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Small-Amplitude Solutions of the Sine–Gordon Equation on an Interval under Dirichlet or Neumann Boundary Conditions

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Summary. We give a complete classification of the small-amplitude finite-gap solutions of the sine–Gordon (SG) equation on an interval under Dirichlet or Neumann boundary conditions. Our classification is based on an analysis of the finite-gap solutions of the boundary problems for the SG equation by means of the Schottky uniformization approach.

Key words: sine–Gordon equation, Dirichlet boundary conditions, Neumann boundary conditions, Schottky uniformization

Introduction

This paper is devoted to analysis of the small-amplitude finite-gap solutions of the sine–Gordon equation on an interval

$$u_{tt} = u_{xx} - \sin u, \qquad 0 < x < \pi. \tag{SG}$$

We consider this equation under Dirichlet or Neumann boundary conditions:

$$u(t, 0) \equiv u(t, \pi) \equiv 0 \tag{D}$$

or

$$u_x(t, 0) = u_x(t, \pi) \equiv 0. \tag{N}$$

The results and the proofs in the (D) and (N) cases are parallel, so we mostly restrict ourselves to the Neumann problem and give a brief reformulation of the main results for the Dirichlet problem.

The equation (SG) + (N) [as well as (SG) + (D)] defines a dynamical system in the phase space Z of pairs $U(t, x) = (u(t, x),\ v = \dot{u}(t, x))$ [Z should be given some

Sobolev norm $\|\cdot\|$, for example, $Z = \mathring{H}^1(0,\pi) \times L_2(0,\pi)$ in the Dirichlet case]. The equations (SG) + (N) and (SG) + (D) are well known to be Hamiltonian: one should supply the phase space Z with the symplectic structure given by the 2-form ω_2,

$$\omega_2((U_1, v_1),\ (u_2, v_2)) = \int_0^\pi (u_1 v_2 - v_1 u_2)\, dx,$$

and consider the Hamiltonian

$$\int_0^\pi \left(\tfrac{1}{2}(v^2 + u_x^2) + \cos u\right) dx.$$

To get an idea of the qualitative behavior of solutions of (SG)+(N), let us consider the linearization of (SG):

$$u_{tt} = u_{xx} - u. \tag{1}$$

The equation (1)+(N) is a linear oscillating system with the frequencies $0^*, 1^*, 2^*, \ldots,$ where we denote

$$j^* = \sqrt{j^2 + 1}.$$

The solutions with frequency j^* have the form (u_j, v_j), where

$$u_j(t,x) = I_j \sin j^*(t + \varphi_j) \cos jx, \qquad I_j \geq 0.$$

Fix any $n \geq 1$ wavenumbers j,

$$j \in \mathbf{V} = \{V_1^0, \ldots, V_n^0\} \subset \mathbb{N} \cup \{0\}, \tag{2}$$

and consider superpositions (=sums) $U^n = (u^n, v^n)$ of solutions (u_j, v_j) with $j \in \mathbf{V}$, $u^n = u_1 + \ldots + u_n$, which are time–quasiperiodic solutions of (1)+(N) with the frequency vector $\omega = (V_1^{0*}, \ldots, V_n^{0*})$. Altogether, the solutions $\tilde{U}^n$ fill the $2n$-dimensional linear subspace E^{2n} of Z:

$$E^{2n} := \operatorname{span}\{(\cos V_j^{0*} x, 0),\ (0, \cos V_j^0 x) \mid j = 1, \ldots, n\}. \tag{3}$$

Each solution $\tilde{U}^n$ lies in an invariant torus $T^n(I)$, where $\dim T^n(I) = n$ if all $I_j > 0$. So the space E^{2n} is foliated into invariant tori and

$$E^{2n} \simeq \mathbb{R}_+^n \times \mathbb{T}^n.$$

The equation (SG) has well-known finite-gap solutions, given by the theta formula

$$u(t,x;\boldsymbol{\lambda}, D) = 2i \log \frac{\theta(i(Vx + Wt + D + \Delta))}{\theta(i(Vx + Wt + D))}, \tag{4}$$

which was first obtained by Kozel and Kotlyarov [KK] and Its (see [M]). The solution (4) defines (and is defined by) its spectral curve X, which is a hyperelliptic Riemann curve with a real involution. In general, any hyperelliptic curve X with a real involution determines a solution of the SG equation. Moreover, there are usually many connected components of the solutions corresponding to the same X, which makes a general picture rather complicated (for details see [BBEIM, EF]).

The picture simplifies if we consider only small-amplitude solutions. In this case, the genus g of the curve equals the number of nontrivial spectral branches of the corresponding L-operator (see [McK, EFM, BBEIM]); the branching points of X are $\{0, \infty\} \cup \{\lambda_1, \bar{\lambda}_1; \cdots; \lambda_g, \bar{\lambda}_g\}$, where $\lambda_j, \bar{\lambda}_j$ $(j = 1, \ldots, g)$ are the edges of the nontrivial spectral branches. The vector $(\lambda_1, \ldots, \lambda_g) \in \mathbb{C}^g$ and $D \in \mathbb{T}^g$ are parameters of the solution.

In Section 1, following [Bo2] and [BiK], we single out among g-gap solutions (4) real-valued 2π-periodic solutions, satisfying the boundary conditions (N) or (D). Moreover, thus obtained solutions $\tilde{U} = (u, \dot{u})$ of (SG)+(N) form $2n$-dimensional analytic varieties $\mathcal{T}^{2n} \subset Z$, $n = [g/2] + 1$, which are similar to the solutions of the Dirichlet problem. The solutions in $\mathcal{T}^{2n}$ of amplitude $< \rho$ form a smooth analytic manifold $\mathcal{T}^{2n}_\rho$, foliated to invariant tori of (SG)+(N):

$$\mathcal{T}^{2n}_\rho = \bigcup_X T^n(X). \tag{5}$$

The solutions with the same curve X lie in the same invariant torus, so the union in (5) is with respect to the curves X giving rise to solutions (4), which meet (N).

The spaces tangent to the manifolds $\mathcal{T}^{2n}_\rho$ at zero are exactly the spaces E^{2n} as in (3), so the spaces E^{2n} [or, equivalently, the vectors $\mathbf{V}$ as in (2)] parametrize the manifolds $\mathcal{T}^{2n}_\rho$.

Despite the fact that the solutions of the sine–Gordon equation are given by the explicit formula (4), their investigation is a serious problem due to a complicated parametrization. To solve this problem in the present paper, we use an approach, suggested in [Bo1] (for details, see Chapter 5 of [BBEIM]) and based on the Schottky uniformization of the spectral curve X. The Schottky uniformization provides a convenient coordinate system $\{\mu\}$ on the manifold formed by the curves X as in (5).

This approach allows us to investigate the manifolds $\mathcal{T}^{2n}_g$, including the nontrivial proof of their smoothness up to zero. Moreover, we obtain representations by convergent series for the frequencies $\lambda_j(\mu)$, the vectors $W(\mu)$, and the "Bloch-like" solutions of the linearized sine–Gordon equation.

We study also the linearization of the equation (LSG) about the solution (4) in the small-amplitude limit (Theorem 7).

The main reason why the manifolds $\mathcal{T}^{2n}_\rho$ and the tori $T^n(X)$ are interesting is their robustness—most of the small-amplitude tori $T^n(X)$ persist in the nonlinear Klein–Gordon (NKG) equation

$$u_{tt} = u_{xx} - mu + f(u), \qquad 0 < x < L, \tag{NKG}$$

where the function $f = o(|u|)$ is odd and nondegenerate,

$$f(u) = -f(-u), \qquad f^{(3)}(0) \neq 0,$$

and the equation is supplemented by (N) or (D) boundary conditions. Indeed, after rescaling the dependent and independent variables, the equation (NKG) can be rewritten in the form

$$u_{tt} = u_{xx} - \sin(u) + O(|u|^5), \qquad 0 < x < L_1; \tag{NKG$'$}$$

i.e., as a higher-order perturbation of the (SG) equation. Detailed information about the tori $T^n(X)$, the manifolds $\mathcal{T}_\rho^{2n}$, and their vicinities that we obtain in this paper allow us in [BoK2] to apply the infinite-dimensional KAM theory [K1, K2] to prove persistence for most of the tori $T^n(X)$ in the equation (NKG′). Jointly the persistent tori form "partial central cantorfolds" of (NKG′), which means the following. Using polar coordinates, a manifold $\mathcal{T}_\rho^{2n}$ can be written as

$$\mathcal{T}_\rho^{2n} = \mathbb{T}^n \times D^n,$$

where D^n is an n-disc of a radius $r(\rho)$. A tangent space to $\mathcal{T}_\rho^{2n}$ at the origin equals E^{2n}—an invariant subspace for the equation linearized at zero and filled with its bounded solutions. So $\mathcal{T}_\rho^{2n}$ is a partial central manifold of the (SG) equation. In [BoK2] we prove that (NKG) has an invariant manifold

$$\tilde{\mathcal{T}}_\rho^{2n} = \mathbb{T}^n \times \tilde{D}^n,$$

where $\tilde{D}^n$ is a Cantor subset of D^n, having density 1 at zero (i.e., its intersection with the ε-disc centered at the origin occupies most—in the measure sense—of the disc when $\varepsilon \to 0$). Besides, $\tilde{\mathcal{T}}_\rho^{2n}$ has a tangent space at the origin that is equal to E^{2n}; $\tilde{\mathcal{T}}_\rho^{2n}$ and is filled by invariant n-tori, formed by time-quasiperiodic solutions of (NKG).

