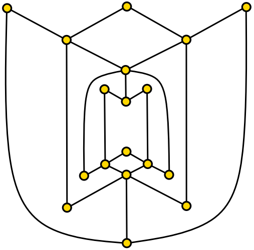
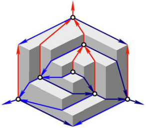
Structural Graph Theory and Dimension

Veit Wiechert





Order and Geometry
Gułtowy 2016

Sparsity Theory for Graphs

Algorithms and Combinatorics 28

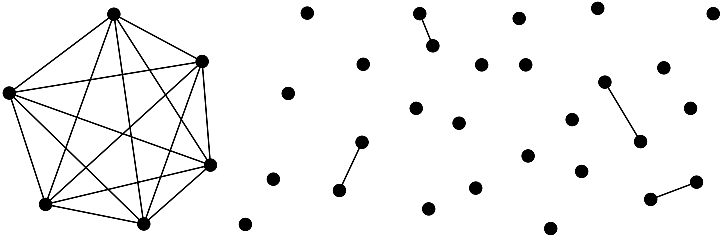
Jaroslav Nešetřil
Patrice Ossona de Mendez

Sparsity

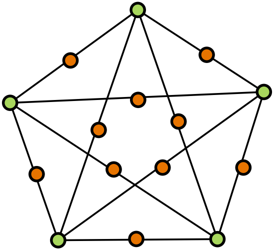
Graphs, Structures, and Algorithms

Springer

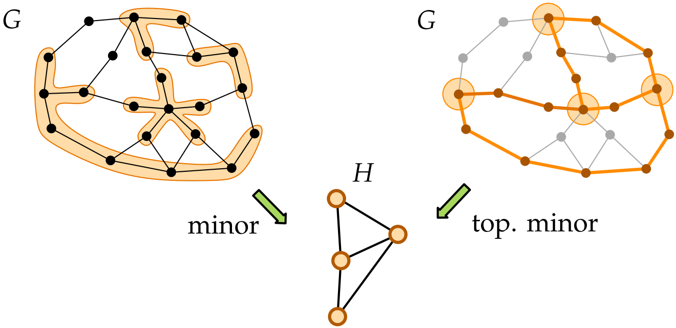
When is a graph sparse?



When is a graph sparse?



Graph Minors



Theorem [Robertson, Seymour]
Let C be a proper minor-closed class of graphs. Then there exists $H_1, \ldots, H_k$ such that

$$C = \{G : H_i \not\preccurlyeq G \text{ for all } i \in [k]\}.$$

Proper minor-closed graph classes

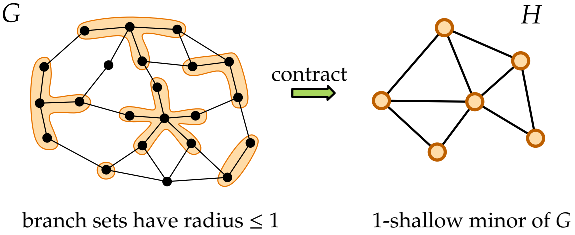
Examples:

- planar graphs
- bounded genus graphs
- graphs of bounded path-width
- graphs of bounded tree-width

Sparse?

- linear many edges
- even their minors have linear many edges!

Shallow Minors



$\mathcal{C}$ class of graphs

$\mathcal{C} \nabla r$ set of r-shallow minors of graphs in $\mathcal{C}$

A class $\mathcal{C}$ has ***bounded expansion*** if there exists a function f s.t. graphs in $\mathcal{C} \nabla r$ have density $\leq f(r)$.

Nowhere dense classes

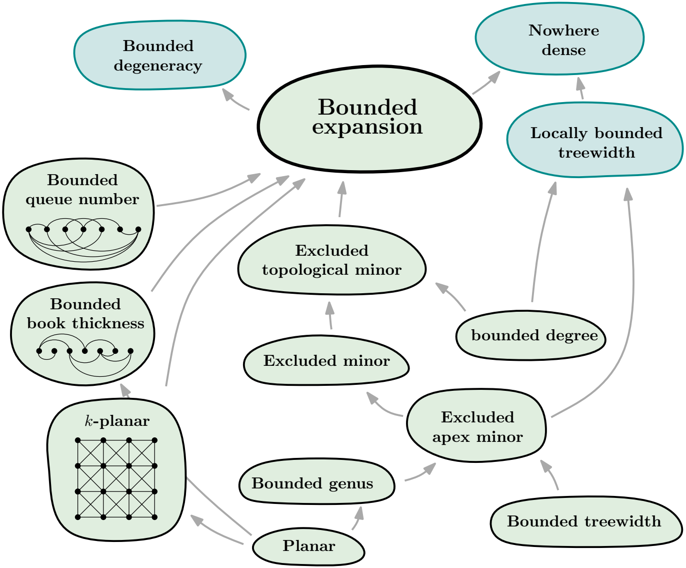
A class C has ***bounded expansion*** if there exists a function f s.t. graphs in $C \nabla r$ have density $\leq f(r)$.

