Докл. Акад. Наук СССР
Том 272 (1983), № 6

Soviet Math. Dokl.
Vol. 28 (1983), No. 2

ON FINITE-ZONE INTEGRATION OF THE LANDAU-LIFSHITS EQUATION

UDC 517

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The purpose of this note is to construct finite-zone solutions of the Landau-Lifshits equation

$$\begin{gathered} \mathbf{s}_t = \mathbf{s} \times \mathbf{s}_{xx} + \mathbf{s} \times J\mathbf{s}, \\ |\mathbf{s}| = 1, \qquad J = \operatorname{diag}(J_1, J_2, J_3), \end{gathered} \tag{1}$$

in the case of uniaxial anisotropy: $J_1 = J_2 < J_3$.

Equation (1) can be imbedded in the scheme of the method of the inverse problem. It is shown in [1] and [2] that the nonlinear relation (1) is a compatibility condition for the two linear equations

$$\psi_x = u(\lambda)\psi, \qquad \psi_t = v(\lambda)\psi,$$

where the 2×2 matrices $u(\lambda)$ and $v(\lambda)$ in the case we consider are rational functions on the Riemann surface Γ of the square root of $\lambda^2 + a^2$, $a^2 = (J_3 - J_1)/4$:

$$\begin{aligned} u(\lambda) &= i \sum_{j=1}^{3} s_j \sigma_j w_j(\lambda), \\ v(\lambda) &= 2i \sum_{j=1}^{3} s_j \sigma_j w_j^{-1}(\lambda) w_1(\lambda) w_2(\lambda) w_3(\lambda) + i \sum_{j=1}^{3} (\mathbf{s} \times \mathbf{s}_x)_j \sigma_j w_j(\lambda). \end{aligned} \tag{2}$$

In (2) the σ_j are the Pauli matrices, $w_3(\lambda) = \lambda$, and $w_1(\lambda) = w_2(\lambda) = \sqrt{\lambda^2 + a^2}$.

The problem we discuss was considered previously in [3] by analyzing the corresponding equation for the monodromy matrix. In particular, the authors of [3] succeeded in obtaining for the conditional eigenvalues the Dubrovin equations and trace identities whose important role in questions of applying the theory of finite-zone integration to problems of ergodic deformations of nonlinear evolution equations was discovered in [4] and [5]. However, in [3] explicit integration in theta functions of (1) was not carried to completion. With this note we hope to fill this gap in the finite-zone theory of equation (1). The method we propose differs considerably from the constructions of [3]. A central feature of our approach is the synthesis of the theory of finite-zone integration, developed by Novikov, Dubrovin, Matveev, and others (for the history of the question and precise references, see the survey [6]) in the form given it by Krichever (see [7]), with the general ideas of the method of the matrix Riemann problem—the modern version of the method of the inverse problem. We very much hope that the experience of our work will be useful in making effective the results of Cherednik [8] devoted to the algebro-geometric analysis of a qualitatively more complex situation—the case of total anisotropy.

1980 *Mathematics Subject Classification*. Primary 35Q20; Secondary 35R30.

1. Main Theorem. *Let $\hat{\Gamma}$ be an arbitrary Riemann surface covering the surface Γ, and suppose that on $\hat{\Gamma}$ there is given a single-valued matrix-valued function $\psi(\lambda, x, t)$ (of dimension 2×2), which is meromorphic*(¹) *on $\hat{\Gamma} \setminus \pi^{-1}\{\infty^{1,2}\}$ ($\pi: \hat{\Gamma} \to \Gamma$ is the covering map) and depends smoothly on the "external" real parameters x and t, such that the following conditions are satisfied*:

