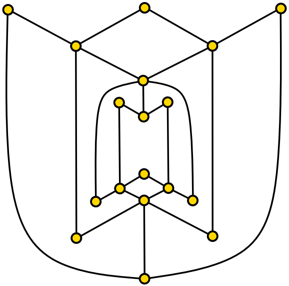
Sparsity and Dimension

Veit Wiechert

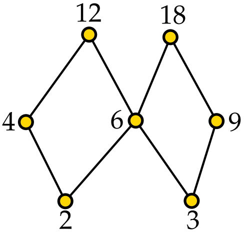
joint work with Gwenaël Joret and Piotr Micek



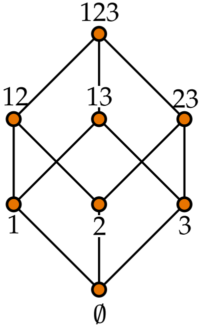
Kolkom 2015, Ilmenau

Introduction

Partial Orders (Posets)

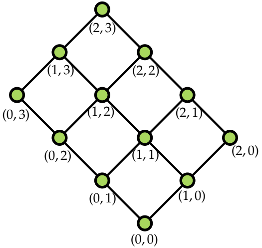


Divisibility order



Boolean lattice B_3

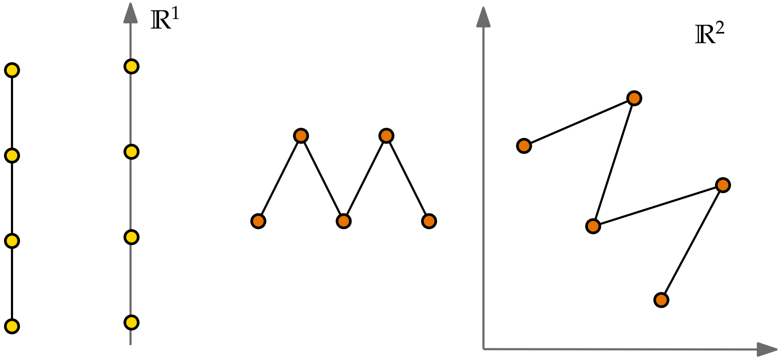
(face lattice of simplex)



Product order

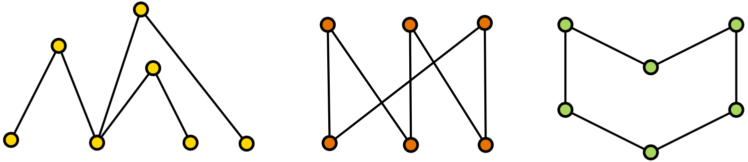
Dimension

The ***dimension*** of a poset **P** is the least d such that **P** is isomorphic to a subposet of $(\mathbb{R}^d, \leq_d)$.



Dimension

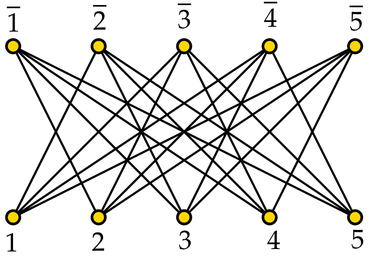
The ***dimension*** of a poset **P** is the least d such that **P** is isomorphic to a subposet of $(\mathbb{R}^d, \leq_d)$.



3-dimensional posets

Standard Examples

S_5



$$\dim(S_n) = n$$

Dimension - Facts

- decid. $\dim(\mathbf{P}) \leq 2$ can be done in poly-time
- decid. $\dim(\mathbf{P}) \leq t$ for fixed $t \geq 3$ is **NPC** **[Yannakakis '82]**
  - → even for height-2 posets **[Felsner, Mustaţă, Pergel]**
- $\dim(\mathbf{P})$ is a hypergraph coloring problem
  - → standard examples correspond to cliques
- S_2-free posets can have arbitrarily large dimension
  - → a.k.a. interval orders

Cover Graphs

Cover Graphs

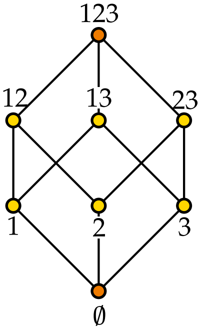
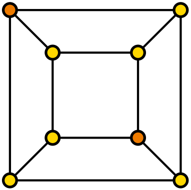
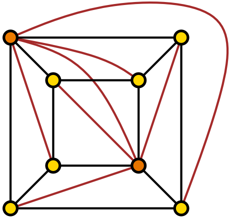


