $0 < R_i < 1$ can be realized. □

*Here constant means $a_k = a_0$, $b_k = b_0$, $c_k = c_0$ for all $k \in \mathbb{Z}$.

7. Lax representation and dual patterns

We start with a general construction of integrable systems on graphs which does not hang on the specific features of the lattice. This notion includes the following ingredients:

- an *oriented graph* $\mathcal{G}$ with the vertices $V(\mathcal{G})$ and the edges $E(\mathcal{G})$;
- a loop group $G[\lambda]$, whose elements are functions from $\mathbb{C}$ into some group G (the complex argument λ of these functions is known in the theory of integrable systems as the spectral parameter);
- a wave function $\Psi : V(\mathcal{G}) \mapsto G[\lambda]$ defined on the vertices of $\mathcal{G}$;
- a collection of transition matrices $L : E(\mathcal{G}) \mapsto G[\lambda]$ defined on the edges of $\mathcal{G}$.

It is supposed that for any oriented edge $\mathfrak{e} = (\mathfrak{z}_{\text{out}}, \mathfrak{z}_{\text{in}}) \in E(\mathcal{G})$ the values of the wave functions in its ends are connected via

$$\Psi(\mathfrak{z}_{\text{in}}, \lambda) = L(\mathfrak{e}, \lambda)\Psi(\mathfrak{z}_{\text{out}}, \lambda). \tag{35}$$

Therefore the following *zero-curvature condition* has to be satisfied. Consider any closed contour consisting of a finite number of edges of $\mathcal{G}$:

$$\mathfrak{e}_1 = (\mathfrak{z}_1, \mathfrak{z}_2), \qquad \mathfrak{e}_2 = (\mathfrak{z}_2, \mathfrak{z}_3), \qquad \ldots, \qquad \mathfrak{e}_p = (\mathfrak{z}_p, \mathfrak{z}_1).$$

Then

$$L(\mathfrak{e}_p, \lambda) \cdots L(\mathfrak{e}_2, \lambda) L(\mathfrak{e}_1, \lambda) = I. \tag{36}$$

In particular, for any edge $\mathfrak{e} = (\mathfrak{z}_1, \mathfrak{z}_2)$ one has $\mathfrak{e}^{-1} = (\mathfrak{z}_2, \mathfrak{z}_1)$ and

$$L(\mathfrak{e}^{-1}, \lambda) = \big(L(\mathfrak{e}, \lambda)\big)^{-1}. \tag{37}$$

Actually, in applications the matrices $L(\mathfrak{e}, \lambda)$ also depend on a point of some set X (the phase space of an integrable system), so that some elements $x(\mathfrak{e}) \in X$ are attached to the edges $\mathfrak{e}$ of $\mathcal{G}$. In this case, the discrete zero-curvature condition (36) becomes equivalent to the collection of equations relating the fields $x(\mathfrak{e}_1), \ldots, x(\mathfrak{e}_p)$ attached to the edges of each closed contour. We say that this collection of equations admits a *zero-curvature representation*. Such representation may be used to apply analytic methods for finding concrete solutions, transformations, or conserved quantities.

In this paper we deal with zero-curvature representations on the hexagonal lattice $\mathcal{HL}$ and especially on a lattice closely related to it: a *special quadrilateral lattice* $\mathcal{QL}$, which is obtained from $\mathcal{HL}$ by deleting from $E(\mathcal{TL})$ the edges of the hexagonal lattice $E(\mathcal{HL})$ (i.e., corresponding to the intersection points). The so-defined lattice $\mathcal{QL}$ has quadrilateral cells, vertices $V(\mathcal{QL}) = V(\mathcal{TL})$, and the edges

$$E(\mathcal{QL}) = E(\mathcal{TL}) \setminus E(\mathcal{HL}),$$

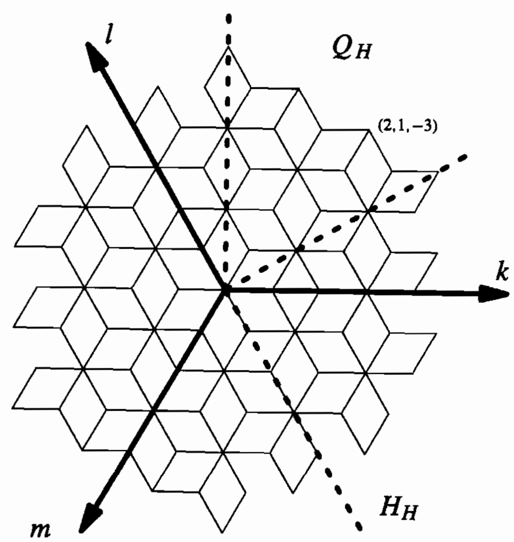

Figure 10. Quadrilateral lattice $\mathcal{QL}$

as shown in Figure 10. The lattice dual to $\mathcal{QL}$ is known as the Kagome lattice (see [B]). As in (5), there are three types of edges in $E(\mathcal{QL})$ distinguished by their directions:

$$\begin{aligned} E_1^Q &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm 1\}, \\ E_2^Q &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega\}, \\ E_3^Q &= \{\mathfrak{e} = [\mathfrak{z}', \mathfrak{z}''] \in E(\mathcal{QL}) : \mathfrak{z}' - \mathfrak{z}'' = \pm \omega^2\}. \end{aligned} \tag{38}$$

There is a natural labelling $(k, \ell, m) \in \mathbb{Z}^3$ of the vertices $V(\mathcal{QL})$ which respects the lattice structure of $\mathcal{QL}$. Let us decompose $V(\mathcal{QL}) = V_0 \cup V_1 \cup V_{-1}$ into three sublattices:

$$\begin{aligned} V_0 &= \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m = 0\}, \\ V_1 &= \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m = 1\}, \\ V_{-1} &= \{\mathfrak{z} = k + \ell\omega + m\omega^2 : \ k, \ell, m \in \mathbb{Z}, \ k + \ell + m = -1\}. \end{aligned} \tag{39}$$

Note that this definition associates a unique triple (k, ℓ, m) to each vertex. Neighboring vertices of $\mathcal{QL}$, that is, those connected by edges, are characterized by the property that their (k, ℓ, m)-labels differ only in one component. Each vertex of V_0 has six edges, whereas the vertices of $V_{\pm 1}$ have only three.

The (k, ℓ, m)-labelling of the vertices of $\mathcal{QL}$ suggests that we consider this lattice to be the intersection of $\mathbb{Z}^3$ with the strip $|k + \ell + m| \leq 1$. This description turns out to be useful, especially for the construction of discrete analogs of z^c and $\log z$ in Section 8. We denote $p = (k, \ell, m) \in \mathbb{Z}^3$ and three different types of edges of $\mathbb{Z}^3$ by

$$E_1 = \big(p, p + (1, 0, 0)\big), \qquad E_2 = \big(p, p + (0, 1, 0)\big),$$
$$E_3 = \big(p, p + (0, 0, 1)\big), \quad p \in \mathbb{Z}^3.$$

The group $G[\lambda]$ which we use in our construction is the *twisted loop group* over $\mathrm{SL}(2, \mathbb{C})$:

$$G[\lambda] = \{L : \mathbb{C} \mapsto \mathrm{SL}(2, \mathbb{C}) \,|\, L(-\lambda) = \sigma_3 L(\lambda) \sigma_3\}, \quad \sigma_3 = \mathrm{diag}(1, -1). \tag{40}$$

To each type E_n, we associate a constant $\Delta_n \in \mathbb{C}$. Let $z_{k,\ell,m}, z^*_{k,\ell,m}$ be fields defined at vertices $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ and satisfying

$$(z_{\mathrm{in}} - z_{\mathrm{out}})(z^*_{\mathrm{in}} - z^*_{\mathrm{out}}) = \Delta_n \tag{41}$$

on all the edges $\mathfrak{e} = (p_{\mathrm{out}}, \ p_{\mathrm{in}}) \in E_n, \ n = 1, 2, 3$. Here we denote $z_{\mathrm{out,in}} = z(p_{\mathrm{out,in}}), \ z^*_{\mathrm{out,in}} = z^*(p_{\mathrm{out,in}})$. We call the mapping z^* *dual* to z. To each oriented edge $\mathfrak{e} = (p_{\mathrm{out}}, p_{\mathrm{in}}) \in E_n$, we attach the following element of the group $G[\lambda]$:

$$L^{(n)}(\lambda) = (1 - \lambda^2 \Delta_n)^{-1/2} \begin{pmatrix} 1 & \lambda(z_{\mathrm{in}} - z_{\mathrm{out}}) \\ \lambda(z^*_{\mathrm{in}} - z^*_{\mathrm{out}}) & 1 \end{pmatrix}. \tag{42}$$

Note that this form, as well as condition (41), is independent of the orientation of the edge. Substituting (41) into (42), one obtains $L(\lambda)$ in terms of the field z only.