Jointly the cantorfolds $\tilde{\mathcal{T}}_\rho^{2n}$ of the equation (NKG) are infinitesimally dense at zero: each nonempty open cone in Z (vertexed at the origin) has a nonempty intersection with the union of all $\tilde{\mathcal{T}}_\rho^{2n}$. This explains why small-amplitude solutions of (NKG) appear to be time-quasiperiodic—a fact, well known from numeric investigations of the φ^4-equation

$$u_{tt} = u_{xx} - mu + u^3$$

[which is an important example of (NKG)]. In particular, for the KAM technique we apply in [BoK2] to the (NKG) equation, the nondegeneracy of the amplitude–frequency modulation is critical for solutions forming the manifold $\mathcal{T}_\rho^{2n}$:

$$\det \partial W^j/\partial \mu_k \,|_{\mu=0} \neq 0,$$

and some nonresonance properties of the frequencies, both of which are proved in [BoK2] based on the analysis of the present paper. Similar nondegeneracy results for the periodic finite-gap solutions of the Korteweg–de Vries (KdV) equation were obtained in [BoK1].

1. Small-Amplitude Finite-Gap Solutions of Boundary-Valued Problems for the Sine–Gordon Equation

We start with some basic facts from the finite-gap theory of the SG equation (see [McK, EF, BBEIM] for the proofs and details). Let $X = \{P = (\lambda, \mu)\}$ be the hyperelliptic Riemann surface of the polynomial

$$\mu^2 = \lambda \prod_{i=1}^{g} (\lambda - \lambda_i)(\lambda - \overline{\lambda_i}), \tag{6}$$

where $\lambda_1, \dots, \lambda_g$ are pairwise different complex numbers from the upper half-plane $\mathbb{C}_+$ (we restrict ourselves to the solutions with complex branching points because the small-amplitude finite-gap solutions we are interested in are of this type). We denote the hyperelliptic involution and the conjugation involution as follows:

$$\tau_1(\lambda, \mu) = (\lambda, -\mu), \qquad \tau_2(\lambda, \mu) = (\overline{\lambda}, -\overline{\mu}).$$

Let us make on X the cut $\gamma_0 = [0, \infty)$ and the cuts γ_i, $i = 1, \dots, g$, where γ_i is a path from $\overline{\lambda_i}$ to λ_i. Let us choose the canonical basis of cycles (a_i, b_i), $i = 1, \dots, g$, on Γ in such a way that the circle a_j surrounds the cut γ_j (see Fig. 1) and fix a basis of holomorphic differentials $d\omega_1, \dots, d\omega_g$ of X normalized by the conditions

$$\oint_{a_m} \omega_j = 2\pi i \delta_{mj}, \qquad j, m = 1, \dots, g.$$

The Riemann matrix $B = (B_{mj})$,

$$B_{mj} = \oint_{b_m} \omega_j, \qquad j, m = 1, \dots, g,$$

defines the theta-function θ,

$$\theta(z|B) = \sum \exp\left(\tfrac{1}{2}\langle Bm, m\rangle + \langle z, m\rangle\right).$$

This function has the matrix of periods $(2\pi i I, B)$.

The function $\sqrt{\lambda}$ is not single-valued on X. To correlate the local parameters $\sqrt{\lambda}$ at the points $\lambda = 0$ and $\lambda = \infty$, we should fix a branch of $\sqrt{\lambda}$ on X. This branch is fixed if a contour $\mathcal{L}$ on X is specified, where $\sqrt{\lambda}$ has a jump alternating its sign ($\sqrt{\lambda}$ is analytic on $X - \mathcal{L}$ and boundary values of $\sqrt{\lambda}$ at two edges of $\mathcal{L}$ differ by a sign $\sqrt{\lambda}_{|\mathcal{L}_+} = -\sqrt{\lambda}_{|\mathcal{L}_-}$). We choose $\mathcal{L}$ to be a union (see Fig. 1) of the contours surrounding the cuts γ_i, which are mapped to γ_j's by the projection $(\lambda, \mu) \to \lambda$. Let

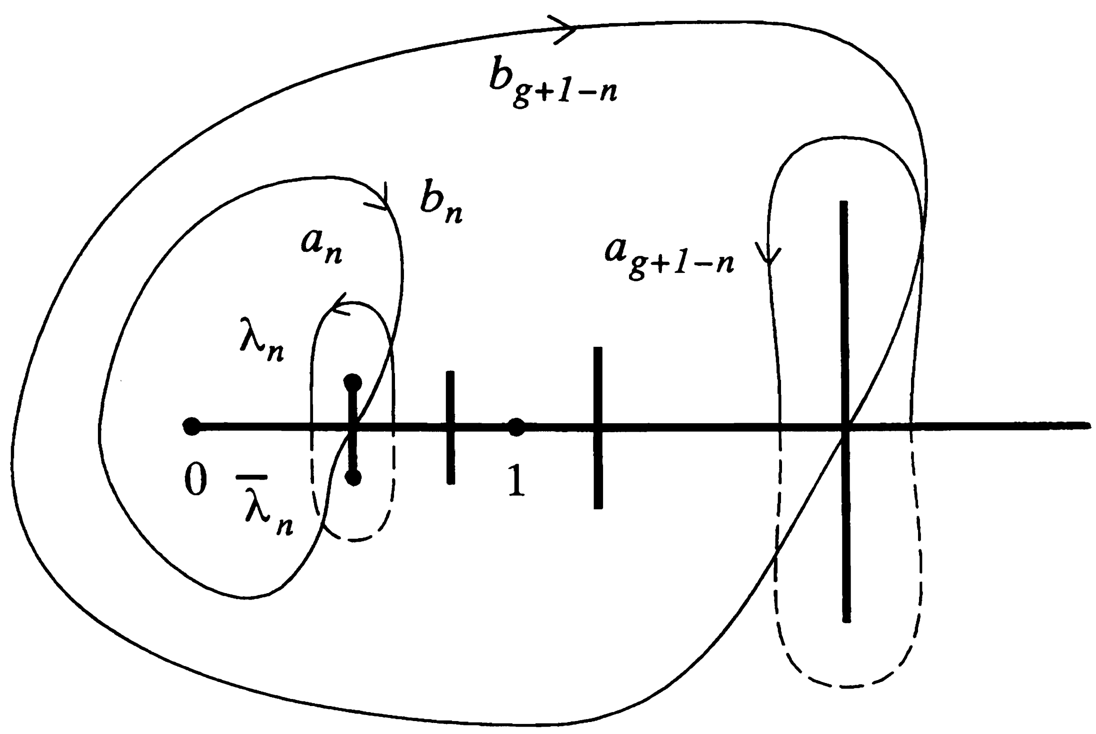

Fig. 1. The spectral curve with the canonical basis.

us consider the Abelian differentials $d\Omega_\infty, d\Omega_0$ with zero a-periods and such that $d\Omega_\infty$ has the only pole in ∞ and $d\Omega_0$ has the only pole in zero:

$$\begin{aligned} d\Omega_\infty(P) &= d(\sqrt{\lambda}), \qquad P \to \infty, \\ d\Omega_0(P) &= d\left(\frac{1}{\sqrt{\lambda}}\right), \qquad P \to 0. \end{aligned} \tag{7}$$

We denote the b-periods of $d\Omega_\infty, d\Omega_0$ as B^∞, B^0,

$$B_n^{\infty,0} = \int_{b_n} d\Omega_{\infty,0},$$

and define the vectors

$$V = \tfrac{1}{4}(B^\infty - B^0), \qquad W = \tfrac{1}{4}(B^\infty + B^0).$$

The antiholomorphic involution τ_2 acts on the basis of the cycles and on the local parameters as follows:

$$\tau_2 a_\kappa = a_\kappa, \qquad \tau_2 b_\kappa = -b_\kappa + a_\kappa, \qquad \tau_2^* \sqrt{\lambda} = -\overline{\sqrt{\lambda}}.$$

These relations imply

$$\tau_2^*\, d\Omega^\infty = -\overline{d\Omega^\infty}, \qquad \tau_2^*\, d\Omega^0 = -\overline{d\Omega^0}$$

and prove the real-valuedness of the g-vectors V, W.

The finite-gap (theta functional) solutions of (SG) are given by the formula (4), where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g)$, $V = V(\boldsymbol{\lambda})$, $W = W(\boldsymbol{\lambda})$; $i\Delta = i(\pi, \dots, \pi)$ is the vector of the half-periods and $D \in \mathbb{T}^g = \mathbb{R}^g/2\pi\mathbb{Z}^g$ is the phase of the solution.

The construction just described corresponds to each vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g) \in \mathbb{M}^g$, where

$$\mathbb{M}^g = \{(\lambda_1, \dots, \lambda_g) \mid \lambda_j \in \mathbb{C}_+,\ \lambda_j \neq \lambda_k\ \forall j \neq k\}, \tag{8}$$

the toroidal family of the finite-gap solutions (1.4), where the phase D varies in the g-torus. [In fact, the construction corresponds to each *set* $\{\lambda_1, \dots, \lambda_g\}$. With some abuse of notation, we do not distinguish a vector $(\lambda_1, \dots, \lambda_g)$ from the set $\{\lambda_1, \dots, \lambda_g\}$.]