A class C is ***nowhere dense*** if for all $r \geq 0$,

$$C \nabla r \neq \text{ set of all graphs.}$$

A class C is ***somewhere dense*** if it is not nowhere dense.

Hierarchy

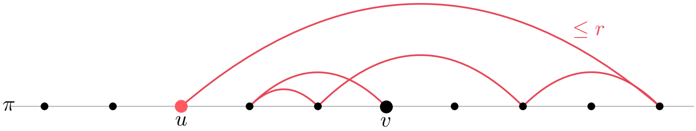


Bounded degeneracy
Nowhere dense
Bounded expansion
Locally bounded treewidth
Bounded queue number
Excluded topological minor
Bounded book thickness
bounded degree
Excluded minor
Excluded apex minor
k-planar
Bounded genus
Bounded treewidth
Planar

Bounded expansion - characterizations

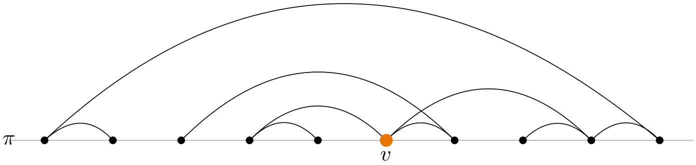
- r-shallow topological minors
- transitive fraternal augmentations
- generalized coloring numbers
- low-treedetph colorings
- neighborhood complexity
- splitter game
- dimension?

Weak Coloring Numbers



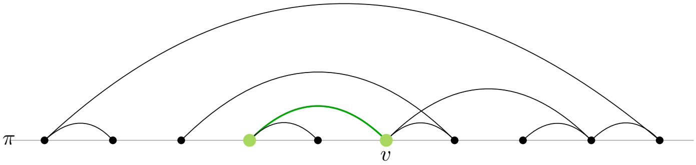
u is weakly r-reachable from v

Weak Coloring Numbers



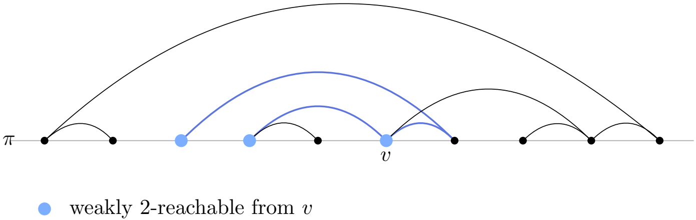
● weakly 0-reachable from v

Weak Coloring Numbers

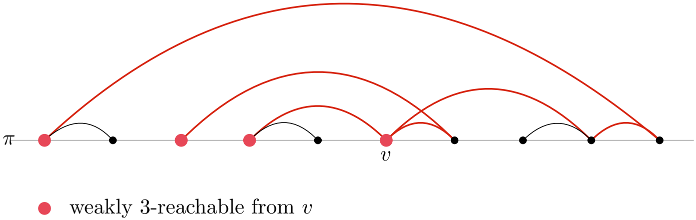


weakly 1-reachable from v

Weak Coloring Numbers



Weak Coloring Numbers



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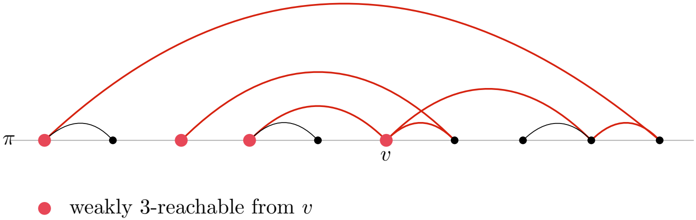
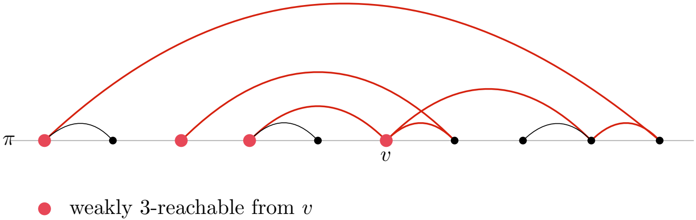


$$\mathrm{wcol}_r(G) := \min_{\pi} \max_{v} |\mathrm{WReach}_r[v, \pi]|.$$

Theorem [Zhu '09]
A class C has bounded expansion iff there exists a function f such that $\mathrm{wcol}_r(G) \le f(r)$ for all $r \ge 0$ and $G \in C$.

