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Lax representation and new formulae for the Goryachev-Chaplygin top

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Abstract. A Lax pair for the Goryachev-Chaplygin top (GCT) is presented. The pair is obtained from that for the Kowalewski top recently found by Reyman and Semenov-Tian-Shansky. Explicit formulae are constructed in terms of theta functions for the dynamical variables.

1. Introduction

The system under consideration is a special case of the motion of a heavy rigid body with a fixed point, discovered by Goryachev and Chaplygin in 1900 [1]. It represents a symmetric top with the principal moments of inertia satisfying $I_1 : I_2 : I_3 = 1 : 1 : \frac{1}{4}$ and the centre of mass located in the equatorial plane. We shall describe the motion of the top in the moving frame. The dynamical variables are the angular momentum $M = (M_1, M_2, M_3)$ and the field strength vector $p = (p_1, p_2, p_3)$ in the moving frame. The fundamental Poisson brackets of these variables are given by

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k \qquad \{M_i, p_j\} = \varepsilon_{ijk} p_k \qquad \{p_i, p_j\} = 0 \qquad i, j, k = 1, 2, 3. \tag{1}$$

The Hamiltonian of the Goryachev-Chaplygin top (GCT) is given by

$$H = \tfrac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1. \tag{2}$$

The system (2) admits an extra integral of motion provided that

$$(M, p) = M_1 p_1 + M_2 p_2 + M_3 p_3 = 0. \tag{3}$$

Note that (M, p) is a Casimir function for the Poisson brackets (1). A more general system described by

$$H = \tfrac{1}{2}(M_1^2 + M_2^2 + 4M_3^2 + 4\gamma M_3) - 2p_1 \tag{4}$$

is called the Goryachev-Chaplygin gyrostat (GCG). It is also integrable if $(M, p) = 0$ [2]. Both GCT and GCG are known to be integrable in the quantum case as well [3]. In passing we mention two recent papers where GCT is studied in a different way. In [4] the R-matrix technique is used to solve both the classical and the quantum problems. In [5] the geometry of the Liouville tori for GCT is thoroughly studied using the general technique developed in [6]. In particular, a close connection is established in [5] between GCT and the periodic Toda lattice with three particles. Our aim in the present paper is to establish a connection between GCT and the Kowalewski top (KT). Our starting point is the Lax pair for KT which was recently found in [7]. Using this result we present a new Lax pair for GCT. This in turn allows us to derive new explicit formulae for the solutions of GCT in terms of hyperelliptic theta functions. These formulae are especially interesting since, unlike most other known cases, the solutions of GCT are not meromorphic in the time variable. Our formulae are of course in complete agreement with the qualitative analysis of the solutions given in [5].

2. The Lax pair for the Goryachev–Chaplygin top

We briefly state the basic results of [7]. Recall that, by definition, the Kowalewski top is a heavy rigid body with a fixed point with the principal moments of inertia satisfying $I_1:I_2:I_3=1:1:\frac{1}{2}$ and the centre of mass located in the equatorial plane. The Hamiltonian of KT is

$$H=\tfrac{1}{2}(M_1^2+M_2^2+2M_3^2)-p_1. \tag{5}$$

A more general system with the Hamiltonian

$$H=\tfrac{1}{2}(M_1^2+M_2^2+2M_3^2+2\gamma M_3)-p_1 \tag{5$'$}$$

is called the Kowalewski gyrostat (KG). The Lax pair for KG constructed in [7] is based on the use of the loop algebra of SO(3, 2). It arises from a general construction due to the same authors [8]. For our present purposes it is more convenient to use a different version of this Lax representation obtained from that presented in [7], by using the four-dimensional (spinor) representation of SO(3, 2). Namely, put