1) *In a neighborhood of any of the points of the set $\pi^{-1}\{\infty^{1,2}\}$ the function ψ has an essential singularity differentiable with respect to x and t of the form*

$$\psi(\lambda, x, t) = \left(\sum_{j=0}^{\infty} \varphi_j(x, t) k^{-j}\right) \exp\{-ix\sigma_3 k - 2it\sigma_3 k^2\} \cdot C, \tag{3}$$

where $\det \varphi_0(x, t) \neq 0$, *$k^{-1}$ is a local parameter, and C is invertible and does not depend on x or t.*

2) *The logarithmic derivatives $\psi_x\psi^{-1}$ and $\psi_t\psi^{-1}$ are single-valued functions on the surface Γ itself and are regular on $\Gamma \setminus \{\infty^{1,2}\}$.*

3) *The reduction condition*

$$\sigma_3\psi(\lambda^\tau, x, t) = \psi(\lambda, x, t)\sigma(\lambda), \tag{4}$$

holds, where $\sigma(\lambda)$ does not depend on x or t, and $\lambda \to \lambda^\tau$ denotes the involution permuting the sheets of the surface Γ ($\sqrt{(\lambda^\tau)^2 + a^2} = -\sqrt{\lambda^2 + a^2}$).

4) *The following gauge conditions are satisfied*:

$$(\psi_{11}(\lambda^+))_x = (\psi_{11}(\lambda^+))_t = (\psi_{21}(\lambda^-))_x = (\psi_{21}(\lambda^-))_t = 0, \quad \forall \lambda^\pm \in \pi^{-1}\{\pm ia\}. \tag{5}$$

Then, up to terms proportional to the identity matrix, the logarithmic derivatives $\psi_x\psi^{-1}$ and $\psi_t\psi^{-1}$ have the form (2), *where*

$$\Sigma s_j\sigma_j = -\varphi_0\sigma_3\varphi_0^{-1}. \tag{6}$$

The functions $s_j(x, t)$ satisfy the relation $\Sigma s_j^2 = 1$ and form a solution of the Landau-Lifshits equation (1).

The assertion of the theorem is equivalent to fixing the matrix Riemann problem (in the formulation of [9]) corresponding to (1) in the scheme of the method of the inverse problem. In the case of total anisotropy an analogous theorem was proved by Mikhailov [10]. The case we consider differs from the situation of [10] only in the following circumstance: σ_1, the reduction (in addition to σ_3—the reduction of (4)) which, according to Mikhailov, it is necessary to impose on the ψ-function in the general case, is replaced in the case of uniaxial anisotropy by the normalization condition (5). The proof of the theorem is here practically unaltered.

2. The construction. Together with the rational curve Γ we consider the hyperelliptic Riemann surface Γ_0 of genus $g > 0$ defined by

$$z^2 = P(\lambda) = (\lambda^2 + a^2)\prod_{j=1}^{2g}(\lambda - e_j), \qquad e_j \neq e_k, e_j \in \mathbf{C}.$$

The two distinct infinities on Γ_0 we denote by $\infty^\pm$ ($z \sim \pm\lambda^{g+1}$, $\lambda \to \pm\infty$), and by the notation $\lambda \to \lambda^*$ we mean the involution permuting the sheets of the surface Γ_0 in its standard realization as a two-sheeted covering of the λ plane. On Γ_0 we fix a vector-valued Baker-Akhiezer function $\vec{\psi}_0 = (\psi_{01}, \psi_{02})$ by the following conditions:

a) $\vec{\psi}_0$ is meromorphic on $\Gamma_0 \setminus \{\infty^\pm\}$ with the nonspecial divisor of poles $D = \Sigma_1^g \mu_k$.

(¹) $\infty^{1,2}$ are the two infinitely distant points on Γ in its standard realization as a two-sheeted covering of the λ plane ($\sqrt{\lambda^2 + a^2} \to \pm\lambda$, $\lambda \to \infty^{1,2}$).

b) $\vec{\psi}_0 \exp(\pm i\lambda x \pm 2i\lambda^2 t)$ *is rational in neighborhoods of the points* $\infty^{\pm}$, *where the first component is regular in these neigborhoods, while the second has simple poles at the points* $\infty^{\pm}$.

c) ψ_{02} *has simple zeros at the branch points* $\pm ia$.

d) $\psi_{01}(ia) = 1$ *and* $(\psi_{02}(\lambda)(\lambda + ia)^{-1/2})|_{\lambda = -ia} = 1$.