Dual fields z, z^* can be characterized in their own terms. The condition that for any quadrilateral of the dual lattice the oriented edges $(z^*_{\mathrm{out}}, z^*_{\mathrm{in}})$ defined by (41) sum up to zero implies the following statements.

THEOREM 7.1

(i) *Let $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ be dual fields, that is, fields satisfying duality condition (41). Then three types of elementary quadrilaterals of $\mathbb{Z}^3$ have the following cross-ratios:*

$$\begin{aligned} q(z_{k,\ell,m}, z_{k+1,\ell,m}, z_{k+1,\ell,m-1}, z_{k,\ell,m-1}) &= \frac{\Delta_1}{\Delta_3}, \\ q(z_{k,\ell,m}, z_{k,\ell,m+1}, z_{k,\ell-1,m+1}, z_{k,\ell-1,m}) &= \frac{\Delta_3}{\Delta_2}, \\ q(z_{k,\ell,m}, z_{k,\ell+1,m}, z_{k-1,\ell+1,m}, z_{k-1,\ell,m}) &= \frac{\Delta_2}{\Delta_1}. \end{aligned} \tag{43}$$

The same identities hold with z replaced by z^*.

(ii) *Given a solution $z : \mathbb{Z}^3 \to \mathbb{C}$ to (43), formula (41) determines (uniquely up to translation) a solution $z^* : \mathbb{Z}^3 \to \mathbb{C}$ of the same system (43).*

It is obvious that the zero-curvature condition (36) is fulfilled for every closed contour in $\mathbb{Z}^3$ if and only if it holds for all elementary quadrilaterals. It is easy to check that the transition matrices (42) satisfy the zero-curvature condition.

THEOREM 7.2
Let $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ be a solution to (41), and let $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4$ be consecutive positively oriented edges of an elementary quadrilateral of $\mathcal{QL}$. Then the zero-curvature condition

$$L(\mathfrak{e}_4, \lambda)L(\mathfrak{e}_3, \lambda)L(\mathfrak{e}_2, \lambda)L(\mathfrak{e}_1, \lambda) = I$$

holds with $L(\mathfrak{e}, \lambda)$ defined by (42). Moreover, let the elements

$$L^{(n)}(\mathfrak{e}, \lambda) = (1 - \lambda^2 \Delta_n)^{-1/2} \begin{pmatrix} 1 & \lambda f \\ \lambda g & 1 \end{pmatrix}, \qquad fg = \Delta_n, \quad n = 1, 2, 3,$$

of $G[\lambda]$ be attached to oriented edges $\mathfrak{e} = (p_{\mathrm{out}}, p_{\mathrm{in}}) \in E_n$. Then the zero-curvature condition on $\mathbb{Z}^3$ is equivalent to the existence of $z, z^* : \mathbb{Z}^3 \to \mathbb{C}$ such that the factorization

$$f(\mathfrak{e}) = z(p_{\mathrm{in}}) - z(p_{\mathrm{out}}), \qquad g(\mathfrak{e}) = z^*(p_{\mathrm{in}}) - z^*(p_{\mathrm{out}})$$

holds. So-defined z, z^* satisfy (41) and (43).

The zero-curvature condition in Theorem 7.2 is a generalization of the Lax pair found in [NC] for the discrete conformal mappings (see [BP1]). The zero-curvature condition implies the existence of the wave function $\Psi : \mathbb{Z}^3 \to G[\lambda]$. The last one can be used to restore the fields z and z^*. There holds the following result having many analogs in the differential geometry described by integrable systems ("Sym formula"; see, e.g., [BP2]).

THEOREM 7.3
Let $\Psi(p, \lambda)$ be the solution of (35) with the initial condition $\Psi(p = 0, \lambda) = I$. Then the fields z, z^* may be found as

$$\left.\frac{d\Psi_{k,\ell,m}}{d\lambda}\right|_{\lambda=0} = \begin{pmatrix} 0 & z_{k,\ell,m} - z_{0,0,0} \\ z^*_{k,\ell,m} - z^*_{0,0,0} & 0 \end{pmatrix}. \tag{44}$$

This simple observation turns out to be useful for analytic constructions of solutions, in particular, in Section 8.

Interpreting the lattice $\mathcal{QL}$ as $\{(k, \ell, m) \in \mathbb{Z}^3 : |k + \ell + m| \leq 1\}$, one arrives at the following.

COROLLARY 7.4
Theorems 7.1, 7.2, and 7.3 hold for the lattice $\mathcal{QL}$ if in the statements one replaces $\mathbb{Z}^3$ by $\mathcal{QL}$, p by $\mathfrak{z}$, E_n by E_n^Q and assumes that $k + \ell + m = 0$ in (43).

Let us return to hexagonal circle patterns and explain their relation to the discrete integrable systems of this section. To obtain the lattice $\mathcal{QL}$, consider the intersection points of a hexagonal circle pattern jointly with the conformal centers of the circles. The image of so-defined mapping $z : V(\mathcal{QL}) \to \hat{\mathbb{C}}$ consists of the intersection points $z(V_1 \cup V_{-1})$ and the conformal centers $z(V_0)$. The edges connecting the points on the circles with their centers correspond to the edges of the quadrilateral lattice $\mathcal{QL}$. Whereas the angles α_n are associated to three types E_n^H, $n = 1, 2, 3$, of the edges of the hexagonal lattice, constants δ_n defined in (12) are associated to three types E_n^Q, $n = 1, 2, 3$, of the edges of the quadrilateral lattice. Identifying them with

$$\Delta_n = e^{-i\delta_n} \tag{45}$$

in the matrices (42), one obtains a zero-curvature representation for hexagonal circle patterns with constant angles.

THEOREM 7.5
The intersection points and the conformal centers of the circles $z : \mathcal{QL} \to \hat{\mathbb{C}}$ of a hexagonal circle pattern with constant angles α_n, $n = 1, 2, 3$, satisfy the cross-ratio system (43) on the lattice $\mathcal{QL}$ with Δ_n, $n = 1, 2, 3$, determined by (12), (45).

Theorem 7.5 follows from the identification of (43) with (11) using (12) and (45).

The duality transformation (41) for arbitrary mappings satisfying the cross-ratio equations preserves the class of such mappings coming from circle patterns with constant angles. The last one is nothing but the dual circle pattern of Definition 3.2.

THEOREM 7.6
Let $z : \mathcal{QL} \to \hat{\mathbb{C}}$ be a hexagonal circle pattern CP with constant angles together with the Euclidean centers of the circles. Then the dual circle pattern CP^*, together with the Euclidean centers of the circles, is given by the dual mapping $z^* : \mathcal{QL} \to \hat{\mathbb{C}}$.

Since Δ_n, $n = 1, 2, 3$, are unitary (see (45)), relation (9) follows directly from (41). The patterns have the same intersection angles since the cross-ratio equations (43) for z and z^* coincide.

8. The z^c and $\log z$ patterns

To construct hexagonal circle pattern analogs of holomorphic functions z^c and $\log z$, we use the analytic point description presented in Section 7. Recall that we interpret the lattice $\mathcal{Q}\mathcal{L}$ of the intersection points and the centers of the circles as a subset $|k+\ell+m| \le 1$ of $\mathbb{Z}^3$ and thus study the zero-curvature representation and Ψ-function on $\mathbb{Z}^3$.

A fundamental role in the presentation of this section is played by a *nonautonomous constraint* for the solutions of the cross-ratio system (43),

$$\begin{aligned} bz_{k,\ell,m}^2 + cz_{k,\ell,m} + d &= 2(k-a_1)\frac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} \\ &\quad + 2(\ell - a_2)\frac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} \\ &\quad + 2(m-a_3)\frac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m+1} - z_{k,\ell,m-1}}, \end{aligned} \tag{46}$$

where $b, c, d, a_1, a_2, a_3 \in \mathbb{C}$ are arbitrary. Note that the form of the constraint is invariant with respect to Möbius transformations.