$$L(\lambda)=\begin{pmatrix} 0 & p_2+\mathrm{i}p_1 & 0 & \mathrm{i}p_3 \\ p_2-\mathrm{i}p_1 & 0 & -\mathrm{i}p_3 & 0 \\ 0 & \mathrm{i}p_3 & 0 & p_2-\mathrm{i}p_1 \\ -\mathrm{i}p_3 & 0 & p_2+\mathrm{i}p_1 & 0 \end{pmatrix}\frac{1}{\lambda}$$
$$+\begin{pmatrix} -\mathrm{i}\gamma & 0 & -M_2-\mathrm{i}M_1 & 0 \\ 0 & \mathrm{i}\gamma & 0 & -M_2+\mathrm{i}M_1 \\ M_2-\mathrm{i}M_1 & 0 & -2\mathrm{i}(M_3+\frac{1}{2}\gamma) & -2\mathrm{i}\lambda \\ 0 & M_2+\mathrm{i}M_1 & 2\mathrm{i}\lambda & 2\mathrm{i}(M_3+\frac{1}{2}\gamma) \end{pmatrix} \tag{6}$$

$$A(\lambda)=\begin{pmatrix} \mathrm{i}(M_3+\frac{1}{2}\gamma) & 0 & -\frac{1}{2}(M_2+\mathrm{i}M_1) & 0 \\ 0 & -\mathrm{i}(M_3+\frac{1}{2}\gamma) & 0 & -\frac{1}{2}(M_2-\mathrm{i}M_1) \\ \frac{1}{2}(M_2-\mathrm{i}M_1) & 0 & -\mathrm{i}(M_3+\frac{1}{2}\gamma) & -\mathrm{i}\lambda \\ 0 & \frac{1}{2}(M_2+\mathrm{i}M_1) & \mathrm{i}\lambda & \mathrm{i}(M_3+\frac{1}{2}\gamma) \end{pmatrix}. \tag{7}$$

Then the Hamiltonian equation of motion defined by (5$'$) is equivalent to the Lax equation

$$\mathrm{d}L/\mathrm{d}t+[L,A]=0.$$

Equations (6) and (7) were communicated to us by Reyman and Semenov-Tian-Shansky.

An important observation is that by removing the first column and the first row of the Lax matrix (6) we get a Lax matrix for GCG. Clearly, we get

$$L(\lambda)=\begin{pmatrix} \frac{2}{3}\mathrm{i}\gamma & -\mathrm{i}p_3/\lambda & -M_2+\mathrm{i}M_1 \\ \mathrm{i}p_3/\lambda & -2\mathrm{i}M_3-\frac{4}{3}\mathrm{i}\gamma & -2\mathrm{i}\lambda+(p_2-\mathrm{i}p_1)/\lambda \\ M_2+\mathrm{i}M_1 & (p_2+\mathrm{i}p_1)/\lambda+2\mathrm{i}\lambda & 2\mathrm{i}M_3+\frac{2}{3}\mathrm{i}\gamma \end{pmatrix}. \tag{8}$$

Put

$$A(\lambda)=\begin{pmatrix} -3\mathrm{i}M_3-\frac{2}{3}\mathrm{i}\gamma & 0 & -M_2+\mathrm{i}M_1 \\ 0 & -2\mathrm{i}M_3-\frac{2}{3}\mathrm{i}\gamma & -2\mathrm{i}\lambda \\ M_2+\mathrm{i}M_1 & 2\mathrm{i}\lambda & 2\mathrm{i}M_3+\frac{4}{3}\mathrm{i}\gamma \end{pmatrix}. \tag{9}$$

Then the Lax equation is equivalent to the Hamiltonian equation with the Hamiltonian (4), provided that the constraint (3) is satisfied.

For future use we introduce the notation

$$L=L_{-1}\lambda^{-1}+L_0+L_1\lambda \tag{10}$$

for the coefficients of the Lax matrix (8).

We note that some formal connection between GCG and KG was already noticed in [9]. Our result shows there is a connection between the two systems on the Lax representation level. By setting $\gamma=0$ in (8) and (9) we get a Lax pair for GCT.