We remember that a solution $U = (u, v)$ of (SG) satisfies the Neumann boundary conditions (N) if it satisfies "even periodic" boundary conditions with the doubled period:

$$U(t,x) \equiv U(t, x + 2\pi), \qquad U(t,x) \equiv U(t, -x). \tag{EP}$$

Similarly, $U(t,x)$ satisfies the Dirichlet boundary conditions (D) if it satisfies the "odd periodic" boundary conditions

$$U(t,x) \equiv U(t, x + 2\pi), \qquad U(t,x) \equiv -U(t, -x). \tag{OP}$$

Lemma 1. *If the set* $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_g\}$ *is symmetric with respect to the inversion*

$$\lambda \to \lambda^{-1}$$

and the divisor D *satisfies*

$$D_j = D_{g+1-j}, \qquad j = 1, \dots, g, \tag{9}$$

then the solution (4) *is even. If the set* $\boldsymbol{\lambda}$ *is as above and the divisor* D *satisfies*

$$D_j = D_{g+1-j} + \pi,$$

then the solution (4) *is odd.*

Proof. Since the set $\boldsymbol{\lambda}$ is invariant with respect to the inversion, the spectral curve X [see (6)] is invariant with respect to the holomorphic involution τ_3, where

$$\tau_3(\lambda, \mu) = \left(\frac{1}{\lambda}, \frac{\mu}{\lambda^{g+1}}\right). \tag{10}$$

The local parameters and the basis of cycles are transformed as follows:

$$\tau_3^* \sqrt{\lambda} = \frac{1}{\sqrt{\lambda}}, \qquad \tau_3 a_n = a_{g+1-n}, \qquad \tau_3 b_n = b_{g+1-n},$$

which implies

$$\tau_3^* \, d\Omega_\infty = d\Omega_0,$$

so for the periods of $d\Omega_0$, $d\Omega_\infty$, and $d\omega$, we have

$$T_3 B^\infty = B^0, \quad T_3 B T_3 = B,$$

where T_3 is the $g \times g$ matrix

$$T_3 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

The vectors V and W are antisymmetric and symmetric, respectively,

$$T_3 V = -V, \qquad T_3 W = W,$$

and the theta-function is symmetric,

$$\theta(T_3 z) = \theta(z) = \theta(-z),$$

so

$$u(t, -x) = 2i \log \frac{\theta(T_3 i(Vx + Wt + D + \Delta) + i(D - T_3 D))}{\theta(T_3 i(Vx + Wt + D) + i(D - T_3 D))}.$$

This function is equal to $u(t, x)$ if $T_3 D = D$ and to $-u(t, x)$ if $T_3 D = D + \Delta$ (because the transformation $D \mapsto D + \Delta$ interchanges the numerator and the denominator of the logarithm's argument). □

Remark. One can show that the symmetries of X, D described above are necessary for the solution $u(x,t)$ to be even or odd. This fact can be proved using the "direct approach" [BBEIM] of constructing the finite-gap solutions.

Due to the complete analogy between the (OP) and (EP) cases in what follows, we prove the results for the (N) boundary conditions only and briefly reformulate them for the (D) case.

For a vector $\boldsymbol{\lambda} \in \mathbb{M}^g$, we denote by $\boldsymbol{\lambda}^{-1} \in \mathbb{M}^g$ the g-vector with the inverse components

$$(\boldsymbol{\lambda}^{-1})_j = (\lambda_j)^{-1}, \qquad j = 1, \dots, g.$$

The vector $\boldsymbol{\lambda} \in \mathbb{M}^g$ gives rise to an (EP) solution, if (possibly, after some renumeration of its components)

$$|\lambda_1| \le 1, \dots, \quad |\lambda_n| \le 1 \quad \text{and} \quad \boldsymbol{\lambda}^{-1} = \boldsymbol{\lambda}, \tag{11}$$

$$(V_1, \dots, V_g)\,(\boldsymbol{\lambda}) \in \mathbb{Z}^g, \tag{12}$$

where

$$n = n(g) = 1 + \left[\frac{g-1}{2}\right].$$

Let $\mathbb{M}^g_{\mathrm{sym}}$ be the set of all $\boldsymbol{\lambda} \in \mathbb{M}^g$ satisfying (11). Since all λ_j's are different,

$$|\lambda_n| = 1 \qquad \forall \boldsymbol{\lambda} \in \mathbb{M}^g \text{ if } g \text{ is odd.}$$

As we have already proved, for $\lambda \in \mathbb{M}^g_{\mathrm{sym}}$ the vectors V, W inherit the symmetries of the surface X. We formulate this fact as the next lemma.

Lemma 2. *If $\boldsymbol{\lambda} \in \mathbb{M}^g_{\mathrm{sym}}$, then the vector W is symmetric and V is antisymmetric with respect to the involution T_3 (i.e., $T_3 W = W$, $T_3 V = -V$). These vectors are given by the formulas*

$$V_k = \tfrac{1}{4}(B_k^\infty - B_{g+1-k}^\infty), \qquad W_k = \tfrac{1}{4}(B_k^\infty + B_{g+1-k}^\infty). \tag{13}$$

By Lemma 1, to extract from the set of even (odd) solutions (4) the solutions of (SG)+(OP) [(SG)+(EP)], we should solve the equation (12) with $\boldsymbol{\lambda} \in \mathbb{M}^g_{\mathrm{sym}}$. We start an analysis of this equation with simple small-gap limits for V and W vectors when $\boldsymbol{\lambda} \in \mathbb{M}^g_{\mathrm{sym}}$ tends to a real vector $\mathbf{l}$ with positive components:

$$V(\boldsymbol{\lambda}) \to V^0(\mathbf{l}), \qquad W(\boldsymbol{\lambda}) \to W^0(\mathbf{l}) \quad \text{as } \boldsymbol{\lambda} \to \mathbf{l} \in \mathbb{R}^g_+,$$

where

$$\begin{aligned} V_j^0(\mathbf{l}) &= V_j^0(l_j) = \frac{1}{2}\left(\sqrt{l_j} - \frac{1}{\sqrt{l_j}}\right), \\ W_j^0(\mathbf{l}) &= W_j^0(l_j) = \frac{1}{2}\left(\sqrt{l_j} + \frac{1}{\sqrt{l_j}}\right) \end{aligned} \tag{14}$$

(see [McK], [EFM], and Theorem 3 below). From (11) we obtain the estimates for components of the limiting vector $\mathbf{l}$:

$$0 < l_1, \dots, l_n \le 1 < l_{n+1}, \dots, l_g.$$

We suppose that all components of the vector $\mathbf{l}$ are different. Then, after unessential reordering of the first and the last n of them, we have

$$0 < l_n < \cdots < l_1 \leq 1 < l_g \cdots < l_{n+1}, \qquad l_j \cdot l_{g+1-j} = 1 \qquad \forall j. \tag{15}$$

After this reordering, the components of the vector V^0 are increasing:

$$V_n^0 < \cdots < V_1^0 \leq 0 < V_g^0 < \cdots < V_{n+1}^0, \qquad V_j^0 = -V_{g+1-j}^0.$$

As $V^0 \in \mathbb{Z}^g$, then we obtain from (14) that

$$\begin{aligned} \{l_1, \ldots, l_n\} \in \Big\{ & n\text{-vectors with components of the form} \\ & \left(\sqrt{j^2+1} - j\right)^2 \Big| \, j = 0, 1, 2, \ldots \Big\}, \\ \{l_{n+1}, \ldots, l_g\} \in \Big\{ & (g-n)\text{-vectors with components of the form} \\ & \left(\sqrt{j^2+1} - j\right)^2 \Big| \, j = -1, -2, \ldots \Big\}. \end{aligned} \tag{16}$$

We shall prove below in Theorem 3 that each vector $\mathbf{l} \in \mathbb{R}_+^g$ satisfying (15) and (16) gives rise to a family of small-amplitude solutions of (SG)+(EP). As a suitable parameter for these families of solutions, we choose the integer n-vector $\mathbf{V} = -(V_1^0, V_2^0, \ldots, V_n^0)$, varying in the set $\mathcal{L}^g$, where

$$\mathcal{L}^g = \{\mathbf{V} = (V_1, \ldots, V_n) \in \mathbb{Z}^{n(g)} \mid V_n > \cdots > V_1 \geq 0,\ V_1 = 0 \text{ iff } g \text{ is odd}\}.$$

For $\mathbf{V} \in \mathcal{L}^g$ fixed, we denote

$$\mathbb{N}_n = \mathbb{N}_n(\mathbf{V}) = (\mathbb{N} \cup \{0\}) \setminus \{-V_1^0, \ldots, -V_n^0\}.$$

We treat $\mathbf{V} = \{-V_1^0, \ldots, -V_n^0\}$ and $\mathbb{N}_n$ as the lists of open and closed gaps of the solution (4).

By (14) and (16), components W_j^0 of the limiting vector W^0 have the form

$$W_j^0 = (V_j^0)^* = \sqrt{(V_j^0)^2 + 1}, \qquad 1 \leq j \leq g.$$

Small-amplitudes solutions we are discussing now correspond to the situation when all the cuts in Figure 1 are small. Our analysis of these solutions is based on the uniformization of Riemann surfaces with small cuts $[\overline{\lambda_j}, \lambda_j]$, given by the Schottky parameters (see Section 2). Below in Theorem 3 we give the final results of this analysis. For the proofs, we refer the reader to Section 2.

Theorem 3. *For every* $\mathbf{V} \in \mathcal{L}^g$ *there exists* $\rho > 0$ *and real-analytic map*

$$\boldsymbol{\lambda}\colon\ M_\rho^C = \{\mu \in \mathbb{C}^n \mid |\mu_j| < \rho\ \forall j\} \to \mathbb{C}^g, \qquad \mu \mapsto \boldsymbol{\lambda}(\mu),$$

such that:

(a) *For* $\mu \in M_\rho^+ = M_\rho^C \cap \mathbb{R}_+^n$, *the vector* $\boldsymbol{\lambda}(\mu)$ *lies in* $\mathbb{M}_{\text{sym}}^g \subset \mathbb{C}_+^g$ *and the Riemann surface* (6) *with* $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mu)$ *satisfies* (11) *and* (12).

(b) *The maps*

$$\mu \mapsto U(t, x; \boldsymbol{\lambda}(\mu), D), \qquad \mu \mapsto W(\boldsymbol{\lambda}(\mu)),$$

are analytic in M_ρ^C *and* $U(t, x; \boldsymbol{\lambda}(0), D) \equiv 0$, $W_j(0) = W_j^0$.

(c) *The vector* $V(\boldsymbol{\lambda}(\mu))$ *equals* V^0 *for all* μ.