Our presentation in this section consists of three parts. First, we explain the origin of the constraint (46), deriving it in the context of isomonodromic solutions of integrable systems. Then we show that it is compatible with the cross-ratio system (43); that is, there exist nontrivial solutions of (43), (46). And finally, we specify parameters of these solutions to obtain circle pattern analogs of holomorphic mappings z^c and $\log z$.

We choose a different gauge of the transition matrices to simplify formulas. Let us orient the edges of $\mathbb{Z}^3$ in the direction of increasing $k+\ell+m$. We conjugate $L^{(n)}(\lambda)$ of positively oriented edges with the matrix $\operatorname{diag}(1,\lambda)$, and then we multiply by $(1-\Delta_n\lambda^2)^{1/2}$ in order to get rid of the normalization of the determinant. Then writing μ for λ^2, we end up with the matrices

$$\mathcal{L}^{(n)}(\mathfrak{e},\mu) = \begin{pmatrix} 1 & z_{\text{in}} - z_{\text{out}} \\ \mu\dfrac{\Delta_n}{z_{\text{in}} - z_{\text{out}}} & 1 \end{pmatrix}, \tag{47}$$

associated to the edge $\mathfrak{e} = (p_{\text{out}}, p_{\text{in}}) \in E_n$ oriented in the direction of increasing $k+\ell+m$. Each elementary quadrilateral of $\mathbb{Z}^3$ has two consecutive positively oriented pairs of edges $\mathfrak{e}_1, \mathfrak{e}_2$ and $\mathfrak{e}_3, \mathfrak{e}_4$. The zero-curvature condition turns into

$$\mathcal{L}^{(n_1)}(\mathfrak{e}_2)\mathcal{L}^{(n_2)}(\mathfrak{e}_1) = \mathcal{L}^{(n_2)}(\mathfrak{e}_4)\mathcal{L}^{(n_1)}(\mathfrak{e}_3).$$

Then the values of the wave function Φ in neighboring vertices are related by the formulas

$$\begin{cases} \Phi_{k+1,\ell,m}(\mu) = \mathcal{L}^{(1)}(\mathfrak{e},\mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_1, \\ \Phi_{k,\ell+1,m}(\mu) = \mathcal{L}^{(2)}(\mathfrak{e},\mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_2, \\ \Phi_{k,\ell,m+1}(\mu) = \mathcal{L}^{(3)}(\mathfrak{e},\mu)\Phi_{k,\ell,m}(\mu), & \mathfrak{e} \in E_3. \end{cases} \tag{48}$$

We call a solution $z : \mathbb{Z}^3 \mapsto \mathbb{C}$ of equations (43) *isomonodromic* (cf. [I]) if there exists a wave function $\Phi : \mathbb{Z}^3 \mapsto \mathrm{GL}(2,\mathbb{C})[\mu]$ satisfying (48) and some linear differential equation in μ:

$$\frac{d}{d\mu}\Phi_{k,\ell,m}(\mu) = \mathcal{A}_{k,\ell,m}(\mu)\Phi_{k,\ell,m}(\mu), \tag{49}$$

where $\mathcal{A}_{k,\ell,m}(\mu)$ are (2×2)-matrices, meromorphic in μ, with poles whose position and order do not depend on k, ℓ, m.

It turns out that the simplest nontrivial isomonodromic solutions satisfy the constraint (46). Indeed, since $\det \mathcal{L}^{(n)}(\mu)$ vanishes at $\mu = 1/\Delta_n$, the logarithmic derivative of $\Phi(\mu)$ must be singular in these points. We assume that these singularities are as simple as possible, that is, simple poles.

THEOREM 8.1
Let $z : \mathbb{Z}^3 \to \mathbb{C}$ be an isomonodromic solution to (43) with the matrix $\mathcal{A}_{k,\ell,m}$ in (49) of the form

$$\mathcal{A}_{k,\ell,m}(\mu) = \frac{C_{k,\ell,m}}{\mu} + \sum_{n=1}^{3} \frac{B_{k,\ell,m}^{(n)}}{\mu - 1/\Delta_n} \tag{50}$$

with μ-independent matrices $C_{k,\ell,m}$, $B_{k,\ell,m}^{(n)}$ and normalized trace $\operatorname{tr}\mathcal{A}_{0,0,0}(\mu) = 0$. Then these matrices have the following form:*

$$\begin{aligned} C_{k,\ell,m} &= \frac{1}{2}\begin{pmatrix} -bz_{k,\ell,m} - c/2 & bz_{k,\ell,m}^2 + cz_{k,\ell,m} + d \\ b & bz_{k,\ell,m} + c/2 \end{pmatrix}, \\ B_{k,\ell,m}^{(1)} &= \frac{k-a_1}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} \\ &\quad \cdot \begin{pmatrix} z_{k+1,\ell,m} - z_{k,\ell,m} & \dfrac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k,\ell,m} - z_{k-1,\ell,m}} \\ 1 & \end{pmatrix} + \frac{a_1}{2}I, \\ B_{k,\ell,m}^{(2)} &= \frac{\ell-a_2}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} \\ &\quad \cdot \begin{pmatrix} z_{k,\ell+1,m} - z_{k,\ell,m} & \dfrac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell,m} - z_{k,\ell-1,m}} \\ 1 & \end{pmatrix} + \frac{a_2}{2}I, \\ B_{k,\ell,m}^{(3)} &= \frac{m-a_3}{z_{k,\ell,m+1} - z_{k,\ell,m-1}} \\ &\quad \cdot \begin{pmatrix} z_{k,\ell,m+1} - z_{k,\ell,m} & \dfrac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m} - z_{k,\ell,m-1}} \\ 1 & \end{pmatrix} + \frac{a_3}{2}I, \end{aligned}$$

and $z_{k,\ell,m}$ satisfies (46).

*As explained in the proof of Theorem 8.1, this normalization can be achieved without loss of generality.

Conversely, any solution $z : \mathbb{Z}^3 \to \mathbb{C}$ to the system (43), (46) is isomonodromic with $\mathscr{A}_{k,\ell,m}(\mu)$ given by the formulas above.

The proofs of Theorems 8.1 and 8.2 are presented in Appendix 9. We prove Theorem 8.1 by computing the compatibility conditions of (48) and (49), and we prove Theorem 8.2 by showing the solvability of a reasonably posed Cauchy problem.

THEOREM 8.2
For arbitrary $b, c, d, a_1, a_2, a_3 \in \mathbb{C}$, the constraint (46) is compatible with the cross-ratio equations (43); that is, there are nontrivial solutions of (43), (46).

Further, we deal with the special case of (46) where

$$b = a_1 = a_2 = a_3 = 0.$$

If $c \neq 0$, one can always assume that $d = 0$ in (46) by shifting $z \to z - \frac{d}{c}$:

$$\begin{aligned} c z_{k,\ell,m} = {} & 2k \frac{(z_{k+1,\ell,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k-1,\ell,m})}{z_{k+1,\ell,m} - z_{k-1,\ell,m}} \\ & + 2\ell \frac{(z_{k,\ell+1,m} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell-1,m})}{z_{k,\ell+1,m} - z_{k,\ell-1,m}} \\ & + 2m \frac{(z_{k,\ell,m+1} - z_{k,\ell,m})(z_{k,\ell,m} - z_{k,\ell,m-1})}{z_{k,\ell,m+1} - z_{k,\ell,m-1}}. \end{aligned} \tag{51}$$

To define a circle pattern analog of z^c, it is natural to restrict the mapping to the following two subsets of $\mathbb{Z}^3$:

$$\begin{aligned} Q &= \{(k,\ell,m) \in \mathbb{Z}^3 \,|\, k \geq 0, \ell \geq 0, m \leq 0\}, \\ H &= \{(k,\ell,m) \in \mathbb{Z}^3 \,|\, m \leq 0\} \subset \mathbb{Z}^3. \end{aligned}$$

COROLLARY 8.3
The solution $z : Q \to \mathbb{C}$ of the system (43) satisfying the constraint (51) is uniquely determined by its values

$$z_{1,0,0},\ z_{0,1,0},\ z_{0,0,-1}. \tag{52}$$

Proof
Using the constraint (51), one determines the values along the coordinate lines $z_{n,0,0}, z_{0,n,0}, z_{0,0,-n}$, $\forall n \in \mathbb{N}$. Then all other $z_{k,\ell,m}$, $k, \ell, -m \in \mathbb{N}$, in consecutive order are determined through the cross-ratios (43). Computations using different cross-ratios give the same result due to the following lemma about the eighth point. □