The Lax representation with a spectral parameter for GCT and GCG permits us to apply the powerful machinery of algebraic geometry [10] to solve the equations of motion. In the following we shall consider only the first case, i.e. put $\gamma=0$. Formulae for the general case may be easily obtained in quite the same way. The technique we use is a modification of the Krichever scheme [11] proposed by Its [12]. We apply it to derive explicit formulae for the dynamical variables of the top (see § 6). As far as we know, such formulae are not available in the literature.

3. The spectral curve

It is well known that dynamical systems given by the Lax equation can be linearised on the Jacobian of the determinant curve of the Lax matrix. Let $\hat{\Gamma}$ denote the curve given by the equation $\det(L(\lambda)-\mu I)=0$. The symmetry relation

$$L(-\lambda)=\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} L(\lambda) \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{11}$$

gives rise to an involution on $\hat{\Gamma}$,

$$\tau:(\lambda,\mu)\to(-\lambda,\mu).$$

It is natural to consider the quotient curve $\Gamma=\hat{\Gamma}/\tau$ given by the equation

$$\mu^3+\mu(2H-4z-1/z)-2\mathrm{i}G=0 \qquad z=\lambda^2$$

where $H=\frac{1}{2}(M_1^2+M_2^2+4M_3^2)-2p_1$ is the Hamiltonian and $G=M_3(M_1^2+M_2^2)+2M_1p_3$ is the Goryachev-Chaplygin integral. It is equivalent to the Chaplygin curve [1]

$$y^2=(\mu^3+2H\mu-2\mathrm{i}G)^2-16\mu^2 \qquad y=8z\mu-\mu^3-2H\mu+2\mathrm{i}G.$$

Note that we always suppose $\Sigma\, M_ip_i=0$, $\Sigma\, p_i^2=1$.

Let $0_1, 0_2$ and ∞_1, ∞_2 denote the two pairs of points of Γ for which $z=0$ and $z=\infty$, respectively. The three-sheeted covering $\Gamma \to \mathbb{C} \ni z$ of the z plane is unramified at $0_1, \infty_1$ and ramified at $0_2, \infty_2$. The covering $\hat{\Gamma} \to \Gamma$ can be described by a suitable cut contour $\mathscr{L}$ running from 0_1 to ∞_1 on Γ: $\hat{\Gamma}$ is obtained by gluing together two copies of Γ along $\mathscr{L}$. $\hat{\Gamma} \to \Lambda \ni \lambda = \sqrt{z}$ is a three-sheeted covering.

Let us choose the sheets in such a way that the points $0_1, \infty_1$ lie on the sheet I, $0_2, \infty_2$ lie on the sheet II and $\tau 0_2, \tau\infty_2$ lie on the sheet III. The function $\lambda = \sqrt{z}$ is double valued on Γ and changes sign when analytically continued along a closed path which intersects $\mathscr{L}$. We fix $\mathscr{L}$ by the condition

$$\mu \to -2\lambda \qquad \lambda \to \infty_2$$
$$\mu \to -1/\lambda \qquad \lambda \to 0_2.$$

4. Properties of the Baker–Akhiezer function

Our main goal is to construct explicitly the Baker–Akhiezer function $\psi(P) = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$ which is analytic on $\hat{\Gamma}$ and satisfies

$$L\psi = \mu\psi \qquad \psi_t = A\psi. \tag{12}$$

We may assume that ψ satisfies the symmetry relation (see (11))

$$\psi(\tau P) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \psi(P) \qquad P \in \hat{\Gamma}.$$

Hence the component ψ_2 may be regarded as a single-valued function on Γ, while ψ_1 and ψ_3 are double valued on Γ and change sign when analytically continued along a closed path intersecting $\mathscr{L}$. We may assume that ψ_i are defined on $\Gamma \backslash \mathscr{L}$ and satisfy the symmetry relation

$$\psi^+_{1,3}(P) = -\psi^-_{1,3}(P) \qquad \psi^+_2(P) = \psi^-_2(P) \tag{13}$$

for P belonging to the cut $\mathscr{L}$.

With a suitable normalisation, the Baker–Akhiezer function ψ has the following properties which characterise it completely.