(d) *The matrix* $\partial W/\partial\mu$ *at the point* $\mu = 0$ *equals*

$$\partial W_j/\partial\mu_k|_{\mu=0} = \begin{cases} -16/W_j^0, & j \neq k, \\ -12/W_j^0, & j = k. \end{cases}$$

(e) *For* $\mu = (0, \dots, \mu_j, \dots 0)$, *where* $\mu_j \geq 0$,

$$U(0, x; \mu, D) = 16\sqrt{\mu_j}\,(\cos V_j^0 x \cos D_j,\ -W_j^0 \cos V_j^0 x \sin D_j) + O(\mu). \tag{17}$$

Corollary 4. *The map*

$$M_\rho^C \to \mathbb{C}^n, \qquad \mu \mapsto (W_1, \dots, W_n)(\mu),$$

is an analytic diffeomorphism on its image, provided ρ *is small enough.*

Proof. We should check that $\det \partial W_j/\partial\mu_k \neq 0$ at $\mu = 0$. This determinant differs by a nonzero factor from the determinant of the matrix $m = (m_{jk})$, where $m_{jj} = 3$ and $m_{jk} = 4$ if $j \neq k$. The matrix m clearly defines an invertible linear map, so $\det m \neq 0$. □

Thus, g-gap solutions $U(t, x; \mu, D)$ of (SG)+(N) analytically depend on μ and D and are parametrized by the discrete parameter $\mathbf{V} \in \mathcal{L}^g$.

Due to the symmetry relations, the vectors V, W, and D are uniquely defined by their first n components (belonging to $\mathbb{R}^n$ and $\mathbb{T}^n$). With some abuse of notation, we denote these n-vectors by the same symbols V, W, and D.

The coordinate system (μ, D) is singular in the points where some μ_j vanish, because for $\mu = 0$ the zone $[\lambda_j, \overline{\lambda_j}]$ shrinks to a point and the solution U does not depend on the phase D_j. This observation and asymptotics (11) hint that the functions $\{(\sqrt{2\mu_j}, D_j) \mid j = 1, \dots, n\}$ form a "good" polar coordinate system and the solution u analytically depends on the corresponding Cartesian coordinates (p, q),

$$p_j = \sqrt{2\mu_j}\cos D_j, \quad q_j = \sqrt{2\mu_j}\sin D_j. \tag{18}$$

Direct calculations, given in Section 3, Lemma 16, prove this conjecture:

Lemma 5. *The map*

$$\Phi_0\colon D_\rho^{2n} := \{(p, q) \mid p_j^2 + q_j^2 < 2\rho\ \forall j\} \to H_s, \qquad \Phi_0(p, q)(x) = U(0, x; p, q),$$

is real-analytic for every $s \in \mathbb{N}$, *and*

$$\frac{\partial}{\partial p_j}\Phi_0(0) = 8\sqrt{2}\,(\cos V_j x, 0), \qquad \frac{\partial}{\partial q_j}\Phi_0(0) = 8\sqrt{2}\,(0, -W_j^0 \cos V_j x). \tag{19}$$

Moreover, the map Φ_0 *is odd*: $\Phi_0(p, q)(x) \equiv -\Phi_0(-p, -q)(x)$.

In the lemma we denote by H_s the Sobolev space of vector-valued even periodic functions $U(x) = (u(x), v(x))$; that is,

$$H_s = \left\{ U(x) \mid U(x) \equiv U(-x) \equiv U(x+2\pi), \quad \int_0^{2\pi} |\partial_x^l U(x)|^2\, dx < \infty\ \forall l \le s \right\}.$$

The formula (19) results from (17). The last statement of the lemma follows directly from the formula (4), since the transformation $D \mapsto D + \Delta$ interchanges the numerator and the denominator of the logarithm's argument in (4).

The following statement (with ρ small enough) is an immediate consequence of the lemma.

Corollary 6. *The set $\mathcal{T}_\rho = \Phi_0(D_\rho^{2n})$ is a $2n$-dimensional analytic submanifold of H_s. This manifold passes through zero, $0 \in H_s$, with the tangent space*

$$T_0\mathcal{T}_\rho = E^{2n} := \operatorname{span}\{(\cos V_j^0 x, 0),\ (0, \cos V_j^0 x) \mid j = 1, \dots, n\}.$$

The manifold is invariant under the flow of (SG)+(N) *and is foliated by the invariant analytic tori of the form*

$$\Phi_0(T^n(\mu)), \qquad T^n(\mu) = \{p_j^2 + q_j^2 = 2\mu_j \ge 0 \mid j = 1, \dots, n\}.$$

The dimension of the torus $T^n(\mu)$ equals n in the general case and drops by 1 if some μ_j vanishes.

Thus, equations (11) and (12) define an n-dimensional analytic subvariety of the g-dimensional domain $\mathbb{M}^g_{\mathrm{sym}}$. Due to Theorem 3, this subvariety has nonempty components $\mathbb{M}^g_{\mathbf{V}}$, parametrized by the vectors $\mathbf{V}$ from $\mathcal{L}^g$. The g-gap solutions of (SG)+(N), corresponding to vectors from $\mathbb{M}^g_{\mathbf{V}}$, form in H_s a $2n$-dimensional variety $\mathcal{T} = \mathcal{T}^{2n}(\mathbf{V})$, diffeomorphic to $\mathbb{M}^g_{\mathbf{V}} \times \mathbb{T}^n$. The intersection of $\mathcal{T}^{2n}(\mathbf{V})$ with small enough neighborhood of zero in the phase space forms a smooth analytic manifold; its closure is a $2n$-dimensional smooth analytic manifold $\mathcal{T}_\rho = \mathcal{T}_\rho(\mathbf{V})$, diffeomorphic to the $2n$-dimensional polydisc D_ρ^{2n}.

Due to Corollary 6, manifold $\mathcal{T}_\rho$ is stratified as follows:

$$\mathcal{T}_\rho = \mathcal{T}_\rho^0 \cup \Big(\bigcup_{g' < g} \mathcal{T}_{\rho, g'} \Big),$$

where $\mathcal{T}_\rho^0 = \mathcal{T}^{2n}(\mathbf{V}) \cap \mathcal{T}_\rho$ is an open part of $\mathcal{T}_\rho$, filled with g-gap solutions, and analytic submanifolds $\mathcal{T}_{\rho,g'}$ are filled with ($g' < g$)-gap solutions of (SG)+(N).

We consider equation (SG) linearized about the g-gap solution $U = (u, v)$,

$$\delta\ddot{u} = \delta u_{xx} - (\cos u(t,x))\,\delta u, \qquad \text{(LSG)}$$

supplemented by (N) [or (D)] boundary conditions.

There is a natural way to construct solutions $\delta u(t,x)$ of (LSG): to write $u(t,x;\mu,D) \equiv u(t,x;\boldsymbol{\lambda}(\mu),D)$ as a degenerate $(g+2)$-zone solution

$$u(t,x;\mu,D) = u^{n+1}(t,x;\mu,\mu_{n+1};D,D_{n+1})|_{\mu_{n+1}=0},$$

where u^{n+1} is a $(g+2)$-gap solution of (SG)+(N), corresponding to a vector $\mathbf{V}^{n+1} = (\mathbf{V}, V^0_{n+1}) \in \mathcal{L}^{g+2}$ ($\mathbf{V} \in \mathcal{L}^g$ corresponds to the solution u and $V^0_{n+1} \in \mathbb{N}_n$). After this we obtain a solution of (LSG) as

$$\lim_{\mu_{n+1}\to 0} \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial u^{n+1}}{\partial D_{n+1}}, \tag{20}$$

where the factor $\mu_{n+1}^{-1/2}$ appears in the formula because not (D_{n+1}, μ_{n+1}) but (p_{n+1}, q_{n+1}) [see (18)] forms a smooth coordinate system near $\mu_{n+1} = 0$. The solution (20) depends on the choice of the phase D_{n+1}. Different solutions are parametrized by elements of the set $\mathbb{N}_n$, which enumerates the closed gaps of the solution U.

Theorem 7. *For each $j \in \mathbb{N}_n$ there exists a linear combination δu_j of the solutions* (2.2) *with $V^0_{n+1} = j$ of the form*

$$\delta u_j(D, t; \mu)(x) = e^{iw_j(\mu)t}\Psi^j(W(\mu)t + D, \mu)(x),$$

where w_j and Ψ^j are analytic functions. The frequency $w_j(\mu)$ equals the $(n+1)$th component of the W-vector of the solution u^{n+1} with $\mu_{n+1} = 0$. It can be analytically extended to some complex polydisc $M^c_\rho = \{|\mu_j| < \rho\}$, where

$$|w_j(\mu) - j^*| \le C \min(|\mu|, (1+j)^{-1}).$$

The function Ψ^j is even in (p, q). It can be analytically extended to some domain

$$\mathcal{O}_\rho = \{(p, q) \in D^c_\rho\} \times \{x \in \mathbb{C} \mid |\operatorname{Im} x| < \rho\},$$

where

$$D^c_\rho = \{(p, q) \in \mathbb{C}^{2n} \mid |p_j|^2 + |q_j|^2 < 2\rho \quad \forall j\}.$$

The function Ψ^j can be represented as

$$\Psi^j = \cos jx + \Psi^{j0}(Wt + D, \mu)(x),$$

where

$$\Psi^{j0}(D, \mu) = \tfrac{1}{2}(e^{ijx}\Psi^{j1}(D, \mu)(x) + e^{-ijx}\Psi^{j1}(D, \mu)(-x)).$$

The function Ψ^{j1} is analytic in x and (p, q)-variables and everywhere in $\mathcal{O}_\rho$,

$$|\Psi^{j1}| \le C|\mu|(1+j)^{-1}. \tag{21}$$

Proof. See Lemma 18 in Section 2. To obtain the estimate (21), one should use the direct and inverse estimates for the norm of an analytic function in a complex strip via its Fourier coefficients (see [A] and [K1], Appendix B to Part 3). □

2. Proofs: Schottky Uniformization of the Spectral Curve and Parameters of the Finite-Gap Solutions

Here we describe a different parametrization of the spectral curve (we restrict ourselves to the case of even genus $g = 2n$), which allows us to investigate the behavior of the solutions in the small-amplitude regime. This parametrization is provided by the Schottky uniformization of the spectral curve. Details of the theory can be found in [Ba, Bu, Sch] (the Schottky uniformization) and in [Bo1, BBEIM, Chapter 5] (application to investigation of the finite-gap solutions).