LEMMA 8.4
Let $H(p_1 + 1/2, p_2 + 1/2, p_3 - 1/2)$, $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$, be the elementary hexahedron (lying in $\mathbb{C}$) with the vertices $z_{p+(k,\ell,-m)}$, $k, \ell, m \in \{0, 1\}$, and let the cross-ratios of the opposite faces of $H(p_1 + 1/2, p_2 + 1/2, p_3 - 1/2)$ be equal to

$$\begin{aligned} & q(z_p, z_{p+(0,0,-1)}, z_{p+(0,1,-1)}, z_{p+(0,1,0)}) \\ & \qquad = q(z_{p+(1,0,0)}, z_{p+(1,0,-1)}, z_{p+(1,1,-1)}, z_{p+(1,1,0)}) =: q_1, \\ & q(z_p, z_{p+(1,0,0)}, z_{p+(1,0,-1)}, z_{p+(0,0,-1)}) \\ & \qquad = q(z_{p+(0,1,0)}, z_{p+(1,1,0)}, z_{p+(1,1,-1)}, z_{p+(0,1,-1)}) =: q_2, \\ & q(z_p, z_{p+(0,1,0)}, z_{p+(1,1,0)}, z_{p+(1,0,0)}) \\ & \qquad = q(z_{p+(0,0,-1)}, z_{p+(0,1,-1)}, z_{p+(1,1,-1)}, z_{p+(1,0,-1)}) =: q_3, \end{aligned}$$

with $q_1 q_2 q_3 = 1$. Then all the vertices of the hexahedron are uniquely determined through four given points, z_p, $z_{p+(1,0,0)}$, $z_{p+(0,1,0)}$, $z_{p+(0,0,-1)}$.

The relation of solutions of (43), (51) to circle patterns is established in the following.

THEOREM 8.5
The solution $z : Q \to \mathbb{C}$ of the system (43), (51) with the initial data

$$z_{1,0,0} = 1, \qquad z_{0,1,0} = e^{i\beta}, \qquad z_{0,0,-1} = e^{i\gamma} \tag{53}$$

and unitary cross-ratios $q_n = e^{-2i\alpha_n}$ determines a circle pattern. For all $(k, \ell, m) \in Q$ with even $k + \ell + m$, the points $z_{k\pm1,\ell,m}, z_{k,\ell\pm1,m}, z_{k,\ell,m\pm1}$ lie on a circle with the center $z_{k,\ell,m}$.

Proof
We say that a quadrilateral is of the *kite form* if it has two pairs of equal adjacent edges (and that a hexahedron is of the kite form if all its six faces are of the kite form). These quadrilaterals have unitary cross-ratios with the argument equal to the angle between the edges (see Lem. 3.3). Our proof of the theorem is based on two simple observations:

- if a quadrilateral has a pair of adjacent edges of equal length and unitary cross-ratio, then it is of the kite form;
- if the cross-ratio of a quadrilateral is equal to $q(z_1, z_2, z_3, z_4) = e^{-2i\alpha}$, where α is the angle between the edges (z_1, z_2) and (z_2, z_3) (as in Fig. 4), then the quadrilateral is of the kite form.

Applying these observations to the elementary hexahedron in Lemma 8.4, one obtains the following.

LEMMA 8.6
Let three adjacent edges of the elementary hexahedron in Lemma 8.4 have equal length, and let the cross-ratios of the face be unitary $q_n = e^{-2i\alpha_n}$. Then the hexahedron is of the kite form.

The constraint (51) with $\ell = m = 0$ implies by induction

$$|z_{2n+1,0,0} - z_{2n,0,0}| = |z_{2n,0,0} - z_{2n-1,0,0}|,$$

and all the points $z_{n,0,0}$ lie on the real axis. The same equidistance and straight-line properties hold true for the ℓ and m axes. Also, by induction, one shows that all elementary hexahedra of the mapping $z : Q \to \mathbb{C}$ are of the kite form. Indeed, due to Lemma 8.6, the initial conditions (53) imply that the hexahedron $H(1/2, 1/2, -1/2)$ is of kite form. Since the axis vertices lie on the straight lines, we get, for example, that the angle between the edges $(z_{1,1,0}, z_{1,0,0})$ and $(z_{1,0,0}, z_{2,0,0})$ is α_3. Now applying the observation (ii) to the faces of $H(3/2, 1/2, -1/2)$, we see that

$$|z_{2,0,0} - z_{1,0,0}| = |z_{2,0,0} - z_{2,1,0}| = |z_{2,0,0} - z_{2,0,-1}|.$$

Lemma 8.6 implies that $H(3/2, 1/2, -1/2)$ is of the kite form. Proceeding further this way and controlling the lengths of the edges meeting at $z_{k,\ell,m}$ with even $k+\ell+m$ and the angles at $z_{k,\ell,m}$ with odd $k+\ell+m$, one proves the statement for all hexahedra. □

Denote the vertices of the hexagonal grid by (see Fig. 10)

$$Q_H = \{(k, \ell, m) \in Q : |k+\ell+m| \le 1\},$$
$$H_H = \{(k, \ell, m) \in H : |k+\ell+m| \le 1\}.$$

The half-plane H_H consists of the sector Q_H and its two images under the rotations with the angles $\pm\pi/3$.

COROLLARY 8.7
The mapping $z : H_H \to \mathbb{C}$ given by (43), (51) with unitary cross-ratios $q_n = e^{-2i\alpha_n}$ and unitary initial data

$$|z_{\pm1,0,0}| = |z_{0,\pm1,0}| = |z_{0,0,0}| = 1$$

is a hexagonal circle pattern with constant angles α_n together with the Euclidean centers of the circles. The centers of the corresponding circles are the images of the points with $k+\ell+m = 0$.

The circle patterns determined by most of the initial data $\beta, \gamma \in \mathbb{R}$ are quite irregular. But for a special choice of these parameters one obtains a regular circle pattern that

we call the hexagonal circle pattern z^c motivated by the asymptotic of the constraint (51) as $k, \ell, m \to \infty$.

Definition 8.8
The hexagonal circle pattern z^c, $0 < c < 2$ (together with the Euclidean centers of the circles), is the solution $z : H_H \to \mathbb{C}$ of (43), (51) with

$$\frac{\Delta_{n+1}}{\Delta_{n+2}} = e^{2i\alpha_n}, \quad n \pmod 3,$$

and the initial conditions

$$z_{1,0,0} = 1, \qquad z_{0,1,0} = e^{ic(\alpha_1+\alpha_2)}, \qquad z_{0,0,-1} = e^{ic\alpha_2},$$
$$z_{-1,0,0} = e^{ic\pi}, \qquad z_{0,-1,0} = e^{-ic\alpha_3}. \tag{54}$$

Motivated by our computer experiments (see, in particular, Fig. 11) and the corresponding result for Schramm's circle patterns with the combinatorics of the square grid (see [AB], [Ag]), we conjecture that the hexagonal z^c is embedded; that is, the interiors of all elementary quadrilaterals $(z(\mathfrak{z}_1), z(\mathfrak{z}_2), z(\mathfrak{z}_3), z(\mathfrak{z}_4))$, $|\mathfrak{z}_{i+1} - \mathfrak{z}_i| = 1$, $i \pmod 4$, are disjoint.

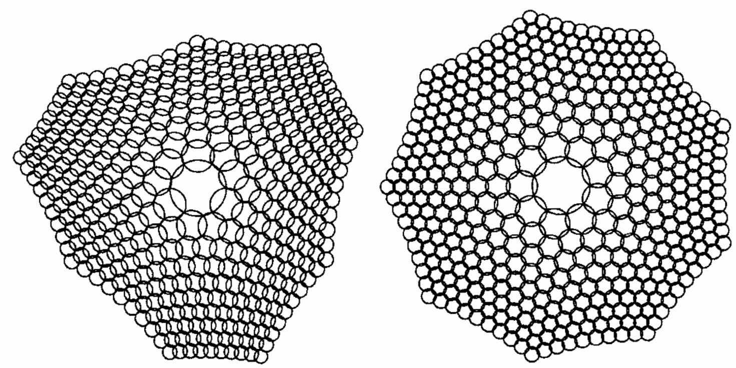

Figure 11. Nonisotropic and isotropic circle patterns $z^{2/3}$

In the isotropic case when all the intersection angles are the same,

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3},$$

one has

$$\Delta_1 = \omega\Delta_2 = \omega^2\Delta_3, \tag{55}$$

and due to the symmetry, it is enough to restrict the mapping to Q_H. The initial conditions (54) become

$$z_{1,0,0} = 1, \qquad z_{0,1,0} = e^{2\pi i c/3}, \qquad z_{0,0,-1} = e^{\pi i c/3}. \tag{56}$$

As for the smooth z^c, the images of the coordinate axes $\arg \mathfrak{z} = \pi/6, \pi/3, \pi/2$ are the axes with the arguments $\pi c/6$, $\pi c/3$, and $\pi c/2$, respectively:

$$\arg z_{n,0,-n} = \frac{\pi c}{6}, \qquad \arg z_{0,n,-n} = \frac{\pi c}{2},$$
$$\arg z_{n,n,m} = \frac{\pi c}{3}, \quad m = -2n;\ -2n \pm 1,\ \forall n \in \mathbb{N}.$$

For $c = 6/q$, $q \in \mathbb{N}$, the circle pattern $z^c(Q_H)$, together with its additional $q-1$ copies rotated by the angles $2\pi n/q$, $n = 1, \ldots, q-1$, comprises a circle pattern covering the whole complex plane. This pattern is hexagonal at all circles other than the central one, which has q neighboring circles. The circle pattern $z^{6/5}$ is shown in Figure 12.