Weak Coloring Numbers



$$\mathrm{wcol}_r(G) := \min_{\pi} \max_{v} |\mathrm{WReach}_r[v,\pi]| .$$

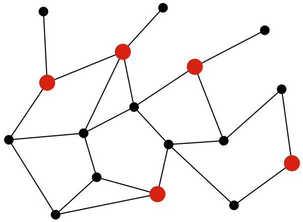
Theorem [Zhu '09]
A class $\mathcal{C}$ is nowhere dense iff for each integer $r \geq 0$ and $\epsilon > 0$, we have $\mathrm{wcol}_r(G) = \mathcal{O}(n^{\epsilon})$ for every $G \in \mathcal{C}$.

Algorithmic Aspects

Dominating Set Problem

Input: Graph G, number k

Problem: Are there k vertices dominating all vertices of G?



NP-complete in general. Is it ***fixed-parameter tractable***? So is there a function f and an algorithm solving the problem in time

$$f(k) \cdot n^{O(1)} \text{ ?}$$

Dominating Set Problem is W[2]-complete

→ unlikely that there exists an FPT for it

NP-complete Graph Problems

- Dominating Set Problem
- k-Colorability
- CLIQUE, INDEPENDET SET
- Steiner tree problem
- k-disjoint paths

General question:
What are the *largest graph classes* on which certain *types of problems* become tractable?

Graph Properties

Goal: Read tractibility of a problem directly off its mathematical description.

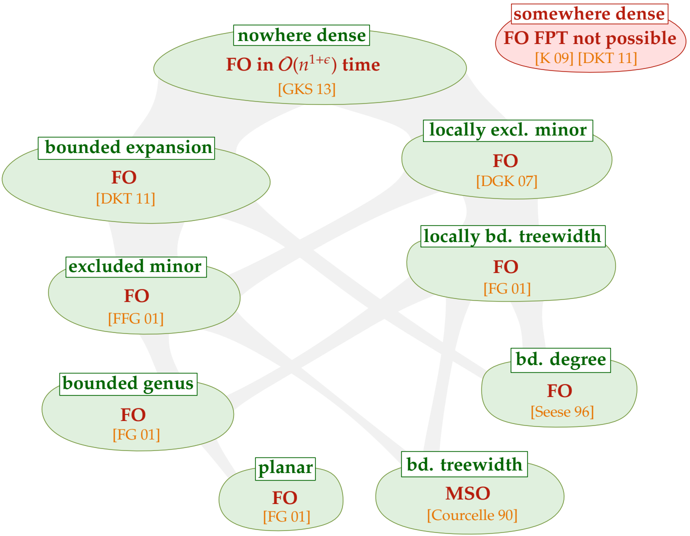
Properties definable in First-Order Logic (FO):

- k-clique, k-independet set
- subgraph containment (for some fixed graph)
- k-dominating set

Properties definable in Monadic Second-Order Logic (MSO):

- connectivity
- hamiltonicity
- k-colorability

META-THEOREMS



Dominating Set Problem

- FPT in linear time on bounded expansion classes
- FPT in almost linear time on nowhere dense classes
- $\exists$ linear kernel on bounded expansion classes
- $\exists$ almost linear kernel on nowhere dense classes
- $\exists$ polynomial kernel for r-DSP on nowhere dense classes

Open Problem

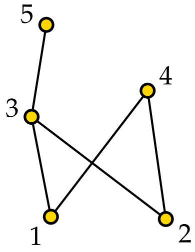
Do there exist (almost) linear kernels for r-DSP on nowhere dense classes?