An arbitrary hyperelliptic curve shown in Figure 1 with the symmetry τ_3 can be uniformized in the following way in the z-plane. Let $C_1, \ldots, C_n$ be disjoint circles (in this section we use the notation z for the uniformizing variable) orthogonal to the real axis and intersecting it only between the points $z = 0$ and $z = 1$. The symmetry

$$\tau_3 z = \frac{1}{z} \tag{22}$$

maps C_k onto a circle C_{g+1-k}, and the symmetry

$$\tau_1 z = -z \tag{23}$$

maps C_k and C_{g+1-k} onto circles $C'_k = \tau_1 C_k$ and $C'_{g+1-k} = \tau_1 C_{g+1-k}$. These circles comprise the boundary of a $2g$-connected domain F (see Fig. 2).

The Moebius transformation σ_k,

$$\frac{\sigma_k z + A_k}{\sigma_k z - A_k} = -\mu_k \frac{z + A_k}{z - A_k}, \qquad 0 < \mu_k < 1,\ 0 < A_k < 1,\ k = 1, \ldots, n,$$

maps the outside of the boundary circle C_k into inside of the boundary circle C'_k, $\sigma_k C_k = C'_k$, and preserves the real axis. The points A_k and $-A_k$ are the fixed points of the transformation σ_k; A_k is inside C_k. The numbers A_k, μ_k are uniquely determined by the circle C_k. (Note that to identify formulas of this section with analogous formulas of [Bo1, BBEIM], one should replace $\mu \to -\mu$.) The radius of C_k is equal to

$$\frac{2A_k}{1/\sqrt{\mu_k} + \sqrt{\mu_k}}$$

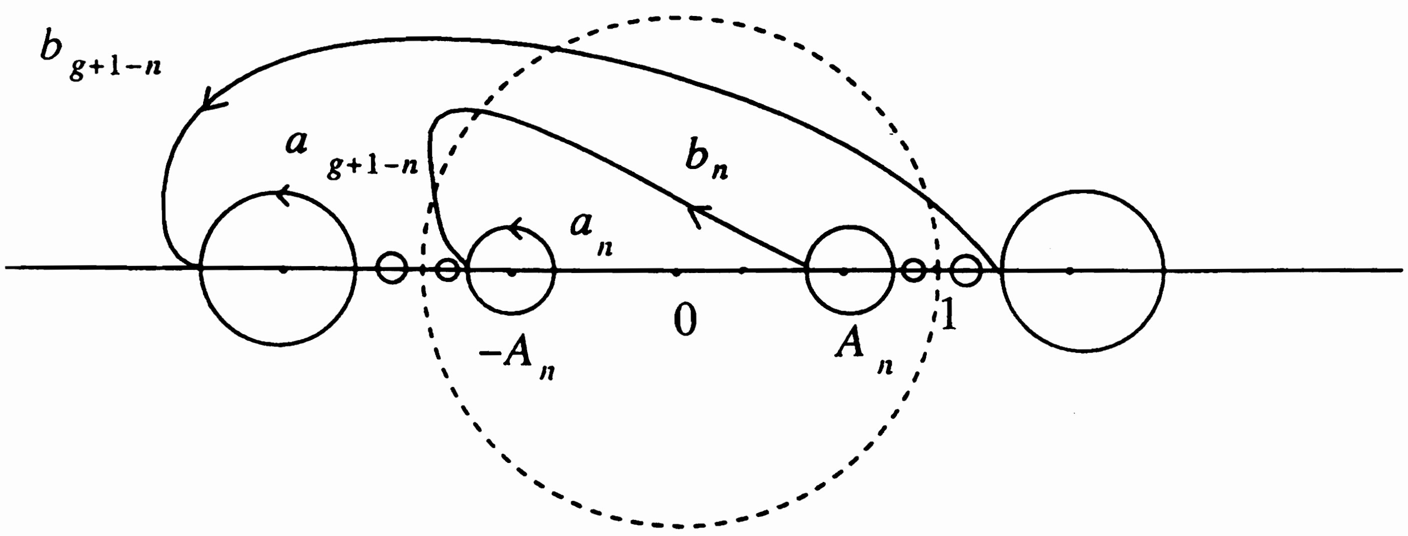

Fig. 2. Schottky uniformization of the spectral curve.

and the center of C_k is given by

$$A_k \frac{1-\sqrt{\mu_k}}{1+\sqrt{\mu_k}}.$$

The transformations σ_{g+1-k} are defined by the rule

$$\sigma_{g+1-k}(z) = \frac{1}{\sigma_k(1/z)}.$$

They map the outside of C_{g+1-k} onto the inside of C'_{g+1-k} and are given by the formula

$$\frac{\sigma_{g+1-k}z + A_{g+1-k}}{\sigma_{g+1-k}z - A_{g+1-k}} = -\mu_{g+1-k}\frac{z + A_{g+1-k}}{z - A_{g+1-k}},$$

$$A_{g+1-k} = A_k^{-1}, \ \mu_{g+1-k} = \mu_k.$$

The elements $\sigma_1, \ldots, \sigma_g$ generate a classical Schottky group [Sch, Bu], which in this case preserves the real axis. F is the fundamental domain of G. Identifying the boundary points of F by the holomorphic transformations $\sigma_1, \ldots, \sigma_g$, we get a real hyperelliptic Riemann surface of genus g, possessing an additional holomorphic involution τ_3. [In a more invariant way X can be defined as $X = \Omega(G)/G$, where $\Omega(G)$ is the discontinuity set of G.] As a matter of fact, τ_1 is a hyperelliptic involution of X with $2g+2$ fixed points, $z=0$ and $z=\infty$, and the pairs of mutually conjugate points given by $\sigma_m z = -\tau_1 z$ or, equivalently,

$$(z - A_m)^2 = -\mu_m(z + A_m)^2 \tag{24}$$

Also we see that

$$\tau_2 z = -\bar{z}$$

is an antiholomorphic involution of X. Fixed points of τ_2 comprise a real oval $i\mathbb{R}$. Another antiholomorphic involution $(\tau_1\tau_2)z = \bar{z}$ has a fixed oval $F \cap \mathbb{R}$ (all the intervals of $F \cap \mathbb{R}$ are glued in one oval of X by the transformations $\sigma_1, \ldots, \sigma_g$). The fundamental domain F is invariant with respect to τ_3, which means that X possesses a holomorphic involution (22) with two fixed points $z = \pm 1$.

Let us choose a canonical basis of cycles of X such that a_m coincides with the positively oriented C'_m, and b_m joins the points $z_m \in C_m$ and $\sigma_m z_m \in C'_m$ (see Fig. 2). This is the same basis that we used before (see Fig. 1) in Section 1.

Abelian differentials on X are well determined by the (-2)-dimensional Poincaré theta series. For the Schottky group with invariant circle (a real axis in our case), these series converge [Bu, Fo]. We denote by G_m the subgroup of G generated by σ_m. Cosets G/G_{m_1} and $G_{m_2}\backslash G/G_{m_1}$ are sets of all elements $\sigma = \sigma_{i1}^{j1} \cdots \sigma_{ik}^{jk}$ such that $i_k \neq m_1$ and, for $G_{m_2}\backslash G/G_{m_1}$, additionally $i_1 \neq m_2$. The following result is classic [Bu, Ba]:

Theorem 8. *The series*

$$d\omega_m = \sum_{\sigma \in G/G_m} \left(\frac{1}{z - \sigma(-A_m)} - \frac{1}{z - \sigma A_m} \right) dz$$

define holomorphic differentials normalized in the basis fixed above. The period matrix is given by

$$
\begin{aligned}
B_{m_1m_2} &= \sum_{\sigma\in G_{m_2}\backslash G/G_{m_1}} \log\{-A_{m_2}, A_{m_2}, \sigma(-A_{m_1}), \sigma(A_{m_1})\}, \qquad m_1 \neq m_2, \\
B_{mm} &= \log(-\mu_m) + \sum_{\substack{\sigma\in G_m\backslash G/G_m \\ \sigma\neq I}} \log\{-A_m, A_m, \sigma(-A_m), \sigma A_m\},
\end{aligned} \tag{25}
$$

where { } *denotes the cross-ratio*

$$\{z_1, z_2, z_3, z_4\} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

G is a discrete subgroup of PLS(2, $\mathbb{C}$). We denote the coefficients of $\sigma \in G$ as follows:

$$\sigma z = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

For the generators σ_m they are equal to

$$
\begin{aligned}
\sigma_m \Longleftrightarrow &\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\left(\sqrt{-\mu_m} + \frac{1}{\sqrt{-\mu_m}}\right) & -\frac{A_m}{2}\left(\frac{1}{\sqrt{-\mu_m}} - \sqrt{-\mu_m}\right) \\ \frac{1}{2A_m}\left(\sqrt{-\mu_m} - \frac{1}{\sqrt{-\mu_m}}\right) & \frac{1}{2}\left(\sqrt{-\mu_m} + \frac{1}{\sqrt{-\mu_m}}\right) \end{pmatrix}.
\end{aligned} \tag{26}
$$

The hyperelliptic involution (23) implies a group involution

$$\sigma \to \sigma^* = \tau_1\sigma\tau_1, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}. \tag{27}$$

Finally, to identify two descriptions of the spectral curve, we construct the function $\lambda(z)$, which has a double zero at $z = 0$ and a double pole at $z = \infty$. Hence it is given by

$$\lambda(z) = \frac{1}{c}\sum_{\sigma\in G}[(\sigma z)^2 - (\sigma 0)^2] \tag{28}$$

with some constant c. Due to (27), the following asymptotics hold:

$$
\begin{aligned}
&\lambda \to \frac{z^2}{c}, \qquad z \to \infty, \\
&\lambda \to \frac{Q}{c}z^2, \qquad Q = \sum_{\sigma\in G}\delta^{-4}(1 - 2\beta\gamma), \qquad z \to 0.
\end{aligned} \tag{29}
$$

