

The Dimension of Posets with Planar Cover Graphs

June 13, 2012

Co-authors

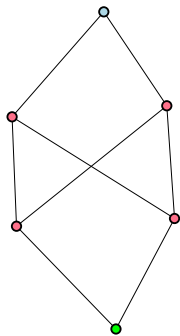


William T. Trotter



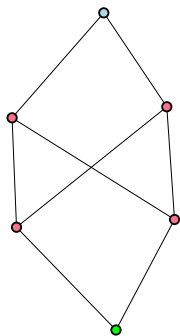
Stefan Felsner

Drawing Posets

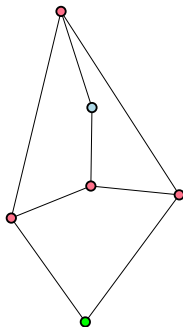


Diagram

Drawing Posets

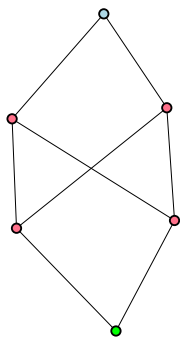


Diagram

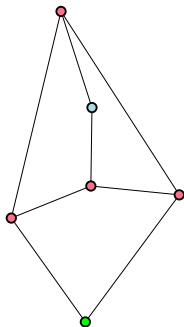


Cover Graph

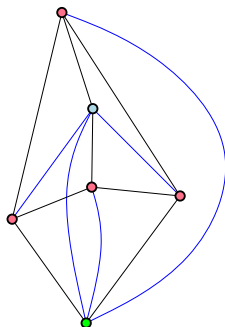
Drawing Posets



Diagram



Cover Graph



Comparability Graph

Dimension and the Standard Example S_n

$R = \{L_1, \dots, L_k\}$ is called a *realizer* if

$$\mathbf{P} = \bigcap_{L \in R} L$$

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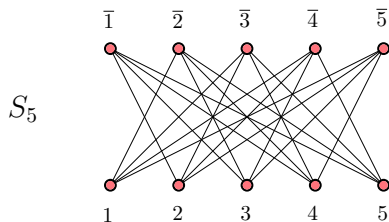
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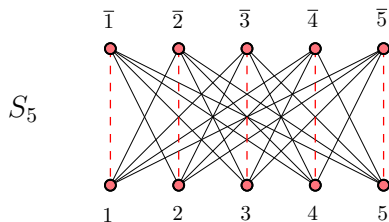


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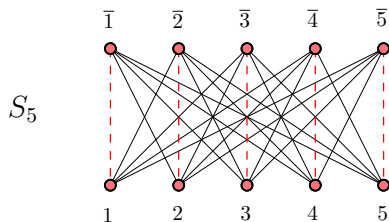


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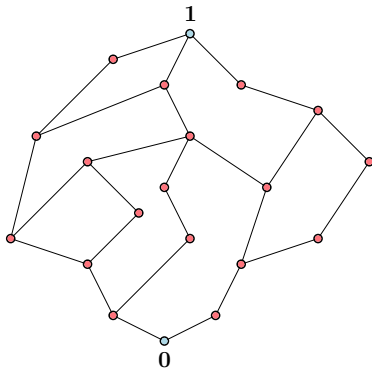


$$\dim(S_n) = n$$

Theorem (Baker, Fishburn, Roberts '71)

If \mathbf{P} is planar and has a zero and a one, then

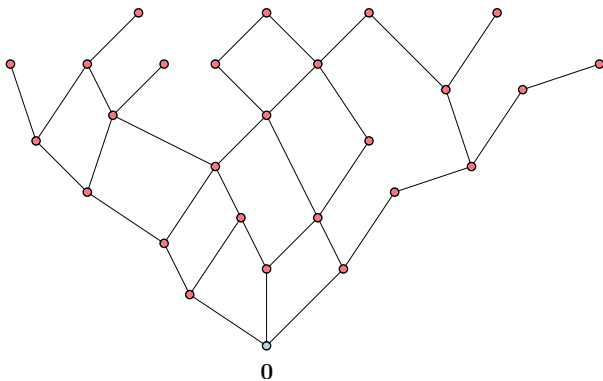
$$\dim(\mathbf{P}) \leq 2$$



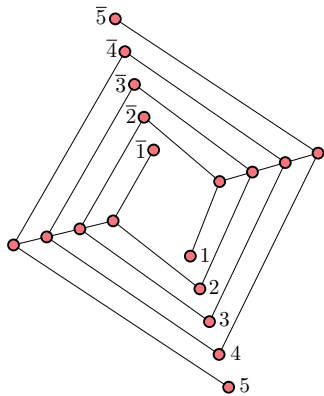
Theorem (Trotter, Moore '77)

