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EULER EQUATIONS IN THE ALGEBRAS e(3) AND so(4). ISOMORPHISMS OF INTEGRABLE CASES

A. I. Bobenko

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As shown in [1], the Hamiltonian system $f_t = \{H, f\}$ with the Hamiltonian $H(M, p)$ and the Poisson bracket

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \ \{M_i, p_j\} = \varepsilon_{ijk} p_k, \ \{p_i, p_j\} = 0, \quad i, j, k = 1, 2, 3, \tag{1}$$

is equivalent to Kirchhoff equations for the motion of a rigid body in an ideal incompressible liquid which is at rest at infinity. The bracket (1) is the Lie–Poisson bracket of the Lie algebra e(3) of the group of motions of the Euclidean space. The integrals $f_1 = p^2$ and $f_2 = \mathbf{pM}$ always exist. Therefore, for the complete integrability of the system, it is sufficient that an additional integral I_4 (besides the Hamiltonian $H = I_3$) exists. The nontrivial integrable cases of Clebsch and Steklov [2], unique when I_3 and I_4 are quadratic, are known.

Another interesting type of system, also having applications to hydrodynamics, is described by Euler equations in $L = so(4) = so(3) \oplus so(3)$. The Lie–Poisson bracket

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k, \ \{S_i, T_j\} = 0, \ \ \{T_i, T_j\} = \varepsilon_{ijk} T_k \quad i, j, k = 1, 2, 3, \tag{2}$$

exists in L^*; it has two integrals: $g_1 = \mathbf{S}^2$ and $g_2 = \mathbf{T}^2$. Consequently, as in the case of e(3), we have a system with two degrees of freedom, for which integration we need one more integral I_4, besides the Hamiltonian $H = I_3$. The integrable cases of Manakov [3] and Steklov (see [4-6]) are known; here I_3 and I_4 are also quadratic. There are no other integrable cases with quadratic integrals.

A contemporary look at the systems of classical mechanics under consideration enables us to connect the integrable cases in so(4) and e(3). The observation of [1] that, under the contraction of so(4) into e(3), the integrals of the Manakov cases are transformed into the Clebsch integrals (analogously, the Steklov second cases transforms into the Steklov first case [5]). Shortly after [6, 7], the Lax representations that are compatible with the contraction so(4) → e(3) and whose spectral parameter varies on an elliptic curve were obtained. However the considered cases of integrability are connected not only by means of the contraction so(4) → e(3). In the present article, we indicate a change of variables that transform them into one another.

Let us define the function $w_\alpha(u)$ in terms of the Jacobi elliptic functions of modulus k: $w_1(u) = \mathrm{sn}^{-1}(u, k)$, $w_2(u) = \mathrm{dn}(u, k)/\mathrm{sn}(u, k)$, $w_3(u) = \mathrm{cn}(u, k)/\mathrm{sn}(u, k)$, $w_\alpha^2 - w_\beta^2 = J_\beta - J_\alpha$, $J_1 = 0$, $J_2 = k^2$, $J_3 = 1$, $w^2 = (w_1^2 + w_2^2 + w_3^2)/3$, $J = (J_1 + J_2 + J_3)/3$, where u varies on the torus $\hat{T}$ with the lattice 4K, 4iK' (K is a complete elliptic integral of first kind of modulus k).

1. The Manakov case

$$\begin{gathered}
L^{(1)}(u) = \sum_{\alpha=1}^{3} \{S_\alpha w_\alpha(u - \varkappa) + T_\alpha w_\alpha(u + \varkappa)\} \sigma_\alpha/2i, \\
A^{(1)}(u) = c_I A_I^{(1)} + c_{II} A_{II}^{(1)}, \quad A_I^{(1)} = \sum S_\alpha w_\alpha(u - \varkappa) \sigma_\alpha/2i, \\
A_{II}^{(1)} = -\sum \left\{ S_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(u - \varkappa) + T_\alpha w_\alpha(2\varkappa) w_\alpha(u - \varkappa) \right\} \sigma_\alpha/2i.
\end{gathered} \tag{3}$$

Here σ_α is a Pauli matrix and $[\alpha_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$. The Lax equation

$$L_t + [L, A] = 0 \tag{4}$$

with the operators (3) describes a Hamiltonian system with the Poisson bracket (2) and the Hamiltonian

$$H = c_I H_I + c_{II} H_{II}, \ H_I = \sum w_\alpha S_\alpha T_\alpha,$$

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$$H_{II} = \frac{1}{2} \sum \left(- w_\alpha^2 (S_\alpha^2 + T_\alpha^2) + 2 \frac{w_1 w_2 w_3}{w_\alpha} S_\alpha T_\alpha \right), \quad w_\alpha \equiv w_\alpha (2\varkappa). \tag{5}$$