(i) ψ is analytic on $\Gamma \backslash \mathscr{L}$, satisfies the symmetry relations (13) on $\mathscr{L}$ and is meromorphic on $\Gamma \backslash \infty_2$.

(ii) In the neighbourhood of the points $0_1, \infty_1, \infty_2$, ψ has the following asymptotic behaviour:

$$\psi \sim \begin{pmatrix} O(\lambda^{-1}) \\ O(1) \\ O(\lambda^{-1}) \end{pmatrix} \qquad \text{for } P \to 0_1$$

$$\psi \sim \begin{pmatrix} q\lambda + O(1) \\ O(1) \\ O(\lambda^{-1}) \end{pmatrix} \qquad \text{for } P \to \infty_1$$

$$\psi \sim \left(\begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} + O(\lambda^{-1}) \right) \exp(-2\lambda t) \qquad \text{for } P \to \infty_2.$$

(iii) The divisor of poles of ψ, $D = \gamma_1 + \gamma_2$, has degree 2 and does not depend on t.

(iv) The normalisation constant q above satisfies the differential equation $q_t/q = -3\mathrm{i}M_3$ whence

$$q = \alpha \exp\left(-3\mathrm{i}\int^t M_3\,\mathrm{d}t\right).$$

The Baker–Akhiezer function ψ with these properties satisfies (12) where L and A are almost the Lax matrices for GCT, with only the condition

$$(L_{-1})_{12} = -(L_{-1})_{21} \tag{14}$$

not being fulfilled automatically. This last condition will be imposed in the last stage of the computation. As we shall see, it amounts to a suitable choice of the integration constant α.

Let $\Psi(\lambda)$ be the 3×3 matrix whose jth column is the branch of ψ on the jth leaf of $\hat{\Gamma}\to\Lambda$ expressed as a function of λ. The involution τ permits leaves II and III. According to (13) we have

$$\Psi(-\lambda) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \Psi(\lambda) \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

and the asymptotics of $\Psi(\lambda)$ are given by

$$\Psi = (\phi + S\lambda^{-1} + \ldots)\begin{pmatrix} 1 & & \\ & \exp(-2\lambda t) & \\ & & \exp(2\lambda t) \end{pmatrix}\begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \phi = \begin{pmatrix} q & & \\ & 1 & 1 \\ & -\mathrm{i} & \mathrm{i} \end{pmatrix}$$

$$\Psi \underset{\lambda\to 0}{=} T\begin{pmatrix} \lambda^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} S \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = S\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

where T satisfies

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} T \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = T\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

The coefficients of $L(\lambda) = L_{-1}\lambda^{-1} + L_0 + L_1\lambda$ are related to the matrices ϕ, S, T in these expansions by

$$L_{-1} = -T\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} T^{-1} \qquad L_0 = -2\left[S\phi^{-1}, \begin{pmatrix} 0 & & \\ & 0 & \mathrm{i} \\ & -\mathrm{i} & 0 \end{pmatrix}\right]$$

which implies

$$\begin{gathered} \mathrm{i}M_3 = S_{22} - \mathrm{i}S_{32} \qquad \tfrac{1}{2}(M_2 - \mathrm{i}M_1) = \mathrm{i}S_{12} \qquad \tfrac{1}{2}(M_2 + \mathrm{i}M_1) = -\mathrm{i}S_{21}/q \\ (L_{-1})_{12} = -T_{12}/T_{22} \qquad (L_{-1})_{21} = T_{22}T_{31}/(T_{11}T_{32} - T_{31}T_{12}) \\ p_2 - \mathrm{i}p_1 = T_{22}T_{11}/(T_{31}T_{12} - T_{11}T_{32}) \qquad p_2 + \mathrm{i}p_1 = -T_{32}/T_{22}. \end{gathered} \tag{15}$$