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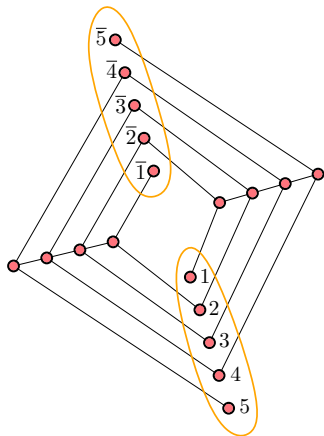
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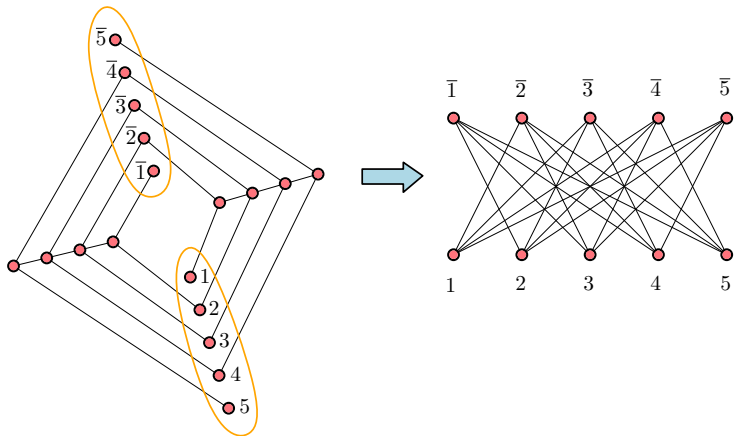
Kelly's example '81



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Which Conditions bound the Dimension?

Theorem (Felsner, Li, Trotter 2010)

If \mathbf{P} has a planar cover graph and $h(\mathbf{P}) \leq 2$ then

$$\dim(\mathbf{P}) \leq 4$$

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Theorem (Streib, Trotter 2011)

For every h there exist a constant c_h , s.t. if \mathbf{P} has height h and a planar cover graph, then

$$\dim(\mathbf{P}) \leq c_h$$

Boxicity and Poset Dimension

$\text{box}(G) := \min k$ such that $\exists k$ -box representation of G .

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If G_P is the comparability graph of P , then

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Corollary

If G_P is planar, then

$$\dim(P) \leq 6$$

Our Results

Theorem

If $G_{\mathbf{P}}$ is planar, then

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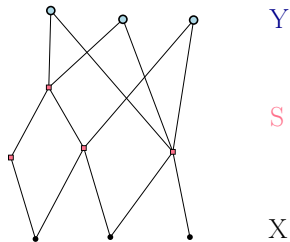
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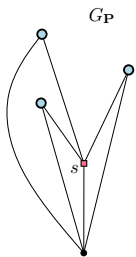
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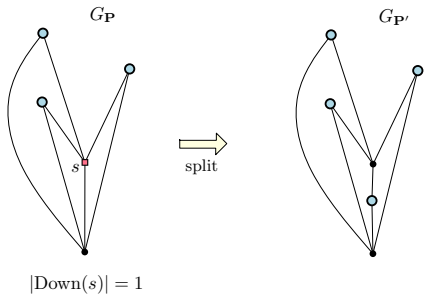
$$\dim(\mathbf{P}) \leq 4$$

