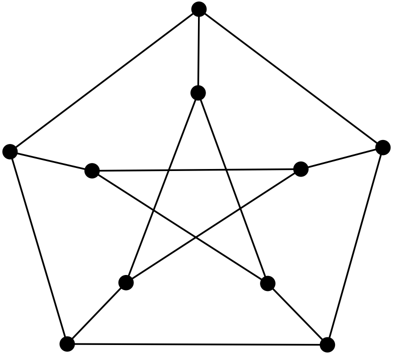
On-line competitive algorithms for coloring bipartite graphs

Veit Wiechert

joint work with Piotr Micek

MDS-Colloquium, Feb. 2015

Off-line Colorings



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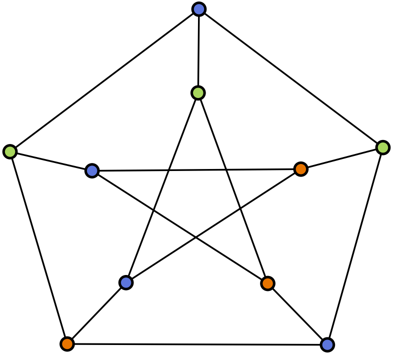

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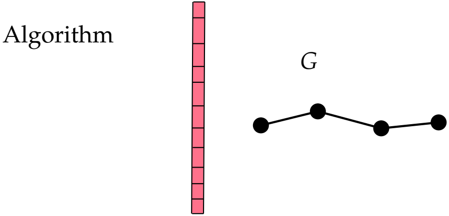

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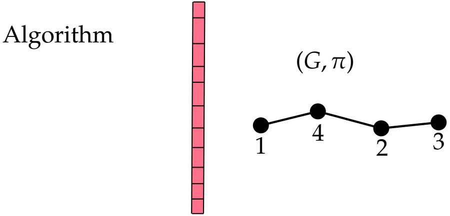

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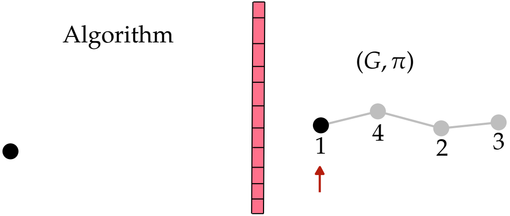

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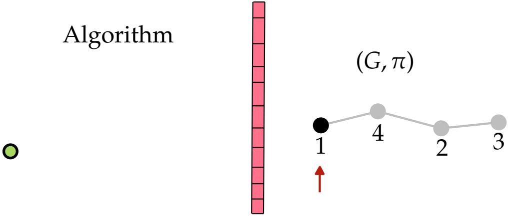

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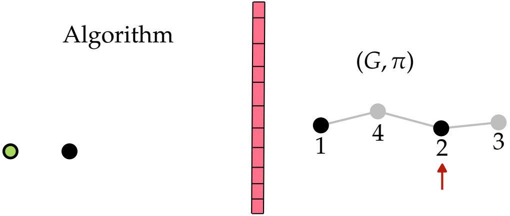

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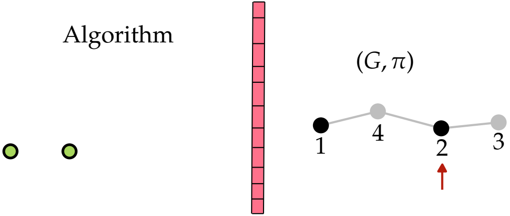

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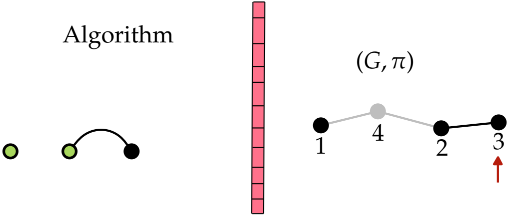

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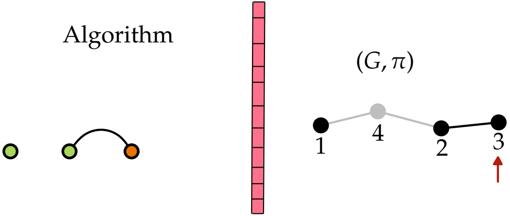

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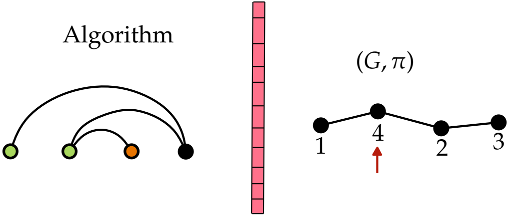

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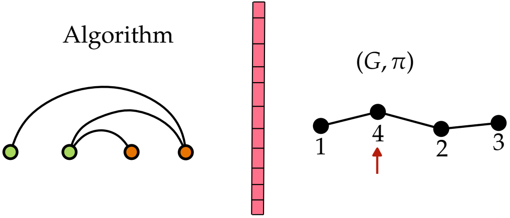