To determine c let us identify the reductions (22) and (10):

$$\lambda\left(\frac{1}{z}\right)\lambda(z) = 1.$$

Using (29), we get

$$c = \sqrt{Q} > 0$$

since $\lambda(z) \to +\infty$, when $z \to \pm\infty$. The contour $\mathscr{L}$ (see Section 1) coincides with the boundary of F; therefore, the branch of $\sqrt{\lambda}$ is single-valued on the fundamental domain F. At the points $z = \infty$ and $z = 0$ it behaves as follows:

$$\begin{aligned} \sqrt{\lambda} &\to \frac{z}{q}, & z &\to \infty; \\ \sqrt{\lambda} &\to qz, & z &\to 0,\ q^2 = c. \end{aligned} \tag{30}$$

The asymptotics (7) and (30) show that the Abelian integral $\Omega^\infty = \int^z d\Omega^\infty$ is equal to

$$\Omega^\infty(z) = \frac{1}{q}\sum_{\sigma\in G}(\sigma z - \sigma 0).$$

Its periods can be calculated directly or by using the reciprocity law, expressing them in terms of the normalized holomorphic differentials

$$B_m^\infty = \frac{1}{q}\sum_{\sigma\in G/G_m}[\sigma(-A_m) - \sigma A_m]. \tag{31}$$

Below we consider the limit of small μs,

$$\mu_m = \varepsilon\nu_m, \qquad \varepsilon \to 0,\ m = 1, \dots, g, \tag{32}$$

with finite νs and calculate all the parameters of the finite-gap solutions as the series with respect to ε. This limit corresponds to small gaps $[\lambda_i, \overline{\lambda}_i]$, since, due to (24) and (28),

$$|\lambda_m - \overline{\lambda}_m| \approx 8A_m^2\sqrt{\mu_m}.$$

As already mentioned, since G preserves the real axis, all the series in this section are absolutely convergent. For small C_ms [which is the case (32)] these series can be rewritten as absolutely convergent series with respect to ε.

Lemma 9. *The series*

$$R_m = \sum_{\sigma\in G/G_m}(\sigma(A_m) - \sigma(-A_m)) \tag{33}$$

converges absolutely and in the limit (32) *has the following leading terms*:

$$\begin{aligned} R_m &= 2A_m + \varepsilon\sum_{l=1}^{g}\nu_l r_{ml} + O(\varepsilon^2), \\ r_{ml} &= \frac{16A_m A_l^2}{A_m^2 - A_l^2}, \qquad l \neq m,\ r_{mm} = 0. \end{aligned} \tag{34}$$

Proof. For the difference $\sigma z_1 - \sigma z_2$ one has

$$\sigma z_1 - \sigma z_2 = \frac{z_1 - z_2}{(\gamma z_1 + \delta)(\gamma z_2 + \delta)}.$$

If σ is one of the generators $\sigma_m^{\pm 1}$ and both z_1, z_2 are outside of C_m and C'_m, then using (26) we see that

$$\sigma_m^{\pm 1} z_1 - \sigma_m^{\pm 1} z_2 = (z_1 - z_2) O(\mu_m).$$

For the terms of (33) corresponding to $\sigma = \sigma_{i1}^{j1} \cdots \sigma_{ik}^{jk}$ this implies

$$\sigma A_m - \sigma(-A_m) = O\big(\varepsilon^{|j_1| + \cdots + |j_k|}\big).$$

The detailed estimates proving absolute convergence of the series are presented in [Ba] and [Bu]. To get the terms of the zero and first order we have to take into account only $\sigma = I$ and the generators $\sigma = \sigma_1, \sigma_1^{-1}, \ldots, \sigma_g, \sigma_g^{-1}$:

$$R_m = 2A_m + \sum_{l=1}^{g} (\sigma_l A_m - \sigma_l(-A_m) + \sigma_l^{-1} A_m - \sigma_l^{-1}(-A_m)) + O(\varepsilon^2).$$

Direct calculation of this sum gives (34). □

Lemma 10. *The following asymptotics for Q hold*:

$$Q = \sum_{\sigma \in G} \delta^{-4}(1 - 2\beta\gamma) = 1 + 32\varepsilon \sum_{k=1}^{n} \nu_k + O(\varepsilon^2). \tag{35}$$

Proof. It is easy to see that the leading terms are given by the generators. Because of (26), for $\sigma = \sigma_k, \sigma_k^{-1}, \sigma_{g+1-k} = \tau_3 \sigma_k \tau_3$, and $\sigma_{g+1-k}^{-1} = \tau_3 \sigma_k^{-1} \tau_3$, we have

$$\frac{1 - 2\beta\gamma}{\delta^4} \approx 8\mu_k,$$

which proves (35). □

Two possible choices of the square root (30) for q correspond to the simple transformation of the solution

$$u(t, x) \to u(-t, -x).$$

We choose q negative. Then (35) implies

$$q^{-1} = -1 + 8\varepsilon \sum_{k=1}^{n} \nu_k + O(\varepsilon^2). \tag{36}$$

Lemma 11. *The components of the vectors V, W are given by absolutely convergent series* ($V_m = -V_{g+1-m}$, $W_m = W_{g+1-m}$):

$$V_m = -\frac{1}{4q}(R_m - R_{g+1-m}), \qquad W_m = -\frac{1}{4q}(R_m + R_{g+1-m}). \tag{37}$$

The asymptotics at $\varepsilon \to 0$ are differentiable and start with the following terms:

$$\begin{aligned}
V_k &= \sinh a_k + \varepsilon \sum_{l=1}^{n} \nu_l v_{kl} + O(\varepsilon^2), \\
W_k &= \cosh a_k + \varepsilon \sum_{l=1}^{n} \nu_l w_{kl} + O(\varepsilon^2), \\
v_{kk} &= -8 \sinh a_k + \frac{2}{\sinh a_k}, \\
w_{kk} &= -8 \cosh a_k - \frac{2}{\cosh a_k}, \\
v_{kl} &= -8 \sinh a_k \left(1 + \frac{\cosh^2 a_l}{\sinh(a_k + a_l)\sinh(a_l - a_k)}\right), \\
w_{kl} &= -8 \cosh a_k \left(1 + \frac{\sinh^2 a_l}{\sinh(a_k + a_l)\sinh(a_l - a_k)}\right), \qquad k \neq l,
\end{aligned} \tag{38}$$

where $e^{a_k} = A_k$.

Proof. The expressions (37) follow from (31), (33), and (13). Combining in (34) the terms with the same νs, we have

$$\begin{aligned}
R_k - R_{g+1-k} = &\left[2A_k + \varepsilon \nu_k r_{k,g+1-k} + \varepsilon \sum_{\substack{l=1 \\ l \neq k}}^{n} \nu_k (r_{kl} + r_{k,g+1-l})\right] \\
&- \left[2A_k^{-1} + \varepsilon \nu_k r_{g+1-k,k} + \varepsilon \sum_{\substack{l=1 \\ l \neq k}}^{n} \nu_l (r_{g+1-k,l} + r_{g+1-k,g+1-l})\right] + O(\varepsilon^2).
\end{aligned}$$

After some simple calculations, we get

$$R_k - R_{g+1-k} = 4 \sinh a_k + \frac{8\varepsilon \nu_k}{\sinh a_k} - 32\varepsilon \sum_{\substack{l=1 \\ l \neq k}}^{n} \nu_k \frac{\sinh a_k \cosh^2 a_l}{\sinh(a_k + a_l)\sinh(a_l - a_k)} + O(\varepsilon^2).$$

Multiplying this expression by (36), we get the first line in (38). The proof of the asymptotics for W is absolutely the same. □

Lemma 12. *For any $V_0(l)$ and sufficiently small parameters $\mu = (\mu_1, \ldots, \mu_n)$ an analytic map*

$$\mu \to a(\mu) = (a_1(\mu), \ldots, a_n(\mu))$$

exists such that $V(\mu, a(\mu)) = V_0(l)$ is independent of μ. Furthermore, W is an analytic function of μ and

$$\begin{aligned}\frac{\partial}{\partial \mu_k} W_m(\mu, a(\mu))|_{\mu=0} &= -\frac{16}{\cosh a_m}, \qquad k \neq m,\\ \frac{\partial}{\partial \mu_m} W_m(\mu, a(\mu))|_{\mu=0} &= -\frac{12}{\cosh a_m}.\end{aligned} \tag{39}$$

Proof. The series (38) is invertible with small ε. The equality

$$V(\mu, a) = V^0(l)$$

due to the implicit function theorem determines $a = a(\mu)$ and

$$\left.\frac{\partial a_m}{\partial \mu_k}\right|_{\mu=0} = -\frac{1}{\cosh a_m} v_{mk},$$

which gives

$$\frac{\partial}{\partial \mu_k} W_m(\mu, a(\mu))|_{\mu=0} = w_{mk} + \sinh a_m \frac{\partial a_m}{\partial \mu_k} = w_{mk} - v_{mk} \tanh a_m$$

and finally (39). The analyticity of $W(\mu)$ follows from the analyticity of $W(\mu, a)$. □

From the general g-gap solution described by the formula (4) we arrive by a limiting procedure at the case of small-amplitude waves. Now we describe asymptotics with respect to ε and also prove facts about the solutions of the linearized equation formulated in Section 1. First we consider general small-amplitude waves of the sine-Gordon equation. We impose symmetries on the spectral data [reduction (22) and symmetry (9) of (D)] later on in final formulas.

So, from now on $C_1, \ldots, C_g$ are disjoint circles orthogonal to the real axis, $C_1', \ldots, C_g'$ are the circles obtained from them by the map $z \to -z$, and $A_1, \ldots, A_g$, $\mu_1, \ldots, \mu_g$ are the uniformization parameters.

