Physics Letters 161 (1991) 274–276
North-Holland

PHYSICS LETTERS A

Finite-gap periodic solutions of the KdV equation are non-degenerate

A.I. Bobenko [1,2] and S.B. Kuksin [3]
Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, W-5300 Bonn 3, Germany

Received 28 August 1991; accepted for publication 29 October 1991
Communicated by A.P. Fordy

We complete the proofs of two statements concerning finite-gap solutions periodic in x of the KdV equation: (i) most of these solutions survive under Hamiltonian perturbations of the KdV equation, (ii) for most of the solutions of the perturbed equation, which were close to some finite-gap potential at $t=0$, the averaging theorem of Bogolyubov–Krylov type is valid.

Various approaches are known for studying weakly perturbed integrable equations of mathematical physics. In particular averaging procedures formulated in terms of the finite-gap solutions of the non-perturbed equation were suggested to solve some initial-value problems. As a rule, an estimate for the disparity is not calculated. The sufficient condition on initial data justifying an application of the averaging procedure was obtained in refs. [1,2], where the case of an integrable system with a discrete spectrum as a non-perturbed equation was considered (the KdV equation with periodic boundary condition is an example of such a system). This condition represents itself some non-degeneracy condition for the initial family of finite-gap solutions (see below for the statement).

We prove the non-degeneracy of all the families of the periodic finite-gap solutions of the KdV equation with zero mean value. Our proof is based on the parametrization of the finite-gap solutions via the Schottky uniformization [3–5].

Real N-gap solutions of the KdV equation

$$4u_t=6uu_x+u_{xxx} \tag{1}$$

are given by the Its–Matseev formula

$$u(t,x)=2\partial_x^2\log\theta(\mathrm{i}(Ux+Wt+D))+2c\,, \tag{2}$$

where θ is the theta function with the period matrix $(2\pi\mathrm{i}I, B)$ and $U, W, D\in\mathbb{R}^N$. The vectors W and D are called the frequency and phase vectors respectively. These solutions are parametrized by the hyperelliptic M-curves of genus N,

$$\mu^2=(\lambda-E_1)...(\lambda-E_{2N+1})\,. \tag{3}$$

Let us denote by $\mathscr{R}$ the variety of these curves. We consider eq. (1) in the space Z_0 of periodic functions with zero mean value:

$$u(t,x)=u(t,x+2\pi),\quad \int_0^{2\pi} u(t,x)\,\mathrm{d}x\equiv 0\,, \tag{4}$$

and denote by $\mathscr{R}_0\subset\mathscr{R}$ a subset, corresponding to the finite-gap solutions with zero mean value. Everywhere below we fix the vector U. By (4),

$$U\in\mathbb{Z}^N,\quad c\equiv 0\,. \tag{5}$$

We denote by $\mathrm{R}(U)$ the variety of the curves corresponding to solutions with fixed vector U and denote $\mathrm{R}_0(U)=\mathrm{R}(U)\cap\mathrm{R}_0$.

The family of solutions (2) and (5) with D varying at the torus $\mathbb{R}^N/2\pi\mathbb{Z}^N$ is called a toroidal family of solutions. The toroidal families of solutions are in one to one correspondence with points of $\mathscr{R}_0(U)$. Let

[1] On leave from the Leningrad Branch of the Steklov Mathematical Institute, Leningrad 191011, USSR.
[2] Supported by the Alexander von Humboldt Foundation.
[3] On leave from the Institute for Information Transmission Problems, Moscow 101447, USSR.

us consider $X \in \mathscr{R}_0(U)$, generating the solution $u_0(t, x)$ of the problem (1) and (4), and the variational equation along $u_0(x)$:

$$4v_t = 3\frac{\partial}{\partial x} v u_0 + v_{xxx} . \tag{6}$$

It is known [1,6,7] that the substitution $v(t)=B(t)V(t)$ (where $B(t)$ is a linear operator in Z_0, quasiperiodic in t) reduces eq. (6) to a linear equation in Z_0: $V_t = A_X V$ with the operator A_X independent of t. Non-zero eigenvalues of A_X are purely imaginary, $\{\pm i\lambda(X)\}$. The numbers $\lambda_j(X)$ are called the fundamental frequencies of the variational equation. The frequency $\lambda_j(X)$ can be found by varying the jth closed gap of the spectrum of $u_0(t, x)$. This means that to find $\lambda_j(X)$ one should ϵ-open the jth gap and calculate the frequency vector $(W_1, ..., W_N, W_j)(X, \epsilon)$ of the obtained $(N+1)$-gap solution. Fundamental frequencies are given by the limit [1,6,7]

$$\lambda_j(X) = \lim_{\epsilon \to 0} W_j(X, \epsilon) . \tag{7}$$

Definition. The family $\mathscr{R}_0(U)$ of N-gap solutions of the problem (1) and (4) is called non-degenerate if