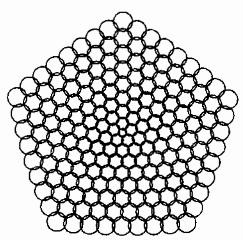

Figure 12. Circle pattern $z^{6/5}$

Definition 8.8 is given for $0 < c < 2$. For $c = 2$, the radii of the circles intersecting the central circle of the pattern become infinite. To get finite radii, the central circle should degenerate to a point

$$z_{0,0,0} = z_{1,0,0} = z_{0,0,-1} = z_{0,1,0} = 0. \tag{57}$$

The mapping $z : Q \to \mathbb{C}$ is then uniquely determined by $z_{2,0,0}$, $z_{0,2,0}$, $z_{0,0,-2}$, $z_{1,1,0}$, $z_{1,0,-1}$, $z_{0,1,-1}$, the values of which can be derived taking the limit $c \to 2-0$. Indeed, consider the quadrant $m = 0$ with the cross-ratios of all elementary quadrilaterals equal to $q = e^{-2i\alpha}$. Choosing $z_{1,0,0} = \epsilon$, $z_{0,1,0} = \epsilon e^{ic\alpha}$ as above, we get $z_{2,0,0} = 2\epsilon/(2-c)$. Normalization

$$c = 2 - 2\epsilon$$

yields

$$z_{2,0,0} = 1, \qquad z_{0,2,0} = e^{ic\alpha}. \tag{58}$$

Since the angles of the triangle with the vertices $z_{0,0,0}$, $z_{1,0,0}$, $z_{1,1,0}$ are $c\alpha/2$, $\pi - \alpha$, $\alpha\epsilon$, respectively, one obtains

$$z_{1,1,0} = \epsilon + R_\epsilon e^{i\alpha}, \qquad R_\epsilon = \epsilon \frac{\sin(c\alpha/2)}{\sin(\alpha\epsilon)}.$$

In the limit $\epsilon \to 0$ we have

$$z_{1,1,0} = \frac{\sin\alpha}{\alpha} e^{i\alpha}. \tag{59}$$

Observe that in our previous notation $\alpha = \alpha_1 + \alpha_2 = \pi - \alpha_3$. Finally, the same computations for the sectors $\{\ell = 0, k \geq 0, m \leq 0\}$ and $\{k = 0, \ell \geq 0, m \leq 0\}$ provide us with the following data:

$$z_{2,0,0} = 1, \qquad z_{0,0,-2} = e^{2i\alpha_2}, \qquad z_{0,2,0} = e^{2i(\alpha_1+\alpha_2)},$$
$$z_{1,0,-1} = \frac{\sin\alpha_2}{\alpha_2} e^{i\alpha_2}, \qquad z_{0,1,-1} = \frac{\sin\alpha_1}{\alpha_1} e^{i(\alpha_1+2\alpha_2)},$$
$$z_{1,1,0} = \frac{\sin(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2} e^{i(\alpha_1+\alpha_2)}. \tag{60}$$

In the same way, the initial data for other two sectors of H_H are specified. The hexagonal circle pattern with these initial data and $c = 2$ is an analog of the holomorphic mapping z^2.

The duality transformation preserves the class of circle patterns we defined.

THEOREM 8.9
We have

$$(z^c)^* = z^{c^*}, \qquad c^* = 2 - c,$$

where $(z^c)^*$ is the hexagonal circle pattern dual to the circle pattern z^c and normalized to vanish at the origin $(z^c)^*(\mathfrak{z} = 0) = 0$.

Proof
Let us consider the mapping on the whole Q. It is easy to see that on the axes the duality transformation (41) preserves the form of the constraint, with c being replaced by $c^* = 2 - c$. Then the constraint with c^* holds for all points of Q due to the compatibility in Theorem 8.2. Restriction to $|k + \ell + m| \leq 1$ implies the claim. □

The smooth limit of the duality transformation of the hexagonal patterns is the following transformation $f \mapsto f^*$ of holomorphic functions:

$$\left(f^*(z)\right)' = \frac{1}{f'(z)}.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log z$. Motivated by this observation, we define the hexagonal circle pattern $\log z$ as the dual to the circle pattern z^2:

$$\log z := (z^2)^*.$$

The corresponding constraint (see (46))

$$2k\frac{(z_{k+1,\ell,m}-z_{k,\ell,m})(z_{k,\ell,m}-z_{k-1,\ell,m})}{z_{k+1,\ell,m}-z_{k-1,\ell,m}} + 2\ell\frac{(z_{k,\ell+1,m}-z_{k,\ell,m})(z_{k,\ell,m}-z_{k,\ell-1,m})}{z_{k,\ell+1,m}-z_{k,\ell-1,m}} + 2m\frac{(z_{k,\ell,m+1}-z_{k,\ell,m})(z_{k,\ell,m}-z_{k,\ell,m-1})}{z_{k,\ell,m+1}-z_{k,\ell,m-1}} = 1 \tag{61}$$

can be derived as a limit $c \to +0$ (see [AB] for this limit in the square grid case). The initial data for $\log z$ are dual to the ones for z^2. In our model case of the quadrant $m = 0$ factorizing $q = \Delta_1/\Delta_2$ with $\Delta_1 = 1/2$, $\Delta_2 = e^{2i\alpha}/2$, one arrives at the following data dual to (57), (58), and (59):

$$z_{0,0,0} = \infty, \qquad z_{1,0,0} = 0, \qquad z_{2,0,0} = \frac{1}{2},$$
$$z_{0,1,0} = i\alpha, \qquad z_{0,2,0} = \frac{1}{2} + i\alpha, \qquad z_{1,1,0} = \frac{\alpha}{2\sin\alpha}e^{i\alpha}. \tag{62}$$

In the isotropic case, the circle patterns are more symmetric and can be described as mappings of Q_H.

Definition 8.10

The isotropic hexagonal circle patterns z^2 and $\log z$ are the mappings $Q_H \to \mathbb{C}$ with the cross-ratios of all elementary quadrilaterals equal* to $e^{-2\pi i/3}$. The mapping z^2 is determined by the constraint (51) with $c = 2$ and the initial data

$$z_{0,0,0} = \infty, \qquad z_{1,0,0} = 0, \qquad z_{0,0,-1} = 0, \qquad z_{0,1,0} = 0,$$
$$z_{2,0,0} = 1, \qquad z_{0,0,-2} = e^{2\pi i/3}, \qquad z_{0,2,0} = e^{4\pi i/3},$$
$$z_{1,0,-1} = \frac{3\sqrt{3}}{2\pi}e^{\pi i/3}, \qquad z_{0,1,-1} = -\frac{3\sqrt{3}}{2\pi}, \qquad z_{1,1,0} = \frac{3\sqrt{3}}{4\pi}e^{2\pi i/3}.$$

*The first point in the cross-ratio is a circle center, and the quadrilaterals are positively oriented.