2. The Clebsch case

$$\begin{gathered}
L^{(2)}(u) = \sum \left\{ p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} + M_\alpha w_\alpha \right\} \sigma_\alpha / 2i, \\
A^{(2)}(u) = d_I A_I^{(2)} + d_{II} A_{II}^{(2)}, \quad A_I^{(2)} = \sum p_\alpha w_\alpha \sigma_\alpha / 2i, \\
A_{II}^{(2)} = \sum \left\{ p_\alpha w_\alpha (w^2 + J_\alpha - 2J) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \sigma_\alpha / 2i, \quad w_\alpha \equiv w_\alpha (u).
\end{gathered} \tag{6}$$

This is a Hamiltonian system with the Poisson bracket (1) and the Hamiltonian

$$H = d_I H_I + d_{II} H_{II}, \quad H = \frac{1}{2} \sum (J_\alpha p_\alpha^2 + M_\alpha^2), \quad H_{II} = \frac{1}{2} \sum \left(\frac{J_1 J_2 J_3}{J_\alpha} p_\alpha^2 - J_\alpha M_\alpha^2 \right). \tag{7}$$

The corresponding expressions for the Steklov cases appear in the following form.

3. The Steklov second case

$$\begin{gathered}
L^{(3)}(u) = \sum \left\{ S_\alpha w_\alpha (u) + \frac{1}{2} T_\alpha (w_\alpha (u - \varkappa) + w_\alpha (u + \varkappa)) \right\} \sigma_\alpha / 2i, \\
A^{(3)}(u) = c_I A_I^{(3)} + c_{II} A_{II}^{(3)}, \quad A_I^{(3)} = 2 \sum S_\alpha \frac{w_1 w_2 w_3}{w_\alpha} (u) \sigma_\alpha / 2i, \\
A_{II}^{(3)} = \sum T_\alpha (w_\alpha (u - \varkappa) - w_\alpha (u + \varkappa)) \sigma_\alpha / 2i,
\end{gathered} \tag{8}$$

$$\begin{gathered}
H = c_I H_I + c_{II} H_{II}, \quad H_I = \sum \left(w_\alpha^2 (\varkappa) S_\alpha^2 - 2 \frac{w_1 w_2 w_3}{w_\alpha} (\varkappa) S_\alpha T_\alpha \right), \\
H_{II} = \sum \left(- \frac{w_1 w_2 w_3}{w_\alpha^2} (\varkappa) T_\alpha^2 + 2 w_\alpha (\varkappa) S_\alpha T_\alpha \right).
\end{gathered} \tag{9}$$

4. The Steklov first case

$$\begin{gathered}
L^{(4)}(u) = \sum \left\{ p_\alpha w_\alpha \left(w^2 + \frac{J_\alpha - J}{2} \right) + \frac{1}{2} M_\alpha w_\alpha \right\} \sigma_\alpha / 2i, \\
A^{(\ )}(u) = d_I A_I^{(4)} + d_{II} A_{II}^{(4)}, \quad A_I^{(4)} = -2 \sum p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \sigma_\alpha / 2i, \\
A_{II}^{(4)} = \sum \left\{ 2 p_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \left(w^2 + J - \frac{J_\alpha}{2} \right) + M_\alpha \frac{w_1 w_2 w_3}{w_\alpha} \right\} \sigma_\alpha / 2i, \quad w_\alpha \equiv w_\alpha (u),
\end{gathered} \tag{10}$$

$$\begin{gathered}
H = d_I H_I + d_{II} H_{II}, \quad H_I = \frac{1}{2} \sum \left(\left(J_\alpha^2 + 2 \frac{J_1 J_2 J_3}{J_\alpha} \right) p_\alpha^2 + 2 J_\alpha p_\alpha M_\alpha - M_\alpha^2 \right), \\
H_{II} = \frac{1}{2} \sum \left(J_\alpha (J_1^2 + J_2^2 + J_3^2 - J_\alpha^2) p_\alpha^2 + 2 \frac{J_1 J_2 J_3}{J_\alpha} p_\alpha M_\alpha + J_\alpha M_\alpha^2 \right).
\end{gathered} \tag{11}$$

Remark. All the considered cases are six-parametric. The parameters, in addition to those occurring in (5), (7), (9), and (11), arise by virtue of the fact that the constants f_1, f_2, g_1, and g_2 can be added to the Hamiltonian. Moreover, the bracket (1) is invariant with respect to the transformation $p_\alpha \to a p_\alpha$ (simultaneously for all α), which varies the Hamiltonian.