(A) $\{W(X) | X \in \mathscr{R}_0(U)\}$ is an N-dimensional domain;

(B) for any $s \in \mathbb{Z}^N \setminus \{0\}$ and $j, j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$,

$$W(X)\cdot s + 2\lambda_j(X) \not\equiv 0, \quad X \in \mathscr{R}_0(U) , \tag{8}$$

$$W(X)\cdot s + \lambda_{j_1}(X) \pm \lambda_{j_2}(X) \not\equiv 0 . \tag{9}$$

If $\mathscr{R}_0(U)$ is non-degenerate then by the results of refs. [1,2,7] the solutions of the perturbed equation possess the properties formulated in the abstract.

Theorem. All families $\mathscr{R}_0(U)$ of N-gap solutions of the problem (1) and (4) are non-degenerate.

This theorem shows that the theorems of refs. [1,2,7] mentioned above are applicable to the KdV equation. They justify the investigation of perturbations. For details see these papers.

For the proof of the theorem we use the technique of the Schottky uniformization [3–5], which we now briefly review. Let us consider the complex z-plane with $2N$ circles orthogonal to the real axis such that all the discs bounded by these circles are disjoint and are arranged in pairs symmetric with respect to $z \to -z$. Each pair determines a hyperbolic transformation σ_n with the fixed points $\pm A_n$:

$$\frac{\sigma_n z + A_n}{\sigma_n z - A_n} = \mu_n \frac{z + A_n}{z - A_n}, \quad 0 < \mu_n < 1, \quad A_n \in \mathbb{R} .$$

The Schottky space $S = \{(A, \mu)\}$ is a full-dimensional subset in $\mathbb{R}^{2N}$ and is described explicitly [3,4]. The complement of the discs mentioned above is a fundamental domain for the Schottky group G generated by $\sigma_1, ..., \sigma_N$. Let Ω be the region of discontinuity for G. All hyperelliptic M-curves (3) can be uniformized as Ω/G with the point $z=\infty$ as a preimage of $\lambda=\infty$.

Let us denote by g the group of transformations on $\mathscr{R}$: $E_n \to E_n + \text{const}$ with the same constant for all n. The parameters (A, μ) determine an element $\tilde{X} \in \mathscr{R}/\mathrm{g}$. The solutions determined by $\tilde{X}$ are of the form (2) and one of them is with $U = \tilde{U}$, $W = \tilde{W}$, $c = \tilde{c}$ given by the following Poincaré theta series:

$$\tilde{U}_n = \sum_{\sigma \in G/G_n} [\sigma A_n - \sigma(-A_n)] ,$$

$$\tilde{W}_n = \sum_{\sigma \in G/G_n} \{(\sigma A_n)^3 - [\sigma(-A_n)]^3\} ,$$

$$c = \sum_{\sigma \in G,\, G \neq I} \gamma^{-2} . \tag{10}$$

Here G_n is a cyclic group generated by σ_n, and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a PSL$(2, \mathbb{R})$ representation of σ. The factor $\mathscr{R}/\mathrm{g}$ is isomorphic to $\mathscr{R}_0$, the corresponding transformation of the solution (2), (10) is the following:

$$U = \tilde{U}, \quad W = \tilde{W} - 3c\tilde{U}, \quad c = 0 . \tag{11}$$

Lemma 1. There is an analytic isomorphism $S \leftrightarrow \mathscr{R}_0$. It defines the analytic coordinates (A, μ) on $\mathscr{R}_0$.

Below we consider the case of small gaps (small potentials). Let us remark that it corresponds to small μ since $|E_{2n+1} - E_{2n}| \approx \sqrt{\mu_n}$ (see ref. [5]).