Dimension

Dimension

The ***dimension*** of a poset **P** is the least d such that there are ***linear extensions*** $L_1, \ldots, L_d$ of **P** with

$$\mathbf{P} = \bigcap_{i \in [d]} L_i.$$



$1 < 2 < 3 < 4 < 5$

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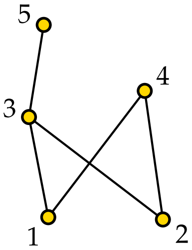

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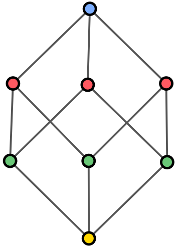
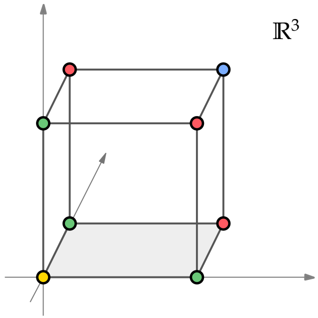
$2 < 1 < 3 < 5 < 4$

DIMENSION

The ***dimension*** of a poset **P** is the least d such that **P** is isomorphic to a subposet of $(\mathbb{R}^d, \leq_d)$.



$\hookrightarrow$



Cover Graphs

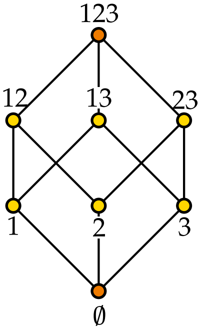
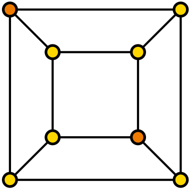
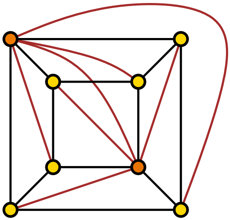


diagram of B_3



cover graph



comparability graph

Width and Dimension

width(**P**) maximum size of an antichain in **P**
height(**P**) maximum size of a chain in **P**

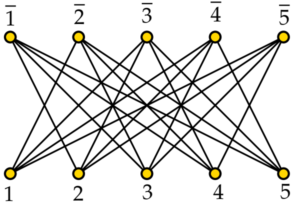
Theorem [Dilworth '50]

$$\dim(\mathbf{P}) \leq \text{width}(\mathbf{P}).$$

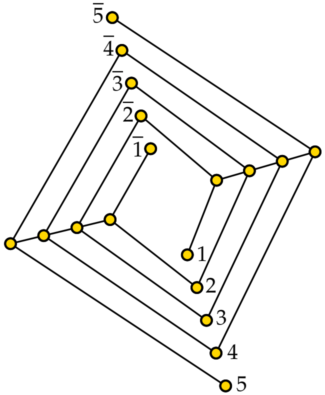
"Large-dimensional posets are *wide*"

but not necessarily *tall*:

S_5



Kelly's examples



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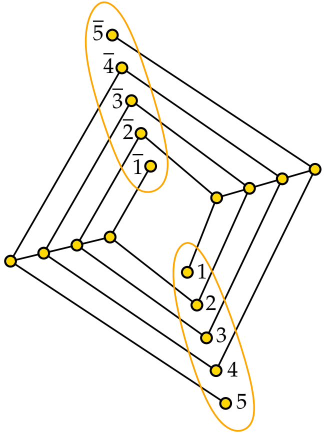

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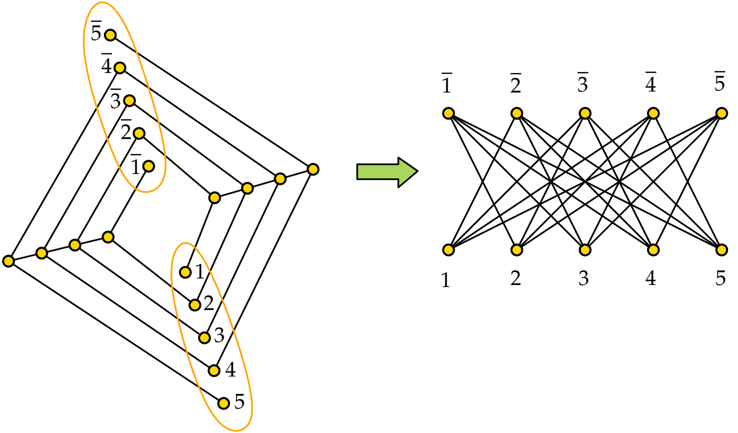