It can be verified in standard fashion that the function $\vec{\psi}_0$ exists and can be simply described by the now-customary explicit formulas in terms of the Riemann θ function.

On the basis of the function $\vec{\psi}_0$ we define the matrix-valued function $\psi_0 = (\vec{\psi}_0, \vec{\psi}_0^*)$ $(f^*(\lambda) \equiv f(\lambda^*))$. The function $\psi_0(\lambda, x, t)$ is a single-valued function on Γ_0 and possesses the following properties:

1) $\psi_0 \exp\{\pm ix\sigma_3\lambda \pm 2it\sigma_3\lambda^2\}$ is rational in neighborhoods of $\infty^{\pm}$.

2) $\det \psi_0(\lambda) = c(x, t)\sqrt{(\lambda^2 + a^2)\prod_{j=1}^{2g}(\lambda - e_j)} \,/\prod_{k=1}^{g}(\lambda - \lambda_k)$;

3) In a neighborhood of each point λ_0 of the set $\{e_j\} \cup \{\pm ia\} \cup \{\mu_k\}$ there is the representation (see [9]) $\psi_0(\lambda) = \hat{\psi}_0(\lambda)(\lambda - \lambda_0)^T C$, $\det \hat{\psi}_0(\lambda) \neq 0$, with a diagonal matrix T and an invertible matrix C not depending on x and t; for the branch points these matrices are, respectively, $\left(\begin{smallmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$, while for the poles they are $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and I (λ a pole of $\vec{\psi}_0$) or σ_1 (λ a pole of $\hat{\vec{\psi}}_0^*$). In all cases the matrix $\hat{\psi}_0(\lambda)$ is holomorphic in $\lambda \in \mathbf{C}$.

From the properties ψ_0 we obtain a result which is important for what follows.

PROPOSITION 1. $\psi_{0x}\psi_0^{-1}$ *and* $\psi_{0t}\psi_0^{-1}$ *are single-valued and regular on* $\mathbf{C}$.

Our objective, which consists in the construction of a function ψ satisfying the conditions of the main theorem, has almost been achieved. We set

$$\psi(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(\lambda^2 + a^2)^{-1/2} \end{pmatrix} \psi_0(\lambda), \qquad \gamma \in \mathbf{C}. \tag{7}$$

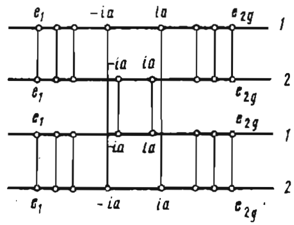

FIGURE 1

The function $\psi(\lambda)$ is single-valued on the two-sheeted covering $\hat{\Gamma}$ of the surface Γ which is constructed in a natural way on the basis of Γ_0. It is convenient to further represent $\hat{\Gamma}$ as a four-sheeted covering of the λ plane. Its genus is equal to $2g - 1$. The surface $\hat{\Gamma}$ is shown schematically in Figure 1. The involution $\lambda \to \lambda^*$ is realized on $\hat{\Gamma}$ as the permutation of sheets $1 \leftrightarrow 2$ and $1' \leftrightarrow 2'$, while the involution $\lambda \to \lambda^\tau$ is realized as the permutation of sheets $1 \leftrightarrow 2'$ and $2 \leftrightarrow 1'$. Because of its definition by (7), the function $\psi(\lambda)$ automatically satisfies the reduction (4) while the matrix σ_1 as $\sigma(\lambda)$. The fact that the logarithmic derivatives $\psi_x\psi^{-1}$ and $\psi_t\psi^{-1}$ are single-valued on Γ and regular on $\Gamma \setminus \{\infty^{1/2}\}$ again follows from (7), Proposition 1, and the absence of zeros of $\det \psi(\lambda)$ at the points $\pm ia$ (compare (7) with 2)). The required normalization and characteristics of the essential singularities of ψ follow from properties d) and 1) of ψ_0.

