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Monodromy of isometric deformations of CMC surfaces

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Abstract. We investgate relationship between the monodromy of isometric deformation of CMC-surfaces in the space forms and that of the canonical deformation of CMC surfaces into the other space forms. As an application, we show that the number of isometric closed CMC-surfaces is finite.

1. Introduction

It is well known that an arbitrary simply connected domain D of CMC-H (i.e. Constant Mean Curvature H) surfaces in the space form $M^3(c)$ of constant curvature c has the following two special properties:

(i) it admits a real-analytic isometric deformation preserving mean curvature function in $M^3(c)$. We call it *H-deformation* of the surface;

(ii) it can be isometrically immersed in other space form $M^3(t)$ as a CMC-$\sqrt{H^2+c-t}$ surface. We call it *t-deformation*.

In fact, t-deformation can be viewed as a deformation of the original surface: We consider the following 1-parameter family of Riemannian metrics

$$g_t = \left(\frac{4}{4+t|x|^2}\right)^2 \sum_{j=1}^{3}(dx_j)^2$$

of constant sectional curvature t, defined on

$$\mathbf{R}^3(t) = \begin{cases} \{x \in \mathbf{R}^3 : |x| < 2/\sqrt{|t|}\} & (\text{if } t<0), \\ \mathbf{R}^3 & (\text{if } t \geq 0) \end{cases},$$

where $|x| = \sqrt{|x_1|^2+|x_2|^2+|x_3|^2}$. When $t=0$, it is just the Euclidean metric and when $t \neq 0$, they can be regarded as the stereographic image of 3-sphere or 3-hyperboloid of radius $1/\sqrt{|t|}$ in Minkowski 4-space. By this parametrization of the metrics, we can view the CMC-surfaces in any space form in $\mathbf{R}^3$ and then t-deformation is a real analytic deformation. If the surface is not simply connected, the H-deformation and t-deformation are not single-valued on the

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surface, but defined on the universal cover of the surface. We shall prove the following:

THEOREM 1. *Let* (Σ^2, ds^2) *be a Riemannian* 2*-manifold and* $F : \Sigma^2 \to M^3(c)$ $(c \in \mathbf{R})$ *an isometric immersion of constant mean curvature* H. *If* t*-deformation is single valued on* Σ^2, *so is* H*-deformation. Moreover the converse is also true unless the original surface is minimal in* $(\mathbf{R}^3, g_0)$.

It should be remarked that the theorem requires neither compactness nor completeness of the surface. When $c \le 0$ and $|H| = \sqrt{|c|}$, the theorem has been proved in [12]. The converse assertion of the theorem for Euclidean minimal surface does not hold in general. (An H-deformable complete minimal surface not preserving t-monodromy is known. See §3 of [12].) On the other hand, we can construct many non-trivial complete minimal surfaces which admits single-valued t-deformation as a dressing up procedure of the Enneper surface. (See [9].) When $c \ge 0$, the construction of CMC surfaces via the Gauss maps is known [1, 2, 4, 5, 10, 11] and the theorem can be proved easily. Thus the essential part of the proof lies in the case $c < 0$.

As an application, we shall prove

COROLLARY 2. *Let* (Σ^2, ds^2) *be a closed Riemannian* 2*-manifold and* $x : \Sigma^2 \to M^3(c)$ $(c \in \mathbf{R})$ *an isometric immersion of constant mean curvature* H, *then the number of congruent classes*

$$N_x := \sharp\{x : (\Sigma^2, ds^2) \to M^3(c);\ \textit{isometric CMC-}H \textit{ immersion}\}$$

is finite. In particular, there exists no global non-trivial isometric deformations of CMC surfaces preserving the mean curvature.

The assertion of the Corollary for $c \ge 0$ has been pointed out in [2]. When H is not constant, Lawson and Tribuzy [8] have shown that the number of isometric immersions with the same mean curvature H on a compact Riemannian 2-manifold is at most two. The existence of closed surfaces with $N_x = 2$ in $M^3(0)$ is an important open problem (see [6]).