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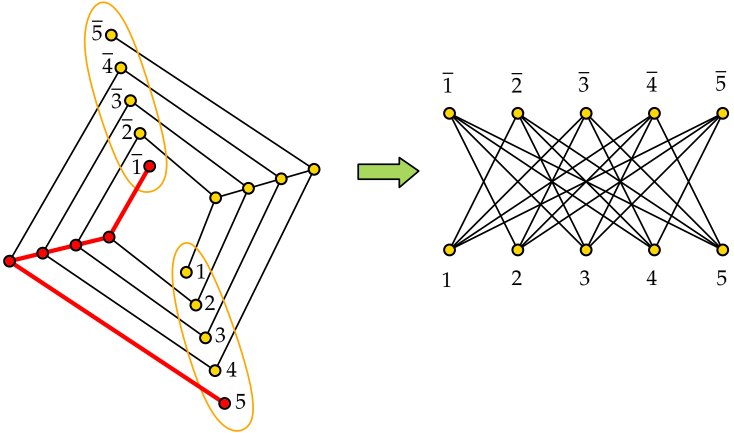


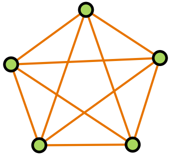
General Question

“Do large-dimensional posets with *sparse* cover graphs have to be *tall*?”

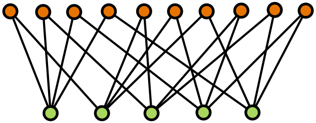
Answer: Yes and No

Theorem [Streib, Trotter, 2014]
The dimension of posets with planar cover graphs is bounded in their height.

Incidence Posets of graphs:



K_5



$\mathbf{P}_{K_5}$

Theorem [Dushnik, Miller, ’41]

$$\dim(\mathbf{P}_{K_n}) \geq \log\log n.$$

Answer: Yes and No

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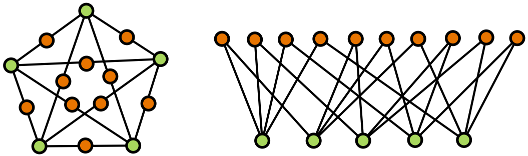

cover($\mathbf{P}_{K_5}$) $\mathbf{P}_{K_5}$

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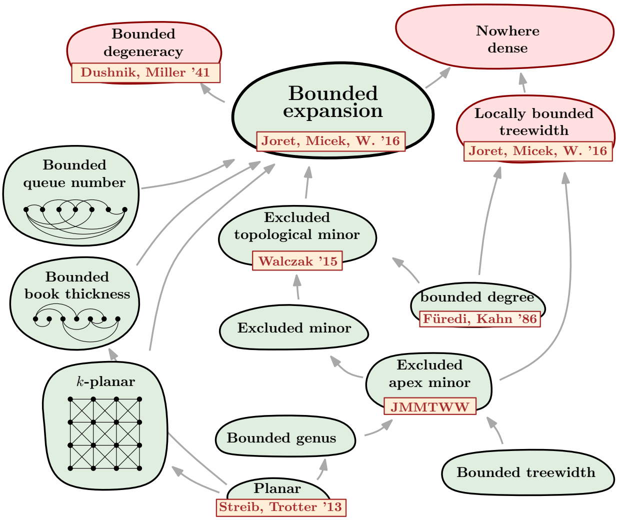
Cover Graphs and Dimension



Weak Coloring Numbers and Dimension

Theorem [Joret, Micek, W., 2016+]
Let $\mathbf{P}$ be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h}(G) \leq c$. Then

$$\dim(\mathbf{P}) \leq 6^c.$$

Graph property	$\mathrm{wcol}_r(G)$	
bounded genus	$\mathcal{O}(r^3)$	**[vH-OdM-Qu-R-S, 2016+]**
treewidth t	$\mathcal{O}(r^t)$	**[GKRSS, 2016]**
no K_n minor	$\mathcal{O}(r^{n-1})$	**[vH-OdM-Qu-R-S, 2016+]**
no K_n top. minor	$2^{\mathcal{O}(r \log r)}$	**[KPRS, 2016]**
bd. expansion	$f(r)$	**[Zhu, 2009]**