5. Construction of the Baker–Akhiezer function

To write down explicit formulae for the functions ψ_1, ψ_2 and ψ_3 we must define a number of objects, most of which are standard in finite-gap integration. Let du_1, du_2 be the normalised Abelian differentials on Γ, so that $\int_{a_i} du_j = 2\pi\sqrt{-1}\delta_{ij}$, $B_{ij} = \int_{b_i} du_j$. The theta function is defined by

$$\theta(x) = \sum_{m\in\mathbb{Z}^2} \exp[\tfrac{1}{2}(Bm, m) + (x, m)].$$

Let $\omega(P)$, $\Delta(P)$ and $\Omega(P)$ be the normalised Abelian integrals of the third and the second kind, respectively†,

$$\begin{array}{llll} P & e^{\omega} & e^{\Delta} & \Omega \\ \infty_1 & a & \lambda^2 + O(1) & f \\ \infty_2 & \lambda + b + O(\lambda^{-1}) & d\lambda^{-1} + O(\lambda^{-2}) & -2\lambda + O(\lambda^{-1}) \\ 0_2 & c\lambda + O(\lambda^2) & e & O(1) \quad . \end{array} \tag{16}$$

Put

$$\omega = \int_{b_i} d\omega \qquad \Delta = \int_{b_i} d\Delta \qquad V = \int_{b_i} d\Omega. \tag{17}$$

There are some useful relations between the different constants in (16) and (17). Comparing the singularities we get

$$\lambda^2/\mu = e^{3\omega} e^{2\Delta} \tag{18}$$

which implies $3\omega + 2\Delta \equiv 0$ modulo the periods. Let us choose the paths $[\infty_2, 0_2]$, $[\infty_1, \infty_2]$ in such a way that an exact equality holds:

$$3\omega + 2\Delta = 0. \tag{19}$$

Also, using (18) we get

$$e^{\Delta} = (d/\lambda)(1 - 3b/\lambda + \ldots) \qquad \lambda \to \infty_2$$

where b is the constant term in the expansion of e^{ω} at ∞_2 (see (16)). Using the general properties of Abelian integrals we obtain

$$f = 3b \qquad d = ae. \tag{20}$$

Let $u = \int_{\infty_2}^{P} du$ be the Abel transform. Choose $D \in \operatorname{Jac}\Gamma$ in such a way that the divisor of zeros of $\theta(u+D)$ on Γ is precisely D, the divisor introduced in the definition of ψ above. We are now in a position to write down explicit formulae for ψ. We have

$$\begin{aligned} \psi_1 &= \frac{q}{a\lambda} \frac{\theta(u + Vt + D - \frac{1}{2}\omega)\theta(D + \frac{3}{2}\omega)}{\theta(u+D)\theta(Vt + D + \omega)} \exp(\Omega t + \omega + \Delta - ft) \\ \psi_2 &= \frac{\theta(u + Vt + D)\theta(D)}{\theta(u+D)\theta(Vt+D)} \exp(\Omega t) \\ \psi_3 &= -\frac{i}{\lambda} \frac{\theta(u + Vt + D + \omega)\theta(D)}{\theta(u+D)\theta(Vt + D + \omega)} \exp(\Omega t + \omega). \end{aligned} \tag{21}$$

† These are uniquely specified by their behaviour in the neighbourhoods of the points $\infty_1, \infty_2, 0_2$.

The expression for the function ψ_1 can also be written in a different form,

$$\psi_1(P)=q(\varphi(P)-\mathrm{i}\varphi(\infty_2)\psi_3(P))$$

$$\varphi=b\lambda\frac{\theta(u+Vt+D-\omega)\theta(D+\frac{3}{2}\omega)}{\theta(u+D)\theta(Vt+D+\frac{1}{2}\omega)}\exp(\Omega t-\omega-ft). \tag{22}$$

6. Formulae for the dynamical variables

Substituting asymptotics of ψ_2, ψ_3 at ∞_2 into (15) we obtain

$$\mathrm{i}M_3=-\frac{1}{2}\frac{\partial}{\partial t}\ln\frac{\theta(Vt+D+\omega)}{\theta(Vt+D)}-b$$

$$q=\alpha\exp(3bt)\left(\frac{\theta(Vt+D+\omega)}{\theta(Vt+D)}\right)^{3/2}$$

where we have used the form $u=(1/2\lambda)V+\dots$ of the Abel transform near ∞_2.