diagram of B_3



cover graph



comparability graph

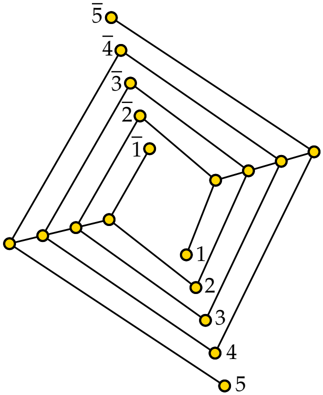
Very sparse Cover Graphs

If the **cover graph** of **P**

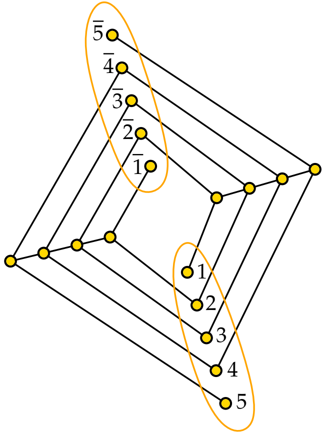
- is a tree, then dim(**P**) ≤ 3. **[Trotter, Moore '77]**
- is outerplanar, then dim(**P**) ≤ 4. **[Felsner, Trotter, W., 12]**
- has pw ≤ 2, then dim(**P**) ≤ 17. **[BKY 14+]**
- has tw ≤ 2, then dim(**P**) ≤ 1276. **[JMTWW 14+]**

Question: Are there constant upper bounds when the cover graph is planar? Or has tw ≥ 3?

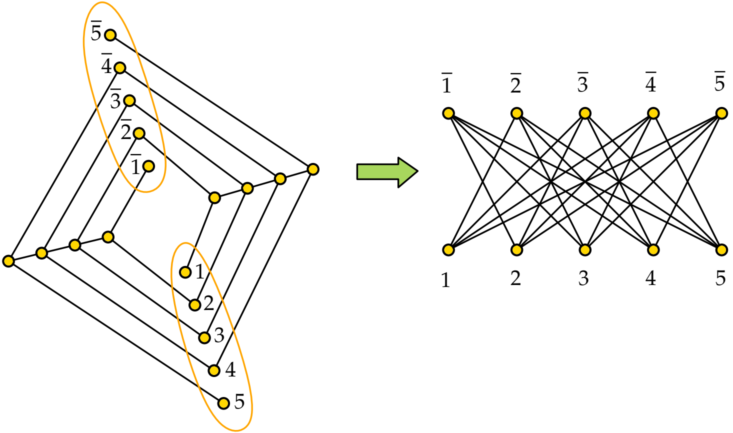
No: Kelly's example '81



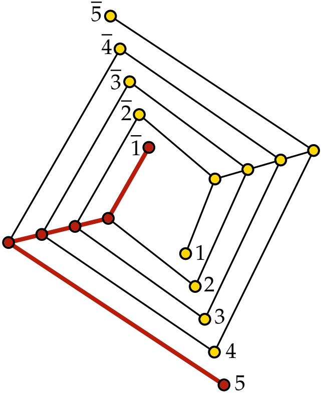
No: Kelly's example '81



No: Kelly's example '81



No: Kelly's example '81



height of **P**:

$h(\mathbf{P})$ = maximum size of a chain in **P**.

Sparse Cover Graphs

Posets of bounded height have bounded dimension if their **cover graphs**

- are planar **[Streib, Trotter]**
- have bounded treewidth **[JMMTWW]**
- exclude some apex as a minor **[JMMTWW]**
- exclude some graph as a minor **[Walczak][Micek, W.]**

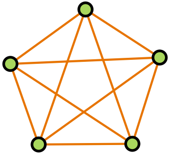
Sparse Cover Graphs

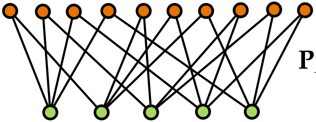
Posets of bounded height have bounded dimension if their **cover graphs**

- are planar **[Streib, Trotter]**
- have bounded treewidth **[JMMTWW]**
- exclude some apex as a minor **[JMMTWW]**
- exclude some graph as a minor **[Walczak][Micek, W.]**

Remark: Graphs in these classes have bounded degeneracy.
But:

K_5





$\mathbf{P}_{K_5}$

$\log\log(n) \leq \dim(\mathbf{P}_{K_n})$ **[Dushnik, Miller '41]**

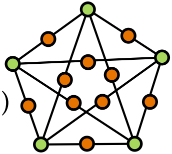
Sparse Cover Graphs

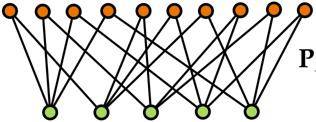
Posets of bounded height have bounded dimension if their **cover graphs**

- are planar **[Streib, Trotter]**
- have bounded treewidth **[JMMTWW]**
- exclude some apex as a minor **[JMMTWW]**
- exclude some graph as a minor **[Walczak][Micek, W.]**