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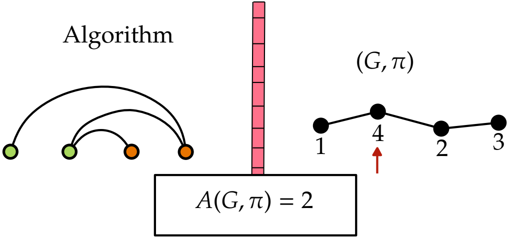

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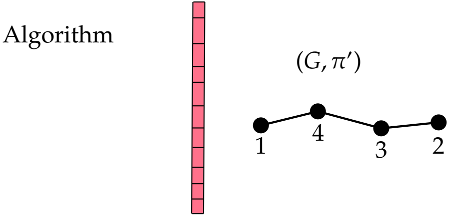

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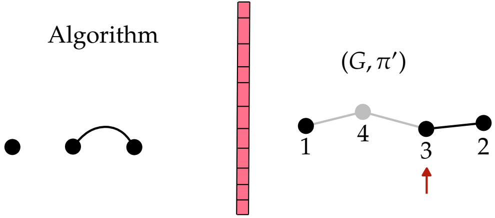

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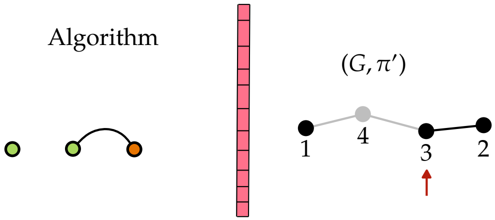

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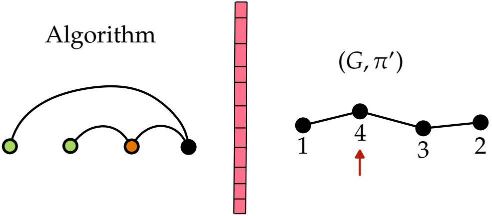

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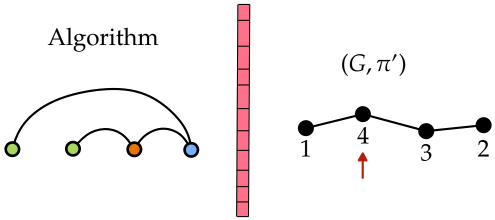

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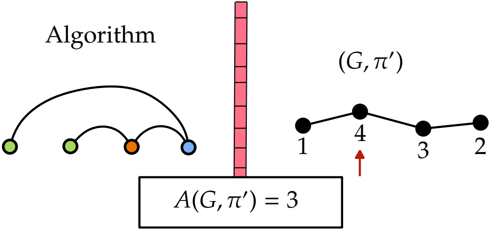

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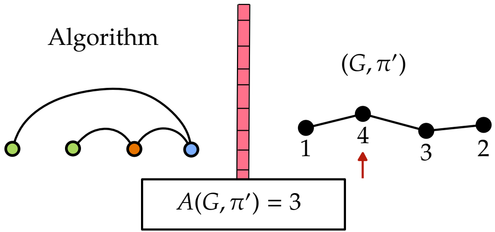

The on-line chromatic number of A on G is defined as

$$A(G) = \max_{\pi} A(G, \pi).$$

On-line Colorings

Folklore:

- there is a forest on n vertices such that any on-line algorithm uses at least $\lfloor \log n \rfloor + 1$ colors in the worst case.
- *First Fit* uses at most $\lfloor \log n \rfloor + 1$ colors on each forest with n vertices.

The on-line chromatic number $\chi^*(G)$ of a graph G is defined as

$$\chi^*(G) = \min_A A(G).$$

An on-line algorithm A is on-line competitive for a class $\mathcal{G}$ of graphs if there is a function f s.t. for each $G \in \mathcal{G}$

$$A(G) \leq f(\chi^*(G)).$$

Example: *First Fit* is on-line competitive on forests

On-line Colorings

Open Problem: Is there an on-line competitive algorithm for bipartite graphs?

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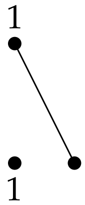

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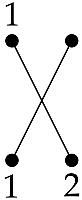

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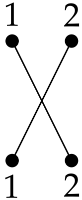

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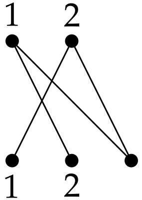

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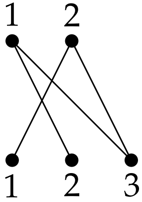

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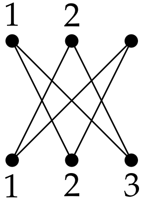

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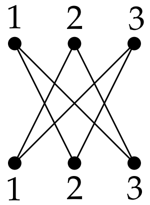