Now we must satisfy the last condition (14). To obtain the expressions for $(L_{-1})_{12}$ and $(L_{-1})_{21}$ we use equations (21) and (22), respectively. We get

$$(L_{-1})_{12}=-\frac{\psi_1(0_2)}{\psi_2(0_2)}=-\frac{ce}{a}q\exp(-ft)\frac{\theta(Vt+D+\frac{1}{2}\omega)\theta(Vt+D)\theta(D+\frac{3}{2}\omega)}{\theta^2(Vt+D+\omega)\theta(D)}$$

$$(L_{-1})_{21}=-\frac{\psi_2(0_2)}{\varphi(0_2)q}=-\frac{c}{a}\frac{1}{q}\exp(ft)\frac{\theta(Vt+D+\omega)\theta(Vt+D+\frac{1}{2}\omega)\theta(D)}{\theta^2(Vt+D)\theta(D+\frac{3}{2}\omega)}.$$

This implies that

$$\alpha^2=-\frac{1}{e}\frac{\theta^2(D)}{\theta^2(D+\frac{3}{2}\omega)}.$$

To compute $p_2+\mathrm{i}p_1$ and $p_2-\mathrm{i}p_1$ we also use both the expressions (21) and (22) for ψ,

$$p_2+\mathrm{i}p_1=-\psi_3(0_2)/\psi_2(0_2)\qquad p_2-\mathrm{i}p_1=-\mathrm{i}\varphi(\infty_2)\psi_2(0_2)/\varphi(0_2).$$

Finally, taking into account (19) and (20) we obtain the following formulae for the physical variables:

$$\begin{aligned}
M_2+\mathrm{i}M_1&=-2\sqrt{e}\,\frac{\theta(Vt+D+\frac{3}{2}\omega)}{\theta(Vt+D+\omega)}\left(\frac{\theta(Vt+D)}{\theta(Vt+D+\omega)}\right)^{1/2}\\
M_2-\mathrm{i}M_1&=-2\sqrt{e}\,\frac{\theta(Vt+D-\frac{1}{2}\omega)}{\theta(Vt+D)}\left(\frac{\theta(Vt+D+\omega)}{\theta(Vt+D)}\right)^{1/2}\\
M_3&=\frac{\mathrm{i}}{2}\frac{\partial}{\partial t}\ln\frac{\theta(Vt+D+\omega)}{\theta(Vt+D)}+b\mathrm{i}\\
p_2+\mathrm{i}p_1&=\mathrm{i}c\frac{\theta(Vt+D+2\omega)\theta(Vt+D)}{\theta^2(Vt+D+\omega)}\\
p_2-\mathrm{i}p_1&=-\mathrm{i}c\frac{\theta(Vt+D-\omega)\theta(Vt+D+\omega)}{\theta^2(Vt+D)}\\
p_3&=\sqrt{e}\,\frac{c}{a}\frac{\theta(Vt+D+\frac{1}{2}\omega)}{[\theta(Vt+D)\theta(Vt+D+\omega)]^{1/2}}.
\end{aligned} \tag{23}$$

The square roots in (23) are quite characteristic and unusual. Their presence is predicted by Painlevé analysis of the equations of motion, which shows that the leading powers of singularities in t are half integers [6]. The sign change of the square root in (23) leads to the transformation $M_1 \to -M_1$, $M_2 \to -M_2$, $p_3 \to -p_3$ preserving the equations of motion.

The paths $[\infty_2, \infty_1]$, $[\infty_2, 0_2]$ are fixed already (see (19)). Constants a, c and e are defined by the integrals upon these very paths.

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Note added: After this paper was completed we discovered that similar but slightly more complicated formulae for the solutions had been obtained in [13].

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