2. Proofs

Let (Σ^2, ds^2) be a Riemannian 2-manifold and

$$F = F_c : \Sigma^2 \to M^3(c)$$

be an isometric immersion with constant mean curvature H. As pointed out in the introduction, the t-deformation

$$F_t : \Sigma^2 \to \mathbf{R}^3(t) \qquad (t \le H^2 + c)$$

is real analytic with respect to t. So is the monodromy

$$\rho_t : \pi_1(\Sigma^2) \to \mathrm{Isom}(\mathbf{R}^3(c)) \subset \mathrm{Conf}(S^3).$$

The real analyticity of ρ_t is of crucial importance in this paper.

The case $c = 0$. When $H = 0$, the assertion has been proved in [12]. So we may assume $H \neq 0$. Then the Gauss map g of F is a non-holomorphic harmonic map into S^2. It is well known that non-holomorphic harmonic map into S^2 has the associated S^1-family $(g^\theta)_{\theta \in S^1}$, which are not-single valued on Σ^2. Let

$$\hat{\rho}_\theta : \pi_1(\Sigma^2) \to SO(3)$$

be the monodromy representation of $(g^\theta)_{\theta \in S^1}$. Since the Gauss maps of H-deformations of F are in the same associated family $(g^\theta)_{\theta \in S^1}$, ρ_θ is identity for all $\theta \in S^1$ if H-deformations of F are single valued. On the other hand if $\rho_\theta = id$ for all θ, then all H-deformations are single valued since $\partial \hat{\rho}_\theta / \partial \theta = 0$. (See §5 of [5] or [2].) Thus F has single valued H-deformation iff $(g^\theta)_{\theta \in S^1}$ are all single valued. The t-deformation F_t $(t > 0)$ of F corresponds to the pair of harmonic maps (g, g^α) where $\alpha = \arg(\sqrt{H^2 - c} + ic)$. Moreover, the monodromy representation ρ_t satisfies $\rho_t(\pi_1(\Sigma^2)) = \{\pm id\}$ iff $(g^\theta)_{\theta \in S^1}$ are all single valued on Σ^2. (See [5] or [2].) Since $\rho_0(\pi_1(\Sigma^2)) = \{id\}$ and ρ_t is real analytic with respect to t, $\rho_t(\pi_1(\Sigma^2)) = \{-id\}$ never holds. So F has single valued H-deformation iff $\rho_t(\pi_1(\Sigma^2)) = \{id\}$ for $t \geq 0$. In this case, we have $\rho_t(\pi_1(\Sigma^2)) = \{id\}$ for $t < 0$ because of the real analyticity of ρ_t with respect to t. This proves the assertion.

The case $c > 0$. In the above discussion, we use the fact that the monodromy of CMC-H surface in $M^3(c)$ for $c > 0$ can be controlled by the pair of harmonic maps in the same associated S^1-family. We also use the fact in this case. Let (g_1, g_2) be the pair of harmonic maps associated with F. Then the H-deformations of F correspond to the pairs $(g_1^\theta, g_2^\theta)_{\theta \in S^1}$. Thus F has single valued H-deformation iff $(g^\theta)_{\theta \in S^1}$ are all single valued. On the other hand, the t-deformations of F correspond to the pairs $(g_1, g_2^\theta)_{\theta \in S^1}$. So F has single valued t-deformation iff $(g^\theta)_{\theta \in S^1}$ are all single valued. This proves the assertion.

The case $c < 0$. Without loss of generality we set $c = -1$. For $|H| = 1$ the assertion has been proved in [12]. So we may assume $H \neq 1$. As a homothetic change of the metric of the ambient space $M^3(t)$, the t-deformation

$$F_t : \Sigma^2 \to M^3(t)$$

of the CMC-H surface $F = F_c$ can be realized as CMC-h surfaces in $M^3(-1)$ whenever $t < 0$, where

$$h = \sqrt{\frac{(H^2 + c - t)}{|t|}}.$$

We denote this normalization of F_t into $M^3(-1)$ by $\hat{F}_h$ where the range of the new parameter h is given by

$$\mathrm{Max}\{H^2 + c, 0\} \leq h < \infty.$$

We call the family $(\hat{F}_h)$ the normalized t-deformation of F in $M^3(-1)$. This normalized t-deformation does not realize t-deformation for $t \geq 0$. However by the real analyticity of the t-deformation, t-deformation is single valued iff so is normalized t-deformation.