For $\log z$ the corresponding constraint is (61) and the initial data are

$$z_{0,0,0} = \infty, \qquad z_{1,0,0} = 0, \qquad z_{0,0,-1} = \frac{\pi}{3}i, \qquad z_{0,1,0} = \frac{2\pi}{3}i,$$
$$z_{2,0,0} = \frac{1}{2}, \qquad z_{0,0,-2} = \frac{1}{2} + \frac{\pi}{3}i, \qquad z_{0,2,0} = \frac{1}{2} + \frac{2\pi}{3}i,$$
$$z_{1,0,-1} = \frac{\pi}{6}\Big(\frac{1}{\sqrt{3}} + i\Big), \qquad z_{0,1,-1} = \frac{\pi}{6}\Big(\frac{1}{\sqrt{3}} + 3i\Big),$$
$$z_{1,1,0} = \frac{\pi}{3}\Big(-\frac{1}{\sqrt{3}} + i\Big).$$

The isotropic hexagonal circle patterns z^2 and $\log z$ are shown in Figure 13.

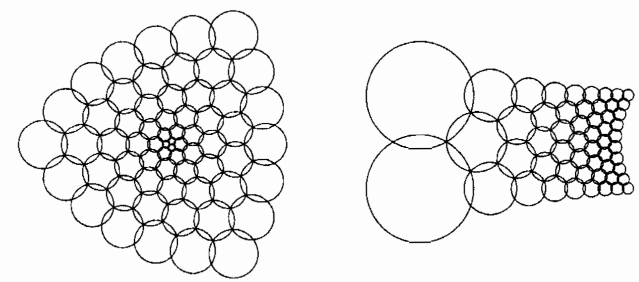

Figure 13. Isotropic circle patterns z^2 and $\log z$

Starting with z^c, $c \in (0, 2]$, one can easily define z^c for arbitrary c by applying some simple transformations of hexagonal circle patterns. The construction here is the same as for Schramm's patterns (see [AB, Sec. 6] for details). Applying the inversion of the complex plane $z \mapsto 1/z$ to the circle pattern z^c, $c \in (0, 2]$, one obtains a circle pattern, satisfying the constraint with $-c$. It is natural to call it the hexagonal circle pattern z^{-c}, $c \in (0, 2]$. Constructing the dual circle pattern, we arrive at a natural definition of z^{2+c}. Intertwining the inversion and the dualization, one constructs circle patterns z^c for any c.

In particular, inverting and then dualizing $z = k + \ell\omega + m\omega^2$ with $\Delta_1 = -3$, $\Delta_2 = -3\omega^2$, $\Delta_3 = -3\omega$, we obtain the circle pattern corresponding to z^3:

$$z_{k,\ell,m} = (k + \ell\omega + m\omega^2)^3 - (k + \ell + m).$$

Note that this is the central extension corresponding to $P_\infty = 0$. The points with even $k+\ell+m$ can be replaced by the Euclidean centers of the circles. As is shown in [AB, Sec. 6], the replacement of $P_\infty = 0$ by $P_\infty = \infty$ preserves the constraint (51).

9. Concluding remarks

We restricted ourselves to the analysis of geometric and algebraic properties of hexagonal circle patterns, leaving the approximation problem beyond the scope of this paper. The convergence of circle patterns with the combinatorics of the square grid to the Riemann mapping has been proven by Schramm in [S]. We expect that his result can be extended to the hexagonal circle patterns defined in this paper.

The entire circle pattern $\mathrm{erf}(z)$ also found in [S] remains rather mysterious. We were unable to find its analog in the hexagonal case, and thus we have no counterexamples to the Doyle conjecture for hexagonal circle patterns with constant angles. It seems that Schramm's $\mathrm{erf}(z)$ is a feature of the square grid combinatorics.

The construction of the hexagonal circle pattern analogs of z^c and $\log z$ in Section 8 was based on the extension of the corresponding integrable system to the lattice $\mathbb{Z}^3$. Theorem 8.5 claims that in this way one obtains a circle pattern labelled by three independent indices k, ℓ, m. Fixing one of the indices, say, $m = m_0$, one obtains a Schramm circle pattern with the combinatorics of the square grid. In the same way, the restriction of this three-dimensional pattern to the sublattice $|k+\ell+m-n_0| \le 1$ with some fixed $n_0 \in \mathbb{Z}$ yields a hexagonal circle pattern. In Section 8 we have defined circle patterns z^c on the sublattices $m_0 = 0$ and $n_0 = 0$. In the same way, the sublattices $m_0 \neq 0$ and $n_0 \neq 0$ can be interpreted as the circle pattern analogs of the analytic function $(z+a)^c$, $a \neq 0$, with the square and hexagonal grid combinatorics, respectively.

It is unknown whether the theory of integrable systems can be applied to hexagonal circle packings. As already mentioned in the introduction, the underdevelopment of the theory of integrable systems on the lattices different from $\mathbb{Z}^n$ may be a reason for this.

We hope that integrable hexagonal circle patterns introduced in this paper and in [BHS] will lead to progress in hexagonal circle packings.

Appendices

Appendix A. A Lax representation for the conformal description

We now give a Lax representation for equations (18). For $\mathfrak{z} \in V(\mathscr{H}\mathscr{L})$, let $[\mathfrak{z}_i, \mathfrak{z}] \in E_i^H$, and let $m_\mathfrak{z}$ be the Möbius transformation that sends $(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)$ to $(0, 1, \infty)$. For $\mathfrak{e} = [\mathfrak{z}, \tilde{\mathfrak{z}}] \in E_i^H$, set $L_i = m_{\tilde{\mathfrak{z}}} \circ m_\mathfrak{z}^{-1}$. The L_i depend only on $S^{(1)}$ and $T_\mathfrak{e}$. They are given in equation (63). Note that we do not need to orient the edges since $L_i L_i^{-1}$ is the identity if we normalize $\det L_i = 1$. The claim is that the closing condition, when multiplying the L_i around one hexagon, is equivalent to equations (18).

THEOREM A.1

Attach to each edge of E_i a matrix $L_i(T, S)$ of the form

$$L_1(T,S) = \begin{pmatrix} -1 & \frac{S-1}{S} \\ \frac{T+S-1}{S-1} & 1 \end{pmatrix},$$
$$L_2(T,S) = \begin{pmatrix} TS & 1-TS \\ S\frac{1+T(S-1)}{S-1} & -TS \end{pmatrix},$$
$$L_3(T,S) = \begin{pmatrix} 1-S & -1+T(\frac{1}{S}-1)+S \\ -S & S-1 \end{pmatrix}; \tag{63}$$

then the zero-curvature condition for each hexagon in $F(\mathscr{H}\mathscr{L})$,

$$L_1(T_4,S)L_2(T_5,S)L_3(T_6,S) = \rho L_3(T_3,S)L_2(T_2,S)L_1(T_1,S) \tag{64}$$

for all S, is equivalent to (18).

Proof

The theorem is proved by straightforward calculations. □

Equations (63) and (64) are a Lax representation for equations (18) with the spectral parameter S.

Appendix B. Discrete equations of Toda type and cross-ratios

In the following we briefly discuss the connection between discrete equations of Toda type and cross-ratio equations for the square grid and the dual Kagome lattice. This interrelationship holds in a more general setting, namely, for discrete Toda systems on graphs. This situation will be considered in a subsequent publication.

We start with the square grid. Let us decompose the lattice $\mathbb{Z}^2$ into two sublattices

$$\mathbb{Z}_k^2 = \{(m,n) \in \mathbb{Z}^2\ (m+n \bmod 2) = k\}, \quad k = 0, 1.$$

THEOREM B.1

Let $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ be a solution to the cross-ratio equation

$$q(z_{m,n}, z_{m+1,n}, z_{m+1,n+1}, z_{m,n+1}) = q. \tag{65}$$

Then, restricted to the sublattices $\mathbb{Z}_0^2$ or $\mathbb{Z}_1^2$, it satisfies the discrete equation of Toda type (16),

$$\frac{1}{z_{m,n}-z_{m+1,n+1}} + \frac{1}{z_{m,n}-z_{m-1,n-1}} = \frac{1}{z_{m,n}-z_{m+1,n-1}} + \frac{1}{z_{m,n}-z_{m-1,n+1}}. \tag{66}$$

Conversely, given a solution $z : \mathbb{Z}_0^2 \to \hat{\mathbb{C}}$ of equation (66) and arbitrary $q, w \in \hat{\mathbb{C}}$, there exists a unique extension of z to the whole lattice $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ with $z_{1,0} = w$ satisfying the cross-ratio condition (65). Moreover, restricted to the other sublattice $\mathbb{Z}_1^2$, the so-defined mapping $z : \mathbb{Z}_1^2 \to \hat{\mathbb{C}}$ satisfies the same discrete equation of Toda type (66).