Lemma 2. The map $(A, \mu) \to (U, W)$ determined by the series (10) and (11) is analytic for $|\mu|$ small enough and may be analytically continued at each point $(A, 0)$ from the closure of S. There this map has the following leading terms:

$$U_n = 2A_n + \sum_{k=1}^{N} u_{nk}\mu_k + O(|\mu|^2) , \tag{12}$$

$$W_n = 2A_n^3 + \sum_{k=1}^{N} w_{nk}\mu_k + O(|\mu|^2) , \tag{13}$$

$$u_{nk} = \frac{16A_n A_k^2}{A_k^2 - A_n^2} \quad (k \neq n), \quad u_{nn} = 0 ,$$

$$w_{nk} = \frac{48A_n^3 A_k^2}{A_k^2 - A_n^2} \quad (k \neq n), \quad w_{nn} = 48A_n^3 .$$

This lemma follows from the explicit formulae (10) and (11). The leading terms are given by the summation in (10) over the elements $\{I, \sigma_1, \sigma_1^{-1}, ..., \sigma_N, \sigma_N^{-1}\} \in G$.

Lemma 3. The sufficiently small parameters $\mu = (\mu_1, ..., \mu_N)$, $\mu_N > 0$, can be taken as coordinates on a subdomain of $\mathscr{R}_0(U^0)$, $U^0 \in \mathbb{Z}^N$. In other words an analytic map $\mu \to A(\mu)$ exists such that the solutions determined by $(A(\mu), \mu)$ form a full-dimensional subdomain in $\mathscr{R}_0(U^0)$. Furthermore,

$$\frac{\partial}{\partial \mu_k} W_n(A(\mu), \mu)\bigg|_{\mu=0} = -48\delta_{nk} A_0^3(0) . \tag{14}$$

Proof. The series (12) is invertible with small μ. The equality $U(A, \mu) = U_0$ due to the implicit function theorem determines $A = A(\mu)$, and $\partial A/\partial \mu = -\frac{1}{2}\partial U/\partial \mu$. It gives

$$\frac{\partial}{\partial \mu_k} W_n(A(\mu), \mu) = w_{nk} - \frac{1}{2}\sum_e \frac{\partial W_n}{\partial A_e} u_{ek}$$

and finally (14). The first statement of the lemma follows from lemma 1.

To complete the proof of the theorem let us suppose that there is an expression of the form (8) identically vanishing on $\mathscr{R}_0(U^0)$. Then in particular it vanishes on the subset of solutions with $|\mu| \ll 1$ discussed in lemma 3. Let us ϵ-open the gap j preserving (5). The obtained $(N+1)$-gap solution is characterized by the parameter $(\mu_1, ..., \mu_N, \mu_j)$ and the frequency vector (W, W_j). The functions W_j can be analytically continued to the point $(\mu, 0)$ and (7) gives

$$\lambda_j(\mu) = W_j(\mu, 0) . \tag{15}$$

Differentiating (8) with respect to μ_n we get

$$\frac{\partial}{\partial \mu_n}(W \cdot s + 2\lambda_j)\bigg|_{\mu=0} = 0 .$$

Together with (14) and (15) it gives $A_n^3 s_n = 0$ for all $n = 1, ..., N$. Vanishing s proves (8). The same arguments prove (9).

The authors are grateful for the hospitality of the Max-Planck-Institut für Mathematik, Bonn, where this work was done.

References

[1] S.B. Kuksin, Math. USSR Sbornik 64 (1989) 397.
[2] S.B. Kuksin, Math. Notes 45 (1989) 373.
[3] A.I. Bobenko, Sov. Math. Dokl. 36 (1988) 38.
[4] A.I. Bobenko and D.A Kubensky, Theor. Math. Phys. 72 (1987) 929.
[5] A.I. Bobenko, Uniformization of Riemann surfaces and effectivization of theta functional formulae, Preprint 257 (1990), Technische Universität Berlin, to be published in: E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its and V.B. Matveev, Algrebraic geometrical approach to nonlinear integrable equations (Springer, Berlin).
[6] I.M. Krichever, Sov. Sci. Rev. Math. Phys. 9 (1991).
[7] I.M. Krichever, Sov. Math. Dokl. 27 (1983) 1312.