In the limit (32) the circles C_n and C_n' collapse to the points A_n and $-A_n$, respectively. In this degenerate case, for all nonidentity mappings σ and for arbitrary $a, b \in F$, the equality $\sigma a = \sigma b$ holds. Therefore, in the series (25) and (33) only the terms corresponding to $\sigma = I$ do not vanish. More detailed calculations as in the proof of Lemma 9 show that the following asymptotics hold:

$$\exp\left(\frac{1}{2} B_{nn}\right) = \sqrt{-\mu_n} \prod_{\substack{\sigma \in G_n \backslash G / G_n \\ \sigma \neq I}} \left(\frac{A_n - \sigma A_n}{A_n - \sigma(-A_n)}\right) = i\sqrt{\varepsilon}\sqrt{v_n}(1 + O(\varepsilon)), \tag{40}$$

$$\begin{aligned}\exp(B_{nm}) &= \left(\frac{A_m - A_n}{A_m + A_n}\right)^2 \prod_{\substack{\sigma \in G_m \backslash G / G_n \\ \sigma \neq I}} \left(\frac{A_m - \sigma A_n}{A_m - \sigma(-A_n)}\right)^2\\ &= \left(\frac{A_m - A_n}{A_m + A_n}\right)^2 (1 + O(\varepsilon)), \qquad n \neq m.\end{aligned} \tag{41}$$

Lemma 13. *For any $A^0 = (A_1^0, \dots, A_g^0)$ with all components different, $A_i^0 \neq A_j^0$, $A_i^0 \neq -A_j^0$, and for*

$$z \in \mathfrak{D}_c \equiv \{z|\ |\operatorname{Re} z_k| < c,\ k = 1, \dots, g\},$$

there is $\varepsilon > 0$ such that, if all $|\mu_k| < \varepsilon$, $k = 1, \dots, g$, the theta function $\theta(z)$ is an analytic function of $A_1, \dots, A_g$, lying in a sufficiently small neighborhood of $A_1^0, \dots, A_g^0$ and of $\mu_1, \dots, \mu_g$. Its asymptotics for $\varepsilon \to 0$ start as follows:

$$\begin{aligned} \theta(z) &= 1 + \theta'(z) + O(\varepsilon), \\ \theta'(z) &= 2i \sum_{k=1}^{g} \sqrt{\mu_k} \cosh z_k, \end{aligned} \tag{42}$$

where $O(\varepsilon) < C\varepsilon$ uniformly for all $z \in \mathfrak{D}_c$.

Proof. When $\varepsilon \to 0$ the theta function can be represented as a rapidly convergent series with respect to ε. The formulas (40) and (41) show that

$$\exp\left(\tfrac{1}{2} B_{kk} m_k^2\right) \sim \varepsilon^{m_k^2/2},$$

whereas B_{kl}, $k \neq l$, remains finite. This shows that the theta function analytically depends on As and μs.

The zero order term ε^0 in the expansion of $\theta(z)$ is given by $m = (0, \dots 0) \in \mathbb{Z}^g$. To calculate the term of order $\sqrt{\varepsilon}$ we should take all $m \in \mathbb{Z}^g$ of the form $m = (0, \dots, 0, \pm 1, 0, \dots 0)$ with only one nonvanishing component equal to ± 1, which gives $\theta'(z)$ as above. Since the theta function is $2\pi i \mathbb{Z}^g$ periodic, z can be taken on the compact set

$$\mathfrak{D}^0 = \{z|\ 0 \le \operatorname{Im} z \le 2\pi,\ |\operatorname{Re} z_k| \le c,\ k = 1, \dots, g\},$$

which proves the uniformity of the estimates with respect to z. □

Lemma 14. *The finite-gap solution* (4) *of the sine–Gordon equation in the limit* (32) *in the first approximation represents a linear superposition of g noninteracting Fourier modes of small amplitudes*

$$u(t, x) = 8\sqrt{\varepsilon} \sum_{k=1}^{g} \sqrt{\nu_k} \cos(V_k x + W_k t + D_k) + O(\varepsilon). \tag{43}$$

Proof. This lemma follows immediately from the first terms of the series (42),

$$\begin{aligned} u(t, x) &= 2i \log \frac{\theta(z + \Delta)}{\theta(z)} \approx 2i \log \frac{1 + \theta'(z + \Delta)}{1 + \theta'(z)} \\ &\approx 2i(\theta'(z + \Delta) - \theta'(z)), \end{aligned}$$

where $z = i(Vx + Wt + D)$. □

Corollary 15. *The solution of the Neumann boundary problem in the small-amplitude case is of the form*

$$u(t,x) = 16\sqrt{\epsilon}\sum_{k=1}^{n}\sqrt{\nu_k}\,\cos(V_k x)\cos(W_k t + D_k) + O(\varepsilon).$$

This corollary follows directly from (43) if we take $g = 2n$, impose the symmetries

$$D_k = D_{g+1-k}, \qquad \nu_k = \nu_{g+1-k}, \qquad k = 1,\ldots,n, \tag{44}$$

explained in this section and in Section 1, and take into account

$$V_{g+1-k} = -V_k, \qquad W_{g+1-k} = W_k. \tag{45}$$

Lemma 16. *For small μ the theta function $\theta(z)$ is an analytic function of A_j, $\sqrt{\mu_j}\exp(\pm z_j)$, and μ_j, $j = 1,\ldots,g$.*

Proof. The formula (40) shows that

$$\exp\tfrac{1}{2}B_{kk} = i\sqrt{\mu_k}\exp\tfrac{1}{2}\tilde{B}_{kk},$$

where $\exp\frac{1}{2}\tilde{B}_{kk}$ is an analytic function of μ and A. For the theta function this implies

$$\begin{aligned}\theta(z) &= \sum_{m\in\mathbf{Z}^g}\exp\left\{\tfrac{1}{2}\sum_{k=1}^{g}B_{kk}m_k^2 + \sum_{k<l}B_{kl}m_k m_l + \sum_{k=1}^{g}m_k z_k\right\}\\ &= \sum_{m\in\mathbf{Z}^g}\prod_{k=1}^{g}\{(i\sqrt{\mu_k})^{m_k^2}e^{z_k m_k}\}\exp\left\{\sum_{k<l}B_{kl}m_k m_l + \tfrac{1}{2}\sum_{k}\tilde{B}_{kk}m_k^2\right\}.\end{aligned} \tag{46}$$

Rewriting the first term in the product above as

$$(i\sqrt{\mu_k})^{m_k^2}e^{z_k m_k} = (i\sqrt{\mu_k}e^{z_k})^{m_k}(-\mu_k)^{(m_k(m_k-1))/2},$$

we see that all the terms in (46) are analytic in A, μ, and $\sqrt{\mu}\exp(\pm z)$. □

In the symmetrical case (44) and (45), taking into account Lemma 12, we get the following lemma.

Lemma 17. *For small μs and for all t and x the solution $u(t,x,\mu_1,\ldots,\mu_n,D_1,\ldots,D_n)$ of the Neumann boundary problem for the sine–Gordon equation is an analytic function of*

$$\sqrt{\mu_k}e^{\pm iD_k}, \qquad k = 1,\ldots,n,$$

or, equivalently, of

$$p_k = \sqrt{2\mu_k}\cos D_k, \qquad q_k = \sqrt{2\mu_k}\sin D_k.$$

Let us pass now to solutions of the linearized SG equation (LSG). As it was explained in Section 1, to obtain them we should take a solution $u_{g+2}(t,x)$ of genus $g+2$ ($g=2n$), parametrized by

$$\mu_1, \dots, \mu_{n+1}, D_1, \dots, D_{n+1},$$

differentiate it by D_{n+1}, and put $\mu_{n+1}=0$. More exactly,

$$\delta u = \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial u_{g+2}(t,x)}{\partial D_{n+1}}\bigg|_{\mu_{n+1}=0}$$

is a solution of the LSG equation.

Let us denote by X_0 the limiting nonsingular Riemann surface of genus g, uniformized by the Schottky group G_0 and generated by $\sigma_1, \dots, \sigma_n$. To simplify the notations we slightly change the numeration of cycles of X. The numeration of cycles and generators for X_0 is the same as above: the cycles $a_k, b_k, a_{g+1-k}, b_{g+1-k}$ correspond to σ_k (or, equivalently, to μ_k); the cycles a_{g+1}, b_{g+1} and a_{g+2}, b_{g+2} correspond to μ_{n+1} (with A_{n+1} and A_{n+1}^{-1}, respectively).

Formulas (40) and the symmetry of the period matrix show that in the limit $\mu_{n+1} \to 0$ the period matrix behaves as follows:

$$B \longrightarrow \begin{pmatrix} B_0 & s & Ts \\ s^T & \log(i\sqrt{\mu_{n+1}}) + 2r & l \\ (Ts)^T & l & \log(i\sqrt{\mu_{n+1}}) + 2r \end{pmatrix}, \tag{47}$$

where B_0, s, r, and l are finite, and the $n \times n$ matrix T is

$$T = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

Here B_0 is the period of X_0, s is a g-dimensional vector with the components

$$s_k = B_{k,g+1} = B_{g+1-k,g+2} = \sum_{\sigma \in G_0} \log\left(\frac{A_{n+1}^{-1} - \sigma(A_k^{-1})}{A_{n+1}^{-1} - \sigma(-A_k^{-1})} \right)^2, \tag{48}$$

and

$$r = \sum_{\substack{\sigma \in G_0 \\ \sigma \neq I}} \log\left(\frac{A_{n+1} - \sigma A_{n+1}}{A_{n+1} - \sigma(-A_{n+1})} \right).$$