We now set

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We shall identify the Lorentz space $\mathbf{R}^{3,1}$ with the space of 2×2 Hermitian matrices $\bar{X}^t = X$

$$X = \sum_{j=0}^{4} X_j \sigma_j \leftrightarrow X = (X_0, X_1, X_3) \in \mathbf{R}^{3,1}$$

with the scalar product

$$\{X, Y\} = -\frac{1}{2}\mathrm{tr}(X\sigma_2 Y^t \sigma_2).$$

The hyperbolic 3-space $M^3(-1)$ consists of the Hermitian matrices X satisfying

$$\det(X) = 1, \qquad \mathrm{tr}(X) > 0. \tag{1}$$

The cases $H > 1$ and $H < 1$ can be treated exactly in the same way using the corresponding formulas for the immersion F and the Gauss map N (see (2) and (3) below and formulas in p. 155 of [3]). We present the proof for the case $H > 1$.

LEMMA 3. *Let $\hat{F}_h$ be the normalized t-deformations of F in $M^3(-1)$ and F_θ $(\theta \in S^1)$ be the H-deformations of F. Suppose the original surface F has* *single valued H-deformations (resp. t-deformations). Then* $\hat{N}_t\hat{F}_t^{-1}$ *(resp.* $N_\theta F_\theta^{-1}$*) are single-valued on* Σ^2 *for all* t *(resp. for all* θ*).*

PROOF. Let

$$\Phi_0 = \Phi_0(x, y, \lambda) : \tilde{\Sigma} \to \mathrm{GL}(2, \mathbf{C}) \qquad (\lambda \in \mathbf{C}\backslash\{0\})$$

be the family of maps defined in Theorem 14.3 in [4], which are the solution of the frame equations (the system (1.8) and (1.9) in [4]) on Σ^2. They have real determinant and are holomorphic with respect to the spectral parameter λ. Then

$$F_\lambda = \Phi_0^{-1}\sigma_2\bar{\Phi}_0\sigma_2 \tag{2}$$

gives a CMC-coth(log$|\lambda|$) immersion defined on $\tilde{\Sigma}^2$. When $\coth(\log|\lambda_0|) = H$, F_{λ_0} coincides with the original surface F. Here F_λ for $\lambda = \lambda_0 e^{i\theta}$ ($\theta \in [0, 2\pi)$) corresponds to the H-deformation of F and F_λ for $\lambda \in \mathbf{R}\backslash\{0\}$ corresponds to the t-deformation of F. The normal vector N of the surface F_λ is given by

$$N_\lambda = \Phi_0^{-1}\sigma_3\sigma_2\bar{\Phi}_0\sigma_2. \tag{3}$$

Thus we have

$$N_\lambda F_\lambda^{-1} = \Phi_0^{-1}\sigma_3\Phi_0. \tag{4}$$

The monodromy of Φ_0 is a holomorphic function of λ. If F possesses single valued H-deformations (resp. t-deformations), the monodromy is trivial for all $|\lambda| = \lambda_0$ (resp. $\lambda \in \mathbf{R}$). Due to the analyticity it is trivial for all $\lambda \in \mathbf{C}\backslash\{0\}$. This proves the assertion. □

To complete the proof of the theorem, it is now sufficient to prove the following

LEMMA 4. *If* $N_\lambda F_\lambda^{-1}$ ($\lambda \in \mathbf{C}\backslash\{0\}$) *is single-valued on* Σ^2, *so is* F_λ.