THEOREM 1. The classical integrable Clebsch and Steklov systems can be considered as Euler equations not only in the algebra e(3), but also in the algebra so(4). In the latter case, they coincide with the Manakov case and the Steklov second case of integrability in the algebra so(4). Let $\mathbf{p}(t)$, $M(t)$ be the solution of the Clebsch system, given by the Hamiltonian (7) and the bracket (1). Then $\mathbf{S}(t)$, $\mathbf{T}(t)$, defined by the identities

$$p_\alpha = w_\alpha (\varkappa) (S_\alpha - T_\alpha), \quad M_\alpha = \frac{w_1 w_2 w_3}{w_\alpha} (\varkappa) (S_\alpha + T_\alpha), \tag{12}$$

is the solution of the system with the Hamiltonian (5) and the bracket (2), where

$$\begin{gathered}
c_I = \frac{w_1^2 w_2^2 + w_1^2 w_3^2 + w_2^2 w_3^2}{2 w_1 w_2 w_3} (d_I - w_1^2 d_{II}) + 2 w_1 w_2 w_3 d_{II}, \\
c_{II} = d_I - w_1^2 d_{II}, \quad w_\alpha \equiv w_\alpha (\varkappa).
\end{gathered} \tag{13}$$

The correspondence between the Steklov cases is given by the equations

$$p_\alpha = S_\alpha, \quad M_\alpha = S_\alpha (w_\alpha^2(\varkappa) - 3w^2(\varkappa)) - 2T_\alpha \frac{w_1 w_2 w_3}{w_\alpha}(\varkappa), \tag{14}$$

$$c_{\mathrm{I}} = d_{\mathrm{I}} - w^2(\varkappa)\, d_{\mathrm{II}}, \quad c_{\mathrm{II}} = -w_1 w_2 w_3(\varkappa)\, d_{\mathrm{II}}. \tag{15}$$

Proof. The L-operators (3) and (6), and also (8) and (10), differ only by the transformation $L(u, t) \to f(u)L(u, t)$ [$f(u)$ is a scalar-valued function that does not depend on t], with respect to which (4) is invariant. Indeed, the identities

$$(w^2(\varkappa) - w^2(u))\, L^{(1)}(u) = L^{(2)}(u), \quad (w^2(u) - w^2(\varkappa))\, L^{(3)}(u) = L^{(4)}(u), \tag{16}$$

where the functions $\mathbf{p}(t)$, $\mathbf{M}(t)$, $\mathbf{S}(t)$, and $\mathbf{T}(t)$ are connected, respectively, by relations (12) and (14), are valid; these relations are proved by equating the coefficients of the singularities in (16). In the Steklov case,

$$2d_{\mathrm{II}} L^{(3)} w_1 w_2 w_3(u) + c_{\mathrm{I}} A_{\mathrm{I}}^{(3)} + c_{\mathrm{II}} A_{\mathrm{II}}^{(3)} = d_{\mathrm{I}} A_{\mathrm{I}}^{(4)} + d_{\mathrm{II}} A_{\mathrm{II}}^{(4)}, \tag{17}$$

where the constants are connected by the relations (15). Equations (13) are proved in analogous, but more cumbersome, manner. The assertion of the theorem is a simple consequence of (16) and (17).

The indicated correspondence between the Clebsch the Manakov Hamiltonian systems has been independently discovered in [8].

All the above-described systems are integrated in terms of theta-functions of two variables. The classical systems of Clebsch (7) and Steklov (11) were investigated even in the last century [9, 10]. The Manakov case has been investigated with the help of an L–A pair with rational dependence on the spectral parameter [11-13]. The article [14] has been devoted to the abstract theory of the algebrogeometric solutions of equations, integrable by the method of the inverse problem with the spectral parameter on an elliptic curve.

Formulas for the solutions of the systems, considered here, are obtained with the help of the technique, suggested in [15] (see [16] for a somewhat more advanced version). The curve Γ, given by the equation $\det(L^{(1)}(u) - zI) = 0$, is a two-sheeted branched (at four points) covering of the torus T, whose sides are 2K and 2iK'. The curve Γ has an involution π that permutes the sheets ($T = \Gamma/\pi$). The Prym manifold $\mathrm{Prym}_\pi(\Gamma)$ is a two-dimensional Abelian torus in our case. The equations of motion for the Manakov system correspond to a conditionally periodic motion in $\mathrm{Prym}_\pi(\Gamma)$ (cf. [12]). In the Steklov cases, the curve Γ, given by the equation $\det(L(u) - zI) = 0$, has, besides the involution π, also the involution $\lambda: u \to -u$. The motion is linearized on the Jacobian $J(\Gamma_0)$ of the curve $\Gamma_0 = \Gamma/\lambda\pi$ of genus 2.