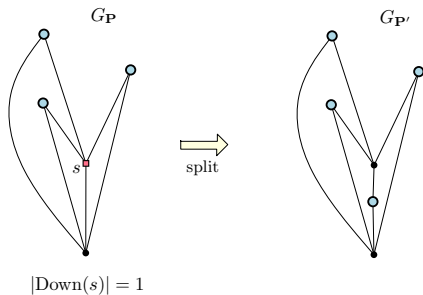
- ▶ case $h(\mathbf{P}) > 4$: $G_{\mathbf{P}}$ can not be planar
- ▶ case $h(\mathbf{P}) \leq 2$: Apply (Felsner, Li, Trotter)
- ▶ case $h(\mathbf{P}) = 3$ or 4 :



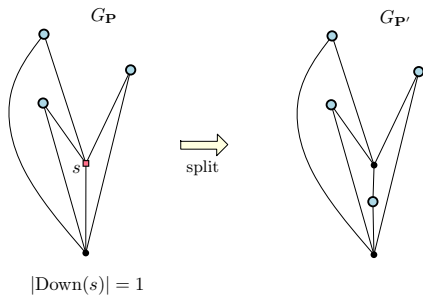


$$|\text{Down}(s)| = 1$$

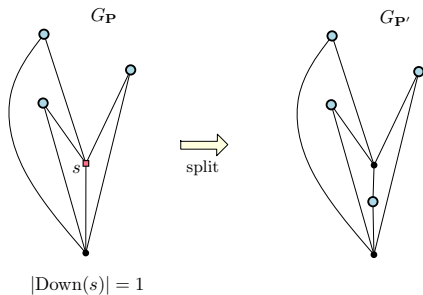




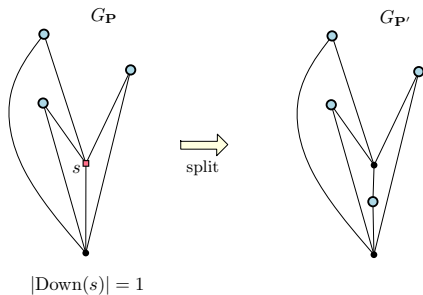
- ▶ $G_{P'}$ is planar



- ▶ $G_{\mathbf{P}'}$ is planar
- ▶ $\dim(\mathbf{P}) \leq \dim(\mathbf{P}')$



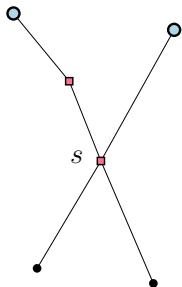
- ▶ $G_{\mathbf{P}'}$ is planar
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- ▶ $G_{P'}$ is planar
- ▶ $\dim(\mathbf{P}) \leq \dim(\mathbf{P}')$
- ▶ similar for the case: $|Up(s)| = 1$
- ▶ this leads to the case:

$$|Down(s)| \geq 2 \quad \text{and} \quad |Up(s)| \geq 2 \quad \text{for every } s \in S$$

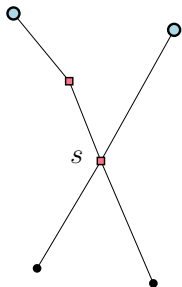
Diagram



$$|\text{Down}(s)| = 2$$

$$|\text{Up}(s)| = 3$$

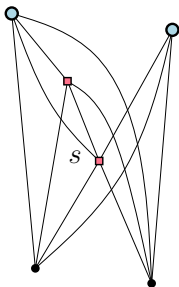
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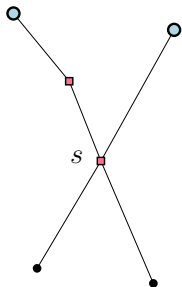
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Comparability Graph