Remark: Graphs in these classes have bounded degeneracy.
But:

cover($\mathbf{P}_{K_5}$)

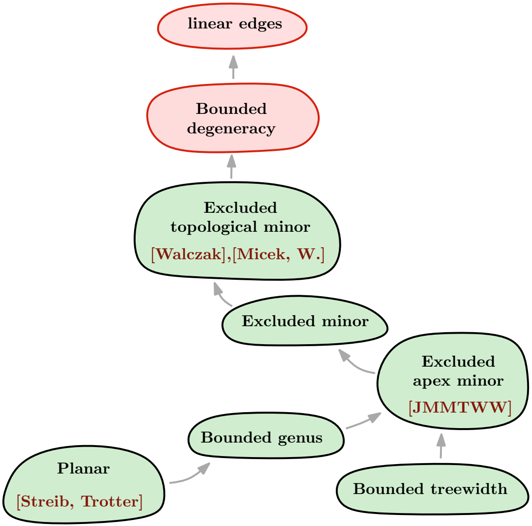




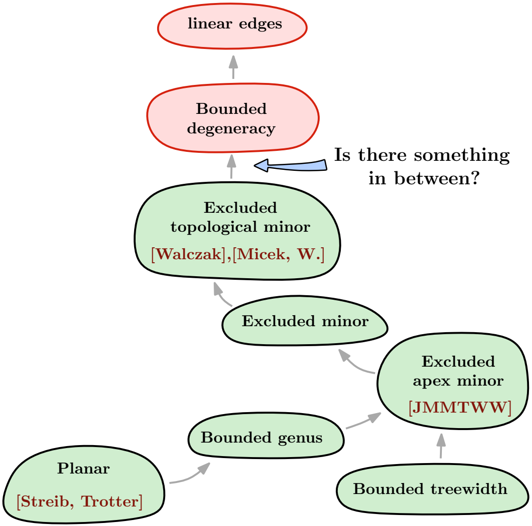
$\mathbf{P}_{K_5}$

$$\log\log(n) \leq \dim(\mathbf{P}_{K_n})$$ **[Dushnik, Miller '41]**

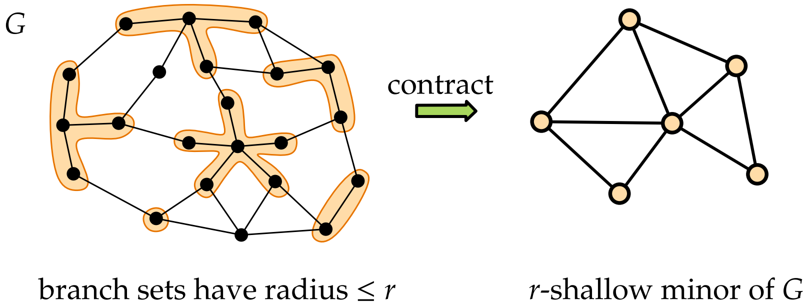
The picture so far



The picture so far

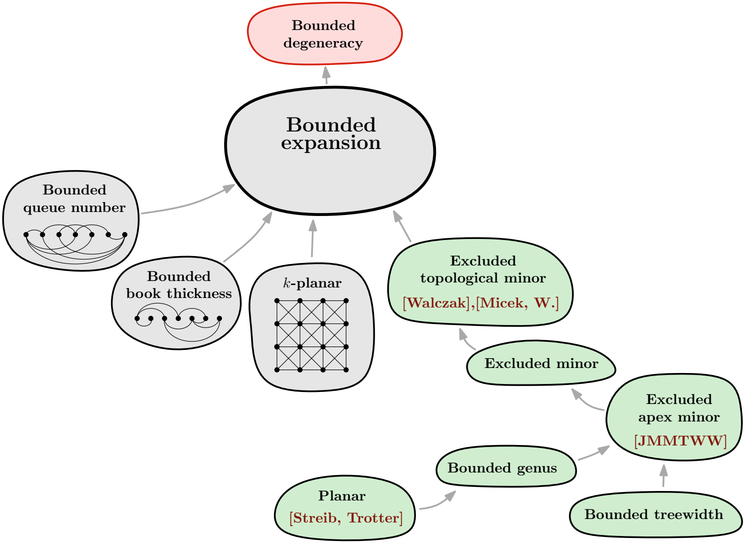


Yes! Classes with Bounded Expansion

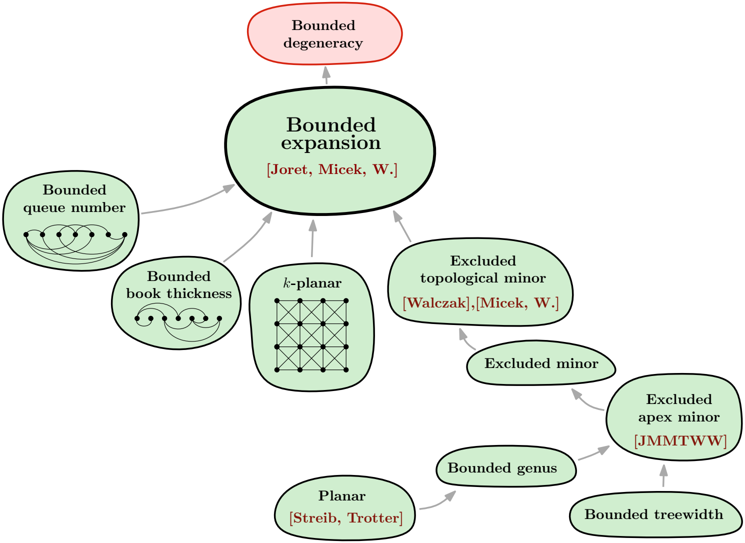


A class C has ***bounded expansion*** if for all $r \geq 0$:
r-shallow minors of $G \in C$ have average degree $\leq f(r)$.

Bounded expansion - Examples



Bounded expansion - Examples



Bounded expansion - Main Theorem

Theorem [Joret, Micek, W., 2015+]
Posets of bounded height whose cover graphs belong to a class with **bounded expansion** have bounded dimension.

Beyond bounded expansion?

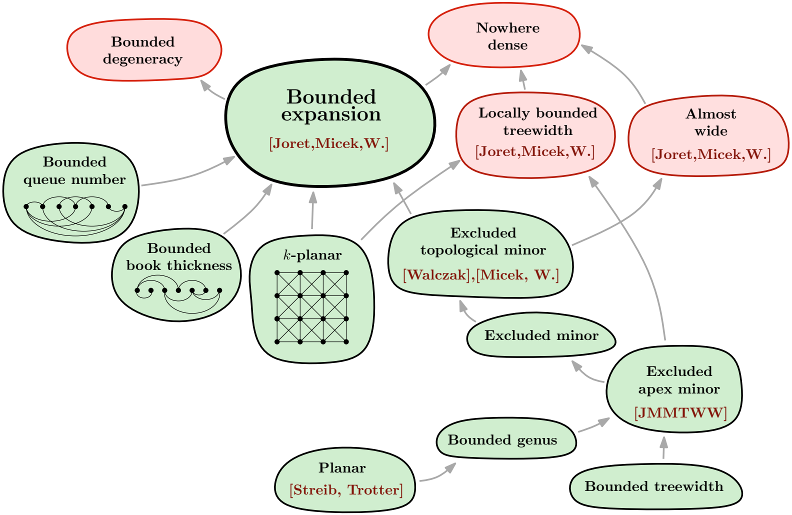
A class C is ***nowhere dense*** if for all $r \geq 0$:
set of r-shallow minors of graphs in $C \neq$ set of all graphs

Examples:

→ graphs with bounded expansion
→ graphs with locally bounded treewidth
→ graphs with $\Delta(G) \leq \text{girth}(G)$

Theorem[Joret, Micek, W., 2015+]