By computing according to explicit formulas the leading coefficient $\varphi_0(x,t)$ of the expansion of ψ at any essential singularity, we arrive, according to (6), at the following expression for the corresponding finite-zone solution of (1):

$$
(8)\quad
\begin{gathered}
s_1 = (y_+y_- + z_+z_-)/(y_-z_+ - y_+z_-),\quad s_2 = i(y_+y_- - z_+z_-)/(y_+z_- - y_-z_+),\\
s_3 = (y_+z_- + y_-z_+)/(y_-z_+ - y_+z_-),\\
y_\pm(x,t) = \theta(\mathbf{g}(x,t) \mp \mathbf{r})\exp\{ix\omega_1(ia) + it\omega_2(ia)\},\\
z_\pm(x,t) = \theta(\mathbf{g}(x,t) + \mathbf{n} \mp \mathbf{r})\exp\{ix\omega_1(-ia) + it\omega_2(-ia)\}\,\omega_\pm\gamma_0,
\end{gathered}
$$

where $\mathbf{n} = \vec{\omega}(-ia)$, $\mathbf{r} = \vec{\omega}(\infty^+)$, $\vec{\omega}(\lambda) = \int_{ia}^{\lambda} d\vec{\omega}$, $d\vec{\omega} = (d\omega_1,\ldots,d\omega_g)$ is a basis of the holomorphic differentials, $\theta(\mathbf{p})$ is the theta function of the surface Γ_0 constructed on the basis of the matrix of B-periods of the basis $d\vec{\omega}$,

$$
\mathbf{g}(x,t) = (1/2\pi)\mathbf{B}_1x + (1/2\pi)\mathbf{B}_2t + \mathbf{g}_0,
$$

$$
\mathbf{g}_0 = \vec{\eta} + \mathbf{K}_R,\qquad \vec{\eta} = \sum_{k=1}^{g}\vec{\omega}(\mu_k).
$$

$\mathbf{K}_R$ is the vector of Riemann constants, $\mathbf{B}_{1,2}$ are the vectors of B-periods of the normalized Abelian integrals of second kind $\omega^{1,2}(\lambda)$ having as poles the points $\infty^\pm$: $\omega^1(\lambda) = \pm(\lambda + \cdots)$, $\omega^2(\lambda) = \pm 2(\lambda^2 + \cdots)$, $\lambda \to \infty^\pm$, $\omega_\pm = \exp\{\lim_{\lambda\to\infty^\pm}(\omega^3(\lambda) - \ln\lambda)\}$, $\omega^3(\lambda)$ is the normalized Abelian integral of third kind with logarithmic singularities at the points $\infty^\pm$ (residues equal to -1) and at the points $\pm ia$ (residues equal to $+1$) and specified by the condition $\omega^3(\lambda) = \ln\sqrt{\lambda^2 + ia} + o(1)$, $\lambda \sim -ia$, and, finally, $\gamma_0 = \gamma\theta(\mathbf{g}_0 - \mathbf{n})/\theta(\mathbf{g}_0)$.

We note that the vector $\mathbf{n}$ is a half period of the lattice generated by the basis $d\vec{\omega}$, and this half period depends on the concrete specification of the basis of the group $H_1(\Gamma_0)$.