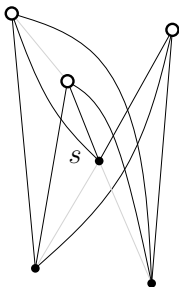
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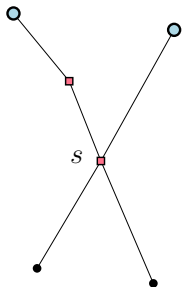
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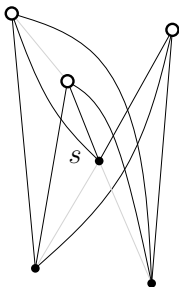
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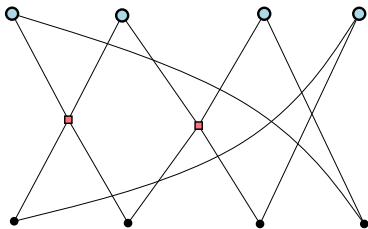
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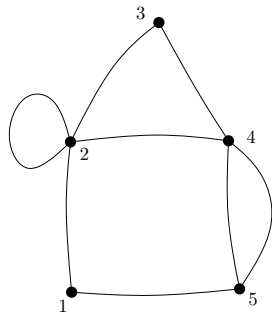
$\Rightarrow G_{\mathbf{P}}$ is not planar

Final Case



$$|\text{Down}(s)| = |\text{Up}(s)| = 2 \quad \text{for every } s \in S$$

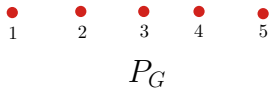
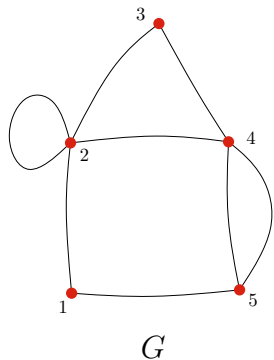
Brightwell-Trotter-Theorem '96



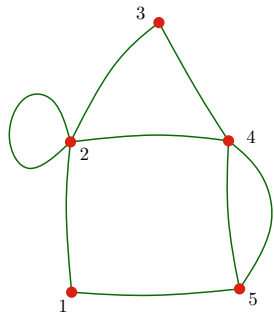
G

P_G

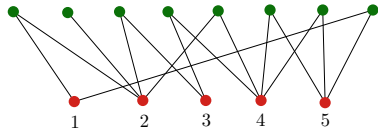
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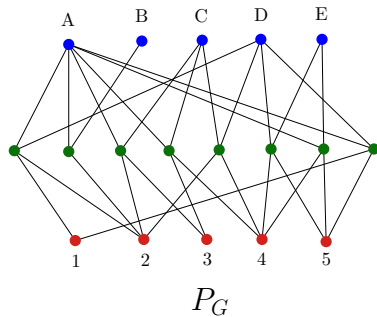
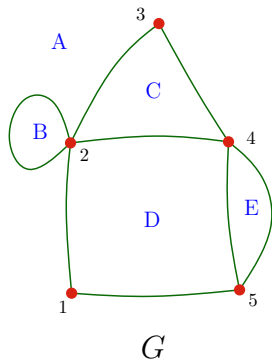


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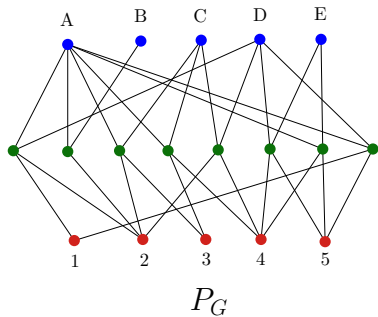
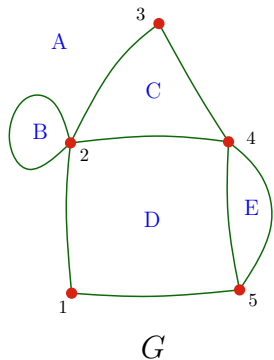


P_G

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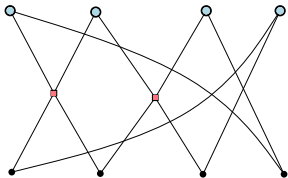


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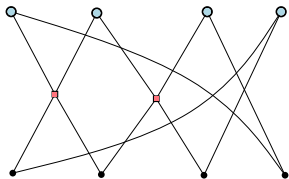


$$\dim(P_G) \leq 4$$

Posets with Planar Comparability Graphs



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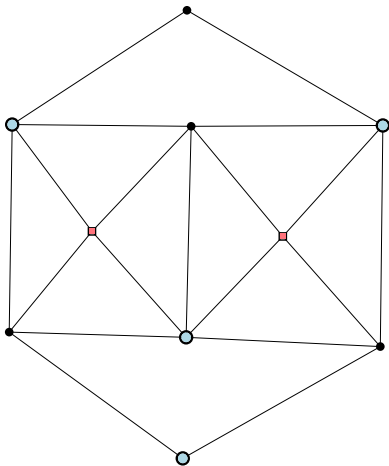