Proof

For any four complex numbers, the cross-ratio equation can be written in a simple fraction form:

$$q(u_1, u_2, u_3, u_4) = q \Leftrightarrow \frac{1}{u_3 - u_2} - \frac{q}{u_3 - u_4} + \frac{q-1}{u_3 - u_1} = 0. \tag{67}$$

To distinguish between the two sublattices, we denote the points belonging to $\mathbb{Z}_0^2$ by z_i, while w_i are the points associated with $\mathbb{Z}_1^2$. Now (67) gives four equations for a

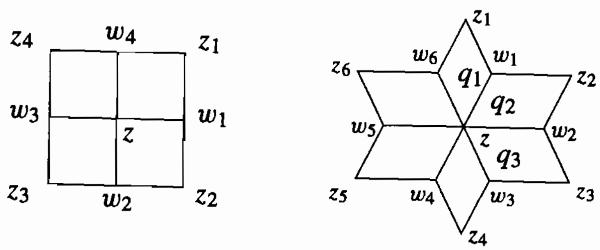

Figure 14. Nine points of $\mathbb{Z}^2$ and 13 points of $V(\mathscr{TL})$

point z and its eight neighbors, as shown in Figure 14:

$$\begin{aligned}
(q-1)\frac{1}{z-z_1} &= q\frac{1}{z-w_1} - \frac{1}{z-w_4},\\
(q^{-1}-1)\frac{1}{z-z_2} &= q^{-1}\frac{1}{z-w_2} - \frac{1}{z-w_1},\\
(q-1)\frac{1}{z-z_3} &= q\frac{1}{z-w_3} - \frac{1}{z-w_2},\\
(q^{-1}-1)\frac{1}{z-z_4} &= q^{-1}\frac{1}{z-w_4} - \frac{1}{z-w_3}.
\end{aligned} \tag{68}$$

Multiplying the second and fourth equation by q and taking the sum over all four equations, (68) yields equation (66). Conversely, given w_1, the first three equations of (68) determine w_2, w_3, and w_4 uniquely. The so-determined w_3 and w_4 satisfy the fourth equation of (68) (for any choice of w_1) if and only if (66) holds. This proves

the second part of the theorem. Obviously, interchanging w and z implies the claim for the other sublattice. □

Now we pass to the dual Kagome lattice shown in Figure 10, where a similar relation holds for the discrete equation of Toda type on the hexagonal lattice. However, the symmetry between the sublattices is lost in this case.

Here we decompose $V(\mathscr{TL})$ into $V(\mathscr{HL})$ and $V(\mathscr{TL}) \setminus V(\mathscr{HL}) \cong F(\mathscr{HL})$.

THEOREM B.2

Given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$ and a solution $z : V(\mathscr{TL}) \to \hat{\mathbb{C}}$ of the cross-ratio equations

$$\begin{aligned}
q(z_{k,\ell,m-1}, z_{k,\ell,m}, z_{k,\ell+1,m}, z_{k,\ell+1,m-1}) &= q_1,\\
q(z_{k+1,\ell,m}, z_{k,\ell,m}, z_{k,\ell,m-1}, z_{k+1,\ell,m-1}) &= q_2,\\
q(z_{k,\ell-1,m}, z_{k,\ell,m}, z_{k+1,\ell,m}, z_{k+1,\ell-1,m}) &= q_3,
\end{aligned} \tag{69}$$

we have the following:

(1) *restricted to $F(\mathscr{HL})$, the solution z satisfies the discrete equation of Toda type (13) on the hexagonal lattice*

$$\sum_{k=1}^{3} A_k\Big(\frac{1}{z-z_k} + \frac{1}{z-z_{k+3}}\Big) = 0; \tag{70}$$

(2) *restricted to $V(\mathscr{HL})$, the solution z satisfies*

$$\sum_{k=1}^{3} A_k \frac{1}{z-z_k} = 0, \tag{71}$$

where $A_i = \Delta_{i+2} - \Delta_{i+1}$ with Δ_i defined through $q_i = \Delta_{i+2}/\Delta_{i+1}$.

Conversely, given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$, a solution z to (70) on $F(\mathscr{HL})$ and $z_{0,0,0}$, there is a unique extension $z : V(\mathscr{TL}) \to \hat{\mathbb{C}}$ satisfying (69).

Given $q_1, q_2, q_3 \in \mathbb{C}$ with $q_1q_2q_3 = 1$, a solution z to (71) on $V(\mathscr{HL})$ and $z_{1,0,0}$, there is a unique extension $z : V(\mathscr{TL}) \to \hat{\mathbb{C}}$ satisfying (69).

Proof

First, (67) immediately shows that (71) is equivalent to having constant S as described in Section 5.

Again we distinguish the sublattices notationally by denoting points associated with elements of $V(\mathscr{HL})$ with w_i, and points from $F(\mathscr{HL})$ with z_i. If the cross-ratios and neighboring points of a $z \in F(\mathscr{HL})$ are labelled as shown in Figure 14, equation (67) gives six equations:

$$
\begin{aligned}
(q_1-1)\frac{1}{z-z_1} &= q_1\frac{1}{z-w_1}-\frac{1}{z-w_6},\\
(q_1-1)\frac{1}{z-z_4} &= q_1\frac{1}{z-w_4}-\frac{1}{z-w_3},\\
(q_2-1)\frac{1}{z-z_2} &= q_2\frac{1}{z-w_2}-\frac{1}{z-w_1},\\
(q_2-1)\frac{1}{z-z_5} &= q_2\frac{1}{z-w_5}-\frac{1}{z-w_4},\\
(q_3-1)\frac{1}{z-z_3} &= q_3\frac{1}{z-w_3}-\frac{1}{z-w_2},\\
(q_3-1)\frac{1}{z-z_6} &= q_3\frac{1}{z-w_6}-\frac{1}{z-w_5}.
\end{aligned}
\tag{72}
$$

To prove the first statement, we take a linear combination of the equations (72). Namely, a times the first two plus b times the second two plus c times the third two. It is easy to see that there is a choice for a, b, and c that makes the right-hand side vanish if and only if $q_1q_2q_3=1$. If $q_1=\Delta_3/\Delta_2$, $q_2=\Delta_1/\Delta_3$, and $q_3=\Delta_2/\Delta_1$, choose $a=\Delta_2$, $b=\Delta_3$, and $c=\Delta_1$. The remaining equation is (70).

To prove the third statement, we note that, given w_1, we can compute w_2 through w_6 from the second through sixth equations of (72), but the closing condition, the first equation of (72), is then equivalent to (70).

The proofs of the second and fourth statements are literally the same if we choose $z_i=z_{i+3}$ and $w_i=w_{i+3}$. □

Appendix C. Proofs of Theorems 8.1 and 8.2

Proof of Theorem 8.1

Let $\Phi_{k,\ell,m}(\mu)$ be a solution to (43), (46) with some μ-independent matrices $C_{k,\ell,m}$, $B^{(n)}_{k,\ell,m}$. The determinant identity

$$\det\Phi_{k,\ell,m}(\mu)=(1-\mu\Delta_1)^k(1-\mu\Delta_2)^\ell(1-\mu\Delta_3)^m\det\Phi_{0,0,0}(\mu)$$

implies that

$$\operatorname{tr}\mathscr{A}_{k,\ell,m}(\mu)=\frac{k}{\mu-1/\Delta_1}+\frac{\ell}{\mu-1/\Delta_2}+\frac{m}{\mu-1/\Delta_3}+a(\mu)$$

with $a(\mu)$ independent of k,ℓ,m. Without loss of generality, one can assume $a(\mu)=0$; that is,

$$\operatorname{tr}B^{(1)}_{k,\ell,m}=k,\qquad \operatorname{tr}B^{(2)}_{k,\ell,m}=\ell,\qquad \operatorname{tr}B^{(3)}_{k,\ell,m}=m.$$

This can be achieved by the change $\Phi\mapsto\exp(-1/2\int a(\mu)\,d\mu)\Phi$.