3. The real property. Suppose that Γ_0 is a real curve; then on the surfaces Γ_0 and $\hat{\Gamma}$ the anti-involution $\lambda \to \bar{\lambda}$ of complex conjugation is defined. We shall assume that a basis of the group $H_1(\Gamma_0)$ has been chosen so that the corresponding a-cycles are invariant with respect to this anti-involution:[2] $\bar{a}_j = a_j$, $j = 1,\ldots,g$. We require of the vector $\eta \in J(\Gamma_0)$ defined by the function $\vec{\psi}_0$ that the following condition be satisfied:

$$
(9)\qquad \operatorname{Re}\vec{\eta} = (g+1)\vec{\mathbf{n}}/2.
$$

By Abel's theorem, in terms of the divisor D condition (9) means that on Γ_0 there exists a function (cf. [13]) $m_0(\lambda)$ such that $(m_0) = D - \bar{D}^* + (ia) + (-ia) - \infty^+ - \infty^-$. We normalize the function $m_0(\lambda)$ by the condition $\lim_{\lambda\to -ia}(m_0(\lambda)(\lambda + ia)^{-1/2}) = 1$. The parameter γ_0, which has so far not been specified, we now define by the equality

$$
(10)\qquad \gamma_0 = \sqrt{2ia\delta},\qquad \delta = \lim_{\lambda\to ia}\exp(\omega^3(\lambda) - \ln\sqrt{\lambda - ia}),
$$

and we require that at least one of the branch points e_j be real. It is then not hard to verify that for the function $\vec{\psi}_0$ we have

$$
\begin{pmatrix} 1 & 0 \\ 0 & (\lambda^2 + a^2)/\gamma^2 \end{pmatrix}\sigma_2\bar{\vec{\psi}}{}_0^* = \vec{\psi}_0(\lambda)\frac{2a}{\gamma^2}m_0(\lambda),
$$

which in terms of $\psi(\lambda)$ can be rewritten in the form

$$
(11)\quad \sigma_2\bar{\psi}^*(\bar{\lambda},x,t) = \psi(\lambda,x,t)m(\lambda),\quad m(\lambda) = \frac{2a}{\gamma(\sqrt{\lambda^2 + a^2})}\begin{pmatrix} m_0(\lambda) & 0 \\ 0 & m_0^*(\lambda)\end{pmatrix}.
$$

([2]) The possibility of this choice of a-cycles is discussed in detail in [11], which is devoted to a classification of the real solutions of the sine-Gordon equation (see also [12]).

The identity (11) is equivalent to the assertion regarding the property that the vector s be real. Our main result can thus be formulated in the following manner.

THEOREM 1. *Suppose that* Γ_0 *is a real curve with the choice of a-cycles and branch points indicated above, and suppose that the vector* $\vec{\eta} \in J(\Gamma_0)$ *and the parameter* $\gamma_0 \in \mathbf{C}$ *are subject to conditions* (9) *and* (10). *Then a real (and hence smooth!) finite-zone solution of the Landau-Lifshits equation is given by* (8).

REMARK 1. On the basis of the results of Dubrovin and Natanzon [11], it is not difficult to carry out the final effectivization of the conditions for the real property we have obtained. By final effectivizatioin we mean a complete classification of all real solutions accompanied by a precise specification of the vector of half periods n.

REMARK 2. When written in terms of projections of the points μ_k onto the plane $\mathbf{C}$, condition (9) assumes a form analogous to the "polynomial" criterion of Kozel and Kotlyarov for reality of finite-zone solutions of the sine-Gordon equation (see [14]).

REMARK 3. For final completion of the "finite-zone integration" of (1) it is necessary to present the algebro-geometric Poisson brackets for its real finite-zone solutions. Apparently, on the basis of the corresponding technique of Novikov and Dubrovin [15], it will not be exceedingly difficult to carry out all the necessary computations in the situation we consider.

In conclusion we wish to thank N. N. Bogolyubov Jr., and A. K. Prikarpat·skiĭ for the possibility they provided of becoming familiar with their paper [3] before its publication.

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Received 21/FEB/83

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Translated by J. R. SCHULENBERGER