There are height-2 posets with cover graphs in a **nowhere dense** class C such that their dimension is unbounded.

Full Picture



Open Problems

- characterize classes with bounded expansion in terms of dimension
- improve bounds:
  - → Polynomial bound when cover graphs are planar?
  - → Single-exponential when cover graphs have bounded treewidth?
  - → **Theorem[Joret, Micek, W.]** Posets with planar diagrams have $\dim \in O(h)$.
- **Král:** $\forall \epsilon > 0$, $\dim \in O_h(n^{\epsilon})$ when cover graphs in a nowhere dense class?
- Do $\mathbf{k} + \mathbf{k}$-free posets with sparse cover graphs have bounded dim?
  - → **Theorem[HSTWW]** Yes, when cover graphs are planar.

Open Problems

- characterize classes with bounded expansion in terms of dimension
- improve bounds:
  - → Polynomial bound when cover graphs are planar?
  - → Single-exponential when cover graphs have bounded treewidth?
  - → **Theorem[Joret, Micek, W.]**
    Posets with planar diagrams have dim $\in O(h)$.
- **Král:** $\forall \epsilon > 0$, dim $\in O_h(n^\epsilon)$ when cover graphs in a nowhere dense class?
- Do $\mathbf{k} + \mathbf{k}$-free posets with sparse cover graphs have bounded dim?
  - → **Theorem[HSTWW]**
    Yes, when cover graphs are planar.

Thank You