PROOF. Let M be the monodromy of Φ_0 along a loop on Σ^2

$$\Phi_0 \to \Phi_0 M.$$

Formula (4) implies that $[NF^{-1}, M] = 0$. Suppose that the matrix M is not identity. For the traceless part M_0 of M we have

$$[NF^{-1}, M_0] = 0.$$

By (1), the orthogonality of N and F yields

$$0 = \{N, F\} = -\frac{1}{2}(N\sigma_2 F^t\sigma_2) = -\frac{1}{2}\mathrm{tr}(NF^{-1}).$$

So there exists a smooth function $u \in C^\infty(\Sigma^2)$ such that

$$NF^{-1} = uM_0. \tag{5}$$

Taking the determinant of both sides, we conclude that u is a constant function which can be included into M_0. The relation (5) yields

$$\sigma_3 \Phi_0 = \Phi_0 M_0.$$

Without loss of generality one can normalize $\Phi_0(z_0) = \sigma_0$ at some point $z_0 \in \Sigma^2$. This implies $[\Phi_0, \sigma_3] = 0$, i.e. Φ_0 is diagonal for all points of Σ^2, which is impossible. Thus $M = \sigma_0$. □

(*Proof of Corollary.*) The assertion has been pointed out for the case $c \geq 0$ in [2]. So we assume $c < 0$. There are no closed CMC surface when $H^2 + c \leq 0$. So we may set $H^2 + c > 0$ and suppose $N_x = \infty$. Since H-deformation is real analytic with respect to the deformation parameter, $N_x = \infty$ implies the surface has single valued H-deformations. Then by Theorem 1, the corresponding t-deformation f_0 in $M^3(0)$ is single-valued and H-deformation of f_0 are all single valued on the surface. But it contradicts that the corollary holds for $c \geq 0$. So we can conclude $N_x < \infty$. □

References

[1] R. Aiyama and K. Akutagawa, Kenmotsu type representation formula for surfaces with prescribed mean curvature in the 3-sphere, Tohoku Math. J. **52** (2000), 95–105.

[2] R. Aiyama, K. Akutagawa, R. Miyaoka and M. Umehara, A global correspondence between CMC-surfaces in S^3 and the pairs of Non-conformal Harmonic maps into S^2, Proc. Amer. Math. Soc. **128** (2000), 939–941.

[3] M. Babich and A. I. Bobenko, Willmore Tori with umbilic lines and minimal surfaces in hyperbolic space, Duke Math. J. **72** (1993), 151–185.

[4] A. I. Bobenko, Constant mean curvature surfaces and integrable equations, Russian Math. Survey **46** (1991), 1–45.

[5] A. I. Bobenko, Surfaces in terms of 2 by 2 matrices: Old and new integrable cases, Harmonic maps and integrable systems (Eds. P. Fordy and J. C. Wood), Vieweg, (1994), 83–127.

[6] A. I. Bobenko, Exploring Surfaces through Methods from the Theory of Integrable systems. Lectures on the Bonnet Problem., SFB288 Preprint No. 403 (1999) (http://xxx.lanl.gov/abs/math/math.DG/9909003).

[7] D. A. Hoffman, Jr. and R. Osserman, On the Gauss map of surfaces in $\mathbf{R}^3$ and $\mathbf{R}^4$, Proc. London Math. Soc. **50** (1985), 27–56.

[8] H. B. Lawson, Jr. and R. A. Tribuzy, On the mean curvature function for compact surfaces, J. Differential Geom. **16** (1981), 179–183.

[9] C. McCune and M. Umehara, An analogue of the UP-iteration for constant mean curvature one surfaces in Hyperbolic 3-space, to appear in Differential Geometry and its Applications.

[10] R. Miyaoka, The splitting and deformations of the generalized Gauss map of compact CMC surfaces, Tôhoku Math. J. **51** (1999), 35–53.
[11] W. Seaman, On surfaces in $\mathbf{R}^4$, Proc. Amer. Math. Soc. **94** (1985), 467–470.
[12] M. Umehara and K. Yamada, Surfaces of constant mean curvature c in $H^3(-c^2)$ with prescribed hyperbolic Gauss map, Math. Ann. **304** (1996), 203–204.

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