The compatibility conditions of (48) and (49) read

$$
\begin{aligned}
\frac{d}{d\mu}\mathscr{L}^{(1)} &= \mathscr{A}_{k+1,\ell,m}\mathscr{L}^{(1)}-\mathscr{L}^{(1)}\mathscr{A}_{k,\ell,m},\\
\frac{d}{d\mu}\mathscr{L}^{(2)} &= \mathscr{A}_{k,\ell+1,m}\mathscr{L}^{(2)}-\mathscr{L}^{(2)}\mathscr{A}_{k,\ell,m},\\
\frac{d}{d\mu}\mathscr{L}^{(3)} &= \mathscr{A}_{k,\ell,m+1}\mathscr{L}^{(3)}-\mathscr{L}^{(3)}\mathscr{A}_{k,\ell,m}.
\end{aligned}
\tag{73}
$$

The principal parts of these equations in $\mu=1/\Delta_1$ imply

$$B^{(1)}_{k+1,\ell,m}\begin{pmatrix}1 & f_1\\ \dfrac{1}{f_1} & 1\end{pmatrix}=\begin{pmatrix}1 & f_1\\ \dfrac{1}{f_1} & 1\end{pmatrix}B^{(1)}_{k,\ell,m},\quad f_1=z_{k+1,\ell,m}-z_{k,\ell,m},$$

$$B^{(1)}_{k,\ell+1,m}\begin{pmatrix}1 & f_2\\ \dfrac{\Delta_2}{f_2\Delta_1} & 1\end{pmatrix}=\begin{pmatrix}1 & f_2\\ \dfrac{\Delta_2}{f_2\Delta_1} & 1\end{pmatrix}B^{(1)}_{k,\ell,m},\quad f_2=z_{k,\ell+1,m}-z_{k,\ell,m},$$

$$B^{(1)}_{k,\ell,m+1}\begin{pmatrix}1 & f_3\\ \dfrac{\Delta_3}{f_3\Delta_1} & 1\end{pmatrix}=\begin{pmatrix}1 & f_3\\ \dfrac{\Delta_3}{f_3\Delta_1} & 1\end{pmatrix}B^{(1)}_{k,\ell,m},\quad f_3=z_{k,\ell,m+1}-z_{k,\ell,m}.$$

The solution with $\operatorname{tr}B^{(1)}_{k,\ell,m}=k$ is

$$B^{(1)}_{k,\ell,m}=\frac{k-a_1}{z_{k+1,\ell,m}-z_{k-1,\ell,m}}\cdot\begin{pmatrix}z_{k+1,\ell,m}-z_{k,\ell,m} & \dfrac{(z_{k+1,\ell,m}-z_{k,\ell,m})(z_{k,\ell,m}-z_{k-1,\ell,m})}{z_{k,\ell,m}-z_{k-1,\ell,m}}\\ 1 & \end{pmatrix}+\frac{a_1}{2}I.$$

The same computation yields the formulas of Theorem 8.1 for $B^{(2)}_{k,\ell,m}$ and $B^{(3)}_{k,\ell,m}$.

To derive a formula for the coefficient $C_{k,\ell,m}$, let us compare $\Phi_{k,\ell,m}(\mu)$ with the solution $\Psi_{k,\ell,m}(\lambda)$ in Theorem 7.3, more exactly, with its extension to the lattice $\mathbb{Z}^3$:

$$
\begin{aligned}
\Psi_{k+1,\ell,m}(\lambda) &= L^{(1)}(\lambda)\Psi_{k,\ell,m},\\
\Psi_{k,\ell+1,m}(\lambda) &= L^{(2)}(\lambda)\Psi_{k,\ell,m},\\
\Psi_{k,\ell,m+1}(\lambda) &= L^{(3)}(\lambda)\Psi_{k,\ell,m},
\end{aligned}
$$

normalized by $\Psi_{0,0,0}(\lambda)=I$. Here the matrices $L^{(n)}$ are given by (42):

$$L^{(n)}(\lambda)=(1-\lambda^2\Delta_n)^{-1/2}\begin{pmatrix}1 & \lambda f_n\\ \lambda\Delta_n/f_n & 1\end{pmatrix}.$$

Consider

$$\tilde\Psi=h(\lambda)\begin{pmatrix}1/\sqrt{\lambda} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix}\Psi\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & 1/\sqrt{\lambda}\end{pmatrix}$$

with

$$h(\lambda) = (1-\lambda^2\Delta_1)^{k/2}(1-\lambda^2\Delta_2)^{\ell/2}(1-\lambda^2\Delta_3)^{m/2}.$$

So-defined $\tilde{\Psi}$ is a function of μ. Since it satisfies the same difference equations (48) as $\Phi(\mu)$ and is normalized by

$$\tilde{\Psi}_{k,\ell,m}(\mu = 0) = I,$$

we have

$$\Phi_{k,\ell,m}(\mu) = \tilde{\Psi}_{k,\ell,m}(\mu)\Phi_{0,0,0}(\mu). \tag{74}$$

Moreover, $\tilde{\Psi}(\mu)$ is holomorphic in $\mu = 0$ and, due to Theorem 7.3, equal at this point to

$$\tilde{\Psi}_{k,\ell,m}(\mu = 0) = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}, \quad Z = z_{k,\ell,m} - z_{0,0,0}.$$

Taking the logarithmic derivative of (74) with respect to μ,

$$\mathscr{A}_{k,\ell,m} = \frac{d\tilde{\Psi}_{k,\ell,m}}{d\mu}\tilde{\Psi}^{-1}_{k,\ell,m} + \tilde{\Psi}_{k,\ell,m}\mathscr{A}_{0,0,0}\tilde{\Psi}^{-1}_{k,\ell,m},$$

and computing its singularity at $\mu = 0$, we get

$$C_{k,\ell,m} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} C_{0,0,0} \begin{pmatrix} 1 & -Z \\ 0 & 1 \end{pmatrix}.$$

The formula for $C_{k,\ell,m}$ in Theorem 8.1 is the general solution to this equation.

Conversely, by direct computation one can check that the compatibility conditions (73) with $\mathscr{A}_{k,\ell,m}$ computed above are equivalent to (46). □

Proof of Theorem 8.2

First, note that one can assume that $b = 0$ by applying a suitable Möbius transformation. Next, by translating z, one can make $z_{k,l,m} = 0$ for arbitrary fixed $k, l, m \in \mathbb{Z}$. (This changes d, however.) Finally, we can assume that $d = 0$ or $d = 1$ since we can scale z. Now one can show the compatibility by using a computer algebra system like MATHEMATICA as follows: given the points $z_{k,l,m}, z_{k+1,l,m}, z_{k,l\pm1,m}, z_{k,l,m\pm1}$, one can compute $z_{k-1,l,m}$ using the constraint (46). With the cross-ratio equations, one can now calculate all points necessary to apply the constraint for calculating $z_{k+2,l,m}$ and $z_{k,l,m+2}$. (In addition to $z_{k-1,l,m-1}$, one needs all points $z_{k\pm1,l\pm1,m\pm1}$.) Once again using the cross-ratios, one calculates all points necessary to calculate $z_{k+1,l,m+2}$ with the constraint. Finally, one can check that the cross-ratio $q(z_{k,l,m+1}, z_{k+1,l,m+1}, z_{k+1,l,m+2}, z_{k,l,m+2})$ is correct. Since the three directions are equivalent and since all initial data were arbitrary (i.e., symbolic), this suffices to show the compatibility. □

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A POLYTOPE CALCULUS FOR SEMISIMPLE GROUPS

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Abstract
We define a collection of polytopes associated to a semisimple group G. *Weight multiplicities and tensor product multiplicities may be computed as the number of such polytopes fitting in a certain region. The polytopes are defined as moment map images of algebraic cycles discovered by I. Mirković and K. Vilonen. These cycles are a canonical basis for the intersection homology of (the closures of the strata of) the loop Grassmannian.*

1. Introduction

Starting with a semisimple complex algebraic group G, we construct a collection of polytopes. The central result is a method that uses these to decompose the tensor product of two irreducible representations. Each tensor product multiplicity is the number of polytopes in a certain set. Other combinatorial descriptions of these numbers are known, notably by A. Berenstein and A. Zelevinsky [BZ] and P. Littelmann [LI].

The method here is based on the geometry of the loop Grassmannian and builds directly on the work of Mirković and Vilonen [MV]. But all algebraic geometry is deferred until Section 5 since we may state the main result (Theorem 1) without it. We do this in Section 2, and we follow with a lot of examples in Sections 3 and 4. Much may be gained from just these first sections without ever understanding what the loop Grassmannian is.

Some background in geometry and representation theory is discussed in Section 5. The representation theory of G is known to be closely related to the geometry of the loop Grassmannian for the Langlands dual group (see [BD], [G]). This relationship was made more explicit with Mirković and Vilonen's discovery of a collection of singular algebraic varieties in the loop Grassmannian, which we call MV-cycles (see [MV]). In terms of geometry, they provide a canonical basis for the intersection homology of the closure of each stratum of the loop Grassmannian. In terms of representation theory, they provide a canonical basis for each irreducible representation of G.

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