Weak Coloring Numbers and Dimension

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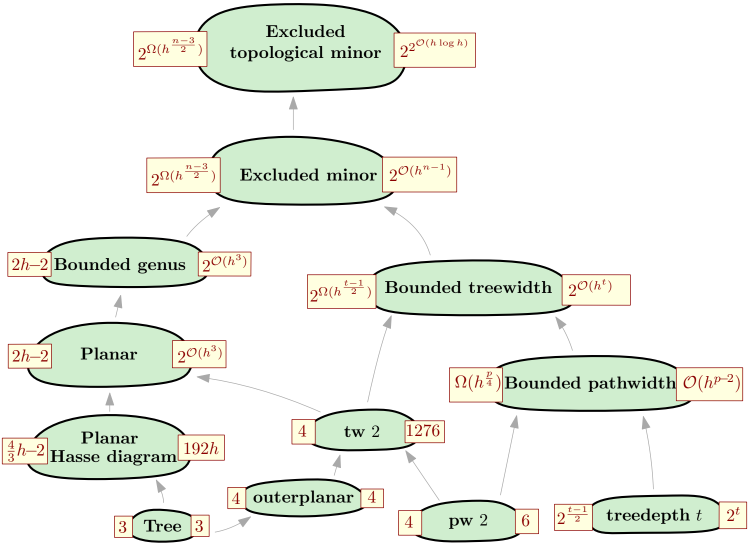
no K_n minor	$\mathcal{O}(r^{n-1})$	$2^{\mathcal{O}(h^{n-1})}$
no K_n top. minor	$2^{\mathcal{O}(r \log r)}$	$2^{2^{\mathcal{O}(h \log h)}}$
bd. expansion	$f(r)$	$g(h)$

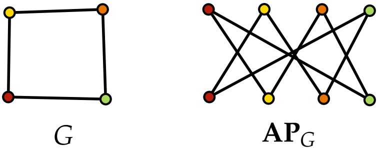
Current best bounds



Nowhere dense cover graphs

Theorem [Joret, Micek, W., 2016]
There are height-2 posets with cover graphs in a **nowhere dense** class $\mathcal{C}$ such that their dimension is unbounded.

Adjacency posets:



Lemma: $\chi(G) \leq \dim(\mathbf{AP}_G)$.

$\mathcal{C}$ = {graphs G with $\Delta(G) \leq \text{girth}(G)$}.

- nowhere dense , unbounded χ
- $\implies$ $\dim(\mathbf{AP}_G)$ is unbounded for $G \in \mathcal{C}$
- $G \in \mathcal{C} \implies$ cover graph of $\mathbf{AP}_G$ in $\mathcal{C}$

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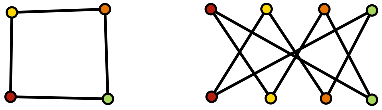


G $\qquad$ $\mathbf{AP}_G$

Lemma: $\chi(G) \leq \dim(\mathbf{AP}_G)$.

$\mathcal{C} = \{\text{graphs } G \text{ with } \Delta(G) \leq \text{girth}(G)\}$.

- has **locally bounded treewidth**, unbounded χ
- $\implies \dim(\mathbf{AP}_G)$ is unbounded for $G \in \mathcal{C}$
- $G \in \mathcal{C} \implies$ cover graph of $\mathbf{AP}_G$ in $\mathcal{C}$

Conjectures

Conjecture
C is a bounded expansion class iff
for each $h \geq 1$, posets of height at most h with cover graphs in C have bounded dimension.

Conjecture
Posets $\mathbf{P}$ of bounded height with cover graphs in a nowhere dense class have dimension

$$\dim(\mathbf{P}) \leq \mathcal{O}(n^{\epsilon}),$$

for each $\epsilon > 0$.

An open dimension problem

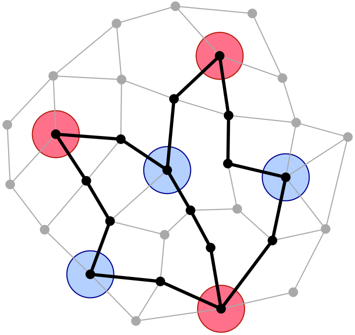
$\mathcal{C}$ does not have bounded expansion

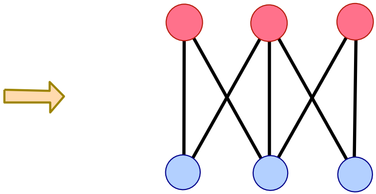
$\Downarrow$

$\exists r$ s.t. r-shallow top. minors have unbounded av. degree

$\Downarrow$

$\exists r$ s.t. bipartite r-shallow top. minors have unbounded av. degree





height-2 poset

An open dimension problem

Problem

Let $\mathcal{P}$ be a class of height-2 posets with unbounded average degree. Is the dimension of subposets of posets in $\mathcal{P}$ necessarily unbounded?

An open dimension problem

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Thank You