• \longleftrightarrow vertices

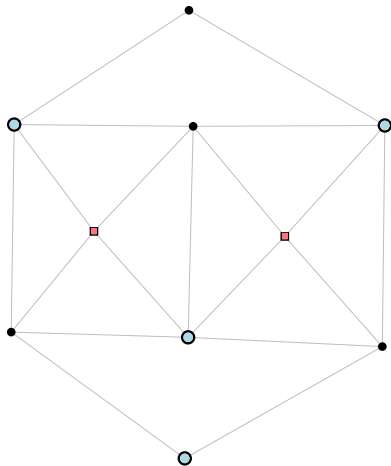
■ \longleftrightarrow edges

○ \longleftrightarrow faces

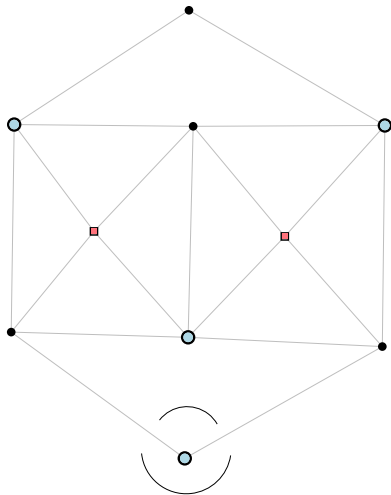
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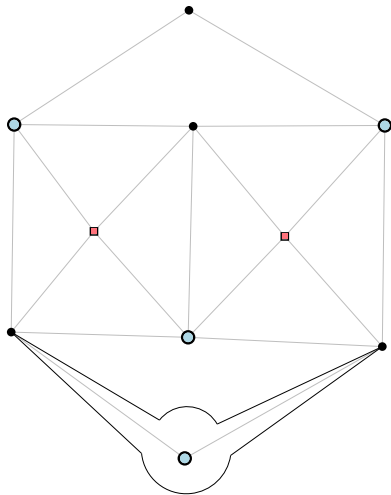
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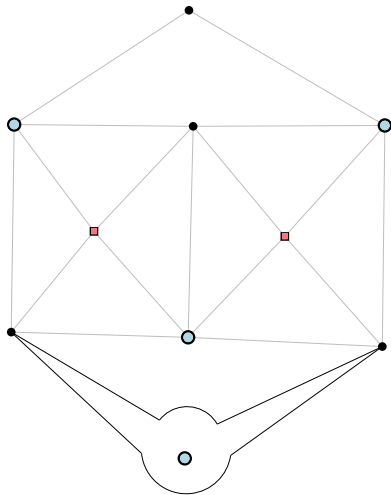
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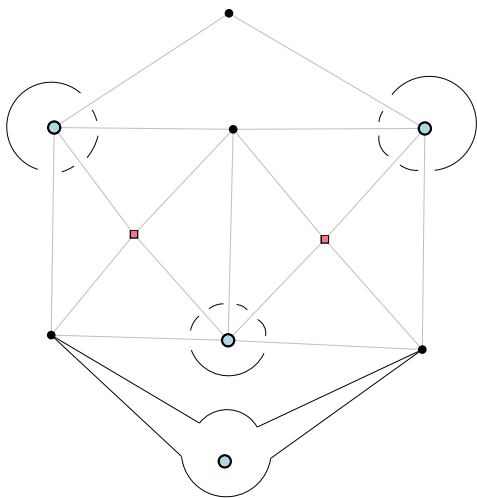
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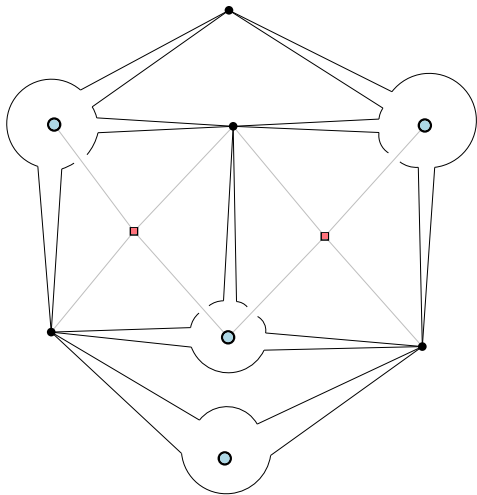
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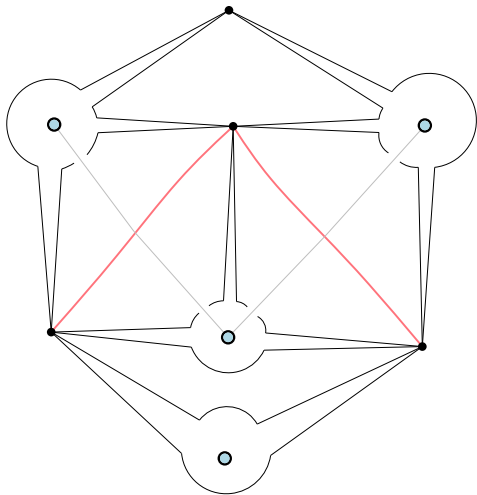
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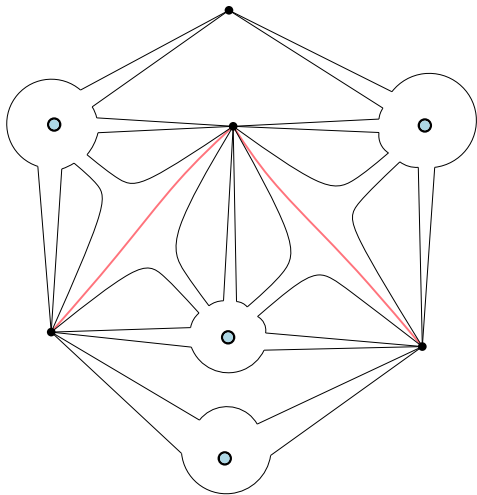
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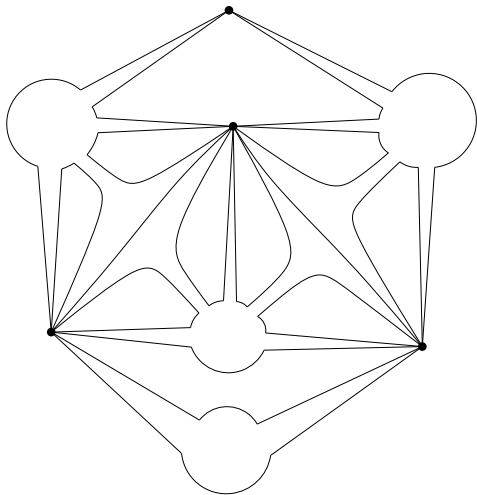
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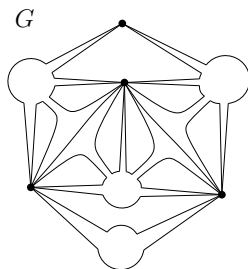
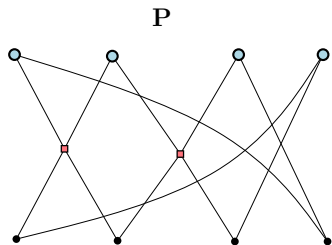
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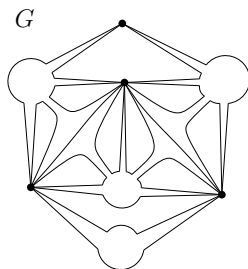
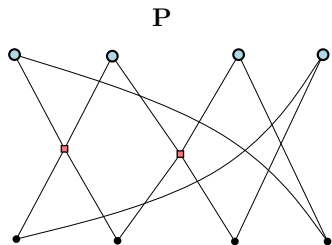


Posets with Planar Comparability Graphs



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$$\Rightarrow \dim(\mathbf{P}) = \dim(\mathbf{P}') \leq \dim(\mathbf{P}_G) \leq 4$$

(BTT)

Posets with Outerplanar Cover Graphs

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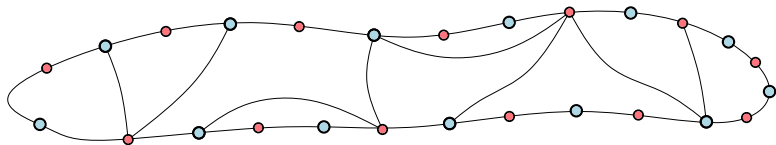
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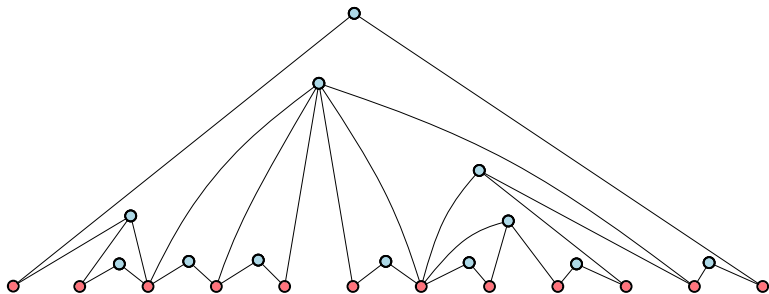
$$\dim(\mathbf{P}) \leq 3$$

Fact: Both inequalities are best possible

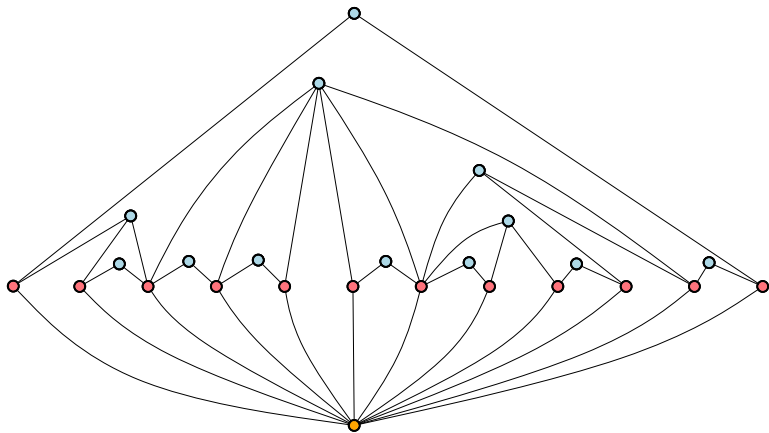
Outerplanar Cover Graph and $h(\mathbf{P}) \leq 2$



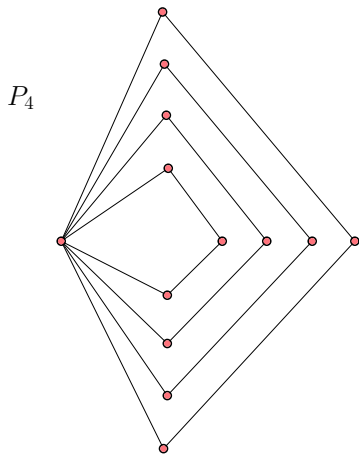
Outerplanar Cover Graph and $h(\mathbf{P}) \leq 2$



Outerplanar Cover Graph and $h(\mathbf{P}) \leq 2$



Lower Bound



If $n \geq 17$ then $\dim(P_n) = 4$

Open Problems

- ▶ better bounds for $\dim(\mathbf{P})$ when \mathbf{P} is planar and $h(\mathbf{P}) \leq k$

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Open Problems

- ▶ better bounds for $\dim(\mathbf{P})$ when \mathbf{P} is planar and $h(\mathbf{P}) \leq k$
- ▶ is it *NP*-complete to decide whether \mathbf{P} is a subposet of a poset with a planar cover graph?
- ▶ Are there t_n s.t. if \mathbf{P} is planar with $\dim(\mathbf{P}) \geq t_n$ then \mathbf{P} contains S_n as a subposet?

Thank you for your attention

Bonusmaterial: nonplanar Poset with outerplanar Cover Graph