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MEAN-PERIODIC EXTENSIONS OF FUNCTIONS

G. M. Gubreev

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Let $\mathfrak{X}$ be the Banach space of the functions that are defined on $[0, 1]$, $\hat{\varphi}$ be a bounded linear functional in $\mathfrak{X}$, and $\mathfrak{L} = \operatorname{Ker} \hat{\varphi}$. A function $y(t)(t \in \mathbf{R}_+)$ is called a mean-periodic right extension of a function $x \in \mathfrak{L}$, if the following conditions are fulfilled: 1) For each $s > 0$ the function $y(t + s)$ belongs to $\mathfrak{X}(t \in [0, 1])$. 2) $x(t) = y(t)$ for $t \in [0, 1]$ (as elements of the space $\mathfrak{X}$). 3) $\hat{\varphi}(y(t+s)) = 0$ for $s \geqslant 0$. Further, the functional $\hat{\varphi}$ will be called a right Delsarte functional if each function x in its kernel has a unique mean-periodic right extension. The notions of a left Delsarte and a two-sided Delsarte functional are defined in the same manner.

Delsarte [1] considered the problem of the mean-periodic extension of functions for the first time and, subsequently, several works were devoted to it (mainly, in the space $C[0, 1]$); we can get aquainted with their main results from the surveys [2-4]. The Sedletskii theorem [4] is the best in this direction. According to this theorem, the functional $\hat{\varphi}(x) = \int_0^1 x(t)\,d\sigma(t)$ $(0, 1 \in \operatorname{Supp} \sigma)$ is a right Delsarte functional in $C[0, 1]$ if $\sigma(t)$ has nonzero jump at $t = 1$. In this note, using the semigroup method, we give complete description of Delsarte functionals in the spaces $L_2[0, 1]$ and $W_2^1[0, 1]$ and also isolate a class of functionals in $C[0, 1]$, in which the above-mentioned result of Sedletskii is final. In all the three cases we will consider an entire functional of exponential type $\varphi(z) \overset{\mathrm{df}}{=} \hat{\varphi}(e^{-izt})$, about which we will assume that the width of its indicator diagram is equal to 1. The last assumption is equivalent to the natural requirement that the initial interval $[0, 1]$ cannot be replaced by a smaller one.

1. Let us observe that under the conditions of the Sedletskii theorem we have

$$\exists b_0 > 0, \text{ что } \forall b > b_0 \ \varphi(t + ib) \nrightarrow 0 \text{ при } t \to +\infty. \tag{1}$$

<u>THEOREM 1.</u> If the functional $\hat{\varphi}(x) = \int_0^1 x(t)\,d\sigma(t)$ satisfies the condition (1), then $\hat{\varphi}$ is a right Delsarte functional in $C[0, 1]$ if and only if $\sigma(t)$ has nonzero jump at the point $t = 1$.

Let us outline the path of proof of the theorem. Each function $x \in \mathfrak{L}$ has a unique mean-periodic extension $\tilde{x}(t)$ $(t \in \mathbf{R}_+)$. The family $(V_s x)(t) = \tilde{x}(t + s)$ $(s \geqslant 0)$ forms a C_0-semigroup of operators in the space $\mathfrak{L}$. Therefore, $V_s = \exp\{-iAs\}$, and the resolvent of the operator A has the form

$$(R_\lambda h)(t) = -i\varphi^{-1}(\lambda)\,\hat{\varphi}\left(\int_t^1 h(u)\,e^{i\lambda u}\,du\,e^{-i\lambda t}\right)e^{-i\lambda t} + i\int_t^1 h(u)\,e^{i\lambda u}\,du\,e^{-i\lambda t}, \quad h \in \mathfrak{L}. \tag{2}$$

It follows from the estimate of the resolvent (the Hille–Yoshida theorem), if, as a preliminary, the second term is dropped in (2) and we set $h = (e^{-izt} - \varphi(z)) \in \mathfrak{L}$ [we normalize the functional $\hat{\varphi}$ by the condition $\hat{\varphi}(1) = 1$], that

$$|\varphi(z)\,y^{-1}(e^{-y}\varphi(iy) - 1) - (iz + y)^{-1}(e^{-iz-y}\varphi(iy) - \varphi(z))| \leqslant K(y - \omega)^{-1}e^{-y}|\varphi(iy)|(\|e^{-izt}\| + |\varphi(z)|), \quad y > \omega.$$

Taking limit along the sequence $\{z_n\}$ $(\operatorname{Im} Z_n = b)$, for which $\lim_{n\to\infty} \varphi(z_n) = a \neq 0$, in this inequality, we get

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