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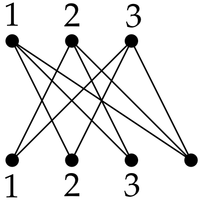

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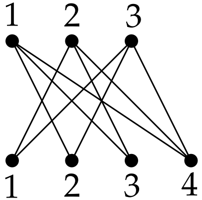

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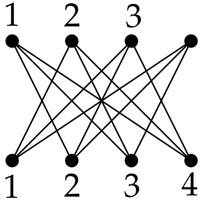

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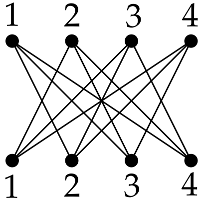

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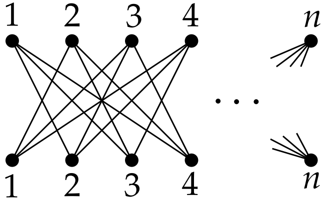

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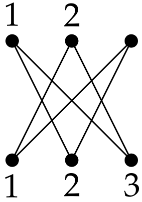

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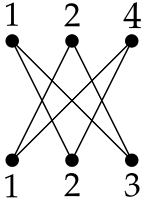

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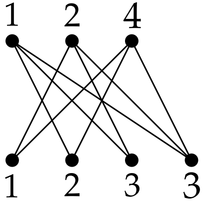

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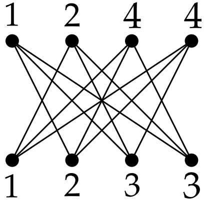

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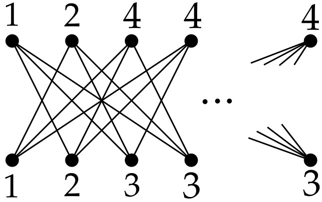


On-line Colorings

Kierstead, Penrice, Trotter '95: There is an on-line algorithm A s.t. for each P_5-free graph G

$$A(G) \leq 4^{\chi(G)}.$$

Gyárfás, Lehel '88: In the worst-case any on-line algorithm uses an abitrary number of colors on P_6-free bipartite graphs.

Broersma, Capponi, Paulusma 2008: There is an on-line competitive algorithm using at most

$$8\chi^*(G) + 8$$

colors on P_7-free bipartite graphs.

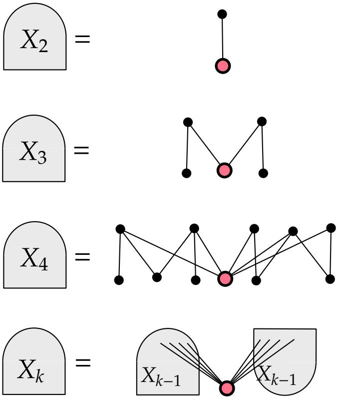
Our Results

Theorem (Micek, W., 2014)

There are on-line coloring algorithms using at most

- $4\chi^*(G) - 2$ colors on each P_7-free bipartite graph G,
- $3(\chi^*(G) + 1)^2$ colors on each P_8-free bipartite graph G,
- $3(2\chi^*(G) + 1)^2$ colors on each P_9-free bipartite graph G.

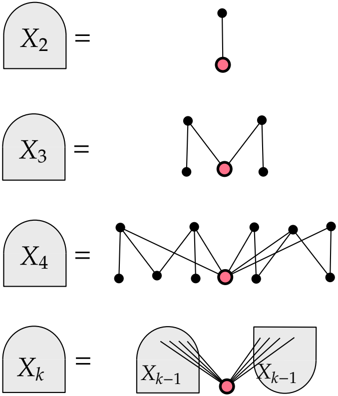
Universal Structure



Universal Structure

Theorem (Broersma, Capponi, Paulusma 2006)

If G contains X_k then $\chi^*(G) \geq k$.



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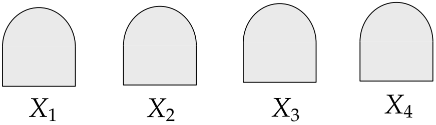

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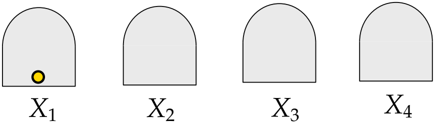

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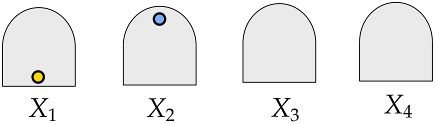

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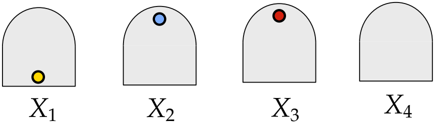

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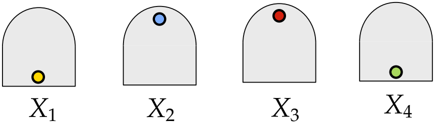