Since iD_{n+1} always enters as an additive term in z_{g+1} and z_{g+2}, we have

$$\begin{aligned} \theta_{(n+1)}(z|B) &\equiv \frac{1}{\sqrt{\mu_{n+1}}} \frac{\partial}{\partial D_{n+1}} \theta(z|B) \mid_{\mu_{n+1}=0} \\ &= \frac{i}{\sqrt{\mu_{n+1}}} \left(\frac{\partial}{\partial z_{g+1}} + \frac{\partial}{\partial z_{g+2}} \right) \theta(z|B) \mid_{\mu_{n+1}=0} \\ &= \frac{i}{\sqrt{\mu_{n+1}}} \sum_{m \in \mathbf{Z}^{g+2}} (m_{g+1} + m_{g+2}) \exp\left(\frac{1}{2} \langle Bm, m \rangle + \langle z, m \rangle \right) \mid_{\mu_{n+1}=0} . \end{aligned}$$

In the limit (47) only the terms with $m_{g+1} = \pm 1$ or $m_{g+2} = \pm 1$ in the series above persist. So $\theta_{(n+1)}(z|B)$ converges to

$$\begin{aligned}
&\left(\exp\left(\frac{1}{2}B_{g+1,g+1}\right)\right)\Big/\sqrt{\mu_{n+1}} \\
&\quad\times\Bigg(\sum_{n\in\mathbf{Z}^g}\exp\left(\tfrac{1}{2}\langle B_0 n, n\rangle + \langle s, n\rangle + \langle z_0, n\rangle\right)e^{z_{g+1}} \\
&\qquad+\sum_{n\in\mathbf{Z}^g}\exp\left(\tfrac{1}{2}\langle B_0 n, n\rangle - \langle s, n\rangle + \langle z_0, n\rangle\right)e^{-z_{g+1}} \\
&\qquad+\sum_{n\in\mathbf{Z}^g}\exp\left(\tfrac{1}{2}\langle B_0 n, n\rangle + \langle Ts, n\rangle + \langle z_0, n\rangle\right)e^{z_{g+2}} \\
&\qquad+\sum_{n\in\mathbf{Z}^g}\exp\left(\tfrac{1}{2}\langle B_0 n, n\rangle - \langle Ts, n\rangle + \langle z_0, n\rangle\right)e^{-z_{g+2}}\Bigg) \\
&\quad= ie^r\big(e^{z_{g+1}}\theta(z_0+s|B_0) + e^{-z_{g+1}}\theta(z_0-s|B_0) \\
&\qquad\quad + e^{z_{g+2}}\theta(z_0+Ts|B_0) + e^{z_{g+2}}\theta(z_0-Ts|B_0)\big),
\end{aligned}$$

where z_0 is a vector consisting of the first g components of

$$z = (z_0, z_{g+1}, z_{g+2}).$$

Applying this formula to δu and substituting

$$\begin{aligned}
z_{g+1} &= i(V_{n+1}x + W_{n+1}t + D_{n+1}), \\
z_{g+2} &= i(-V_{n+1}x + W_{n+1}t + D_{n+1}),
\end{aligned}$$

we get

$$\begin{aligned}
\delta u &= 4e^r(ve^{iD_{n+1}} + \bar{v}e^{-iD_{n+1}}), \\
v &= \frac{1}{2}e^{iW_{n+1}t}\left\{e^{iV_n x}\left(\frac{\theta(z_0+s+\Delta|B_0)}{\theta(z_0+\Delta|B_0)} + \frac{\theta(z_0+s|B_0)}{\theta(z_0|B_0)}\right)\right. \\
&\qquad\left. + e^{-iV_n x}\left(\frac{\theta(z_0+Ts+\Delta|B_0)}{\theta(z_0+\Delta|B_0)} + \frac{\theta(z_0+Ts|B_0)}{\theta(z_0|B_0)}\right)\right\},
\end{aligned} \tag{49}$$

where $z_0 = i(Vx + Wt + D)$ and $V, W \in \mathbb{R}^g$ are determined by x_0. Since D_{n+1} is arbitrary, v is also a solution of the LSG equation.

Lemma 18. *The solution v_j of the LSG equation with the Neumann boundary conditions corresponding to the jth opening gap ($V_{n+1} = j$) is given by the formula*

$$\begin{aligned}
v_j &= e^{iw_j(\mu)t}\big(e^{ijx}F(Vx + Wt + D, \mu) \\
&\qquad + e^{-ijx}F(-Vx + Wt + D, \mu)\big), \\
F(Vx + Wt + D, \mu) &= \frac{1}{2}\left(\frac{\theta(Vx + Wt + D + s + \Delta|B_0)}{\theta(Vx + Wt + D + \Delta|B_0)}\right. \\
&\qquad\left. + \frac{\theta(Vx + Wt + D + s|B_0)}{\theta(Vx + Wt + D|B_0)}\right).
\end{aligned} \tag{50}$$

For

$$x \in \mathfrak{D} = \{x \mid |\operatorname{Im} x| < c\},$$

there is $\varepsilon > 0$ *such that for all* $|\mu_k| < \varepsilon$, $k = 1, \ldots, n$:

(i) $F(Vx + Wt + D, \mu) = \tilde{F}(Vx + Wt, p_1, q_1, \ldots, p_n, q_n)$, *where* $\tilde{F}$ *is an analytic function of all* $p_k = \sqrt{2\mu_k} \cos D_k$, $q_k = \sqrt{2\mu_k} \sin D_k$, *invariant with respect to the transformation* $p_k, q_k \to -p_k, -q_k$ *for all* $k = 1, \ldots, n$, *simultaneously, and having the following asymptotics*:

$$F = 1 + O\left(\frac{\varepsilon}{j}\right), \qquad \varepsilon \to 0, \ j \to \infty, \tag{51}$$

where $O(\varepsilon/j) < C\varepsilon/j$ *uniformly for all* $x \in \mathfrak{D}$.

(ii) $w_j(\mu)$ *is an analytic function of* $\mu_1, \ldots, \mu_n$ *with the asymptotics*

$$w = j^* + O(\varepsilon), \qquad \varepsilon \to 0,$$

$$w_j = j^* + O(j^{-1}), \qquad j \to \infty,$$

where $j^* = \sqrt{j^2 + 1}$.

Proof. Since $T(Vx + Wt + D) = -Vx + Wt + D$ (see Section 1), the formula (49) is equivalent to (50). First we investigate the asymptotics of w_j. We have seen already in (38) that when $\varepsilon \to 0$,

$$V_{n+1} = \sinh a_{n+1} + O(\varepsilon), \qquad W_{n+1} = \cosh a_{n+1} + O(\varepsilon),$$

where $A_{n+1} = e^{a_{n+1}}$. Combined with the equality

$$V_{n+1} = -j,$$

this implies

$$w_j(\mu) = j^* + O(\varepsilon), \qquad j^* = \sqrt{j^2 + 1}.$$

The calculation of the asymptotics with $j \to \infty$ is also straightforward. Formulas (37) (we recall the special numeration for the $n + 1$th gap) imply

$$\frac{w_{n+1}}{v_{n+1}} \to \frac{w_j}{-j} = \frac{R_{g+1} + R_{g+2}}{R_{g+1} - R_{g+2}},$$

where

$$R_{g+1} = \sum_{\sigma \in G_0} \big(\sigma A_{n+1} - \sigma(-A_{n+1})\big),$$

$$R_{g+2} = \sum_{\sigma \in G_0} \big(\sigma(A_{n+1}^{-1}) - \sigma(-A_{n+1}^{-1})\big).$$

In the limit $A_{n+1} \to 0$ we have

$$R_{g+1} = \sum_{\sigma \in G_0} \frac{d}{dz}\big(\sigma z - \sigma(-z)\big)|_{z=0} A_{n+1} + O(A_{n+1}^2),$$

$$R_{g+2} = 2A_{n+1}^{-1} + O(1),$$

which yields

$$w_j = j\left(\frac{1 + R_{g+1}/R_{g+2}}{1 - R_{g+1}/R_{g+2}}\right) = j + O(j^{-1}), \qquad j \to \infty.$$

The asymptotics for F follow from the simple estimate

$$\begin{aligned}\left|\frac{\theta(z+\Delta+s)}{\theta(z+\Delta)} + \frac{\theta(z+s)}{\theta(z)} - 2\right| \\ \leq \sum_{k=1}^{g}\left|\frac{(\partial/\partial z_k)\theta(z+\Delta+\xi_k)}{\theta(z+\Delta)} + \frac{(\partial/\partial z_k)\theta(z+\xi_k)}{\theta(z)}\right||s_k|,\end{aligned} \tag{52}$$

with some ξ_ks in $[z+\Delta, z+\Delta+s]$. Indeed, formula (48) shows that

$$s_k = \sum_{G_0} \log\left(\frac{1 - A_{n+1}\sigma(A_k^{-1})}{1 - A_{n+1}\sigma(-A_k^{-1})}\right)^2 = O(A_{n+1})$$

for all i. Since $A_{n+1} \approx 1/2j$, we get for $|s_i|$ in (52),

$$|s_i| = O(j^{-1}). \tag{53}$$

On the other hand, formula (42) implies

$$\begin{aligned}&\frac{(\partial/\partial z_k)\theta(z+\Delta+\xi_k)}{\theta(z+\Delta)} + \frac{(\partial/\partial z_k)\theta(z+\xi_k)}{\theta(z)} \\ &= \frac{2i\sqrt{\mu_k}\sinh(z_k+\xi_k+i\pi) + O(\varepsilon)}{1 + 2i\sum_{l=1}^{g}\sqrt{\mu_l}\cosh(z_l+i\pi) + O(\varepsilon)} \\ &\quad + \frac{2i\sqrt{\mu_k}\sinh(z_k+\xi_k) + O(\varepsilon)}{1 + 2i\sum_{l=1}^{g}\sqrt{\mu_l}\cosh(z_l) + O(\varepsilon)} \\ &= O(\varepsilon).\end{aligned} \tag{54}$$

These estimates are uniform with respect to $x \in \mathfrak{D}$, since, as in the proof of Lemma 13, we can consider x varying in a compact set. Combining (52)–(54) we get (51).

Finally, the change $(p, q) \to (-p, -q)$ in $\tilde{F}$ is equivalent to the common shift $z \to z + \Delta$ of the arguments of all theta functions. This shift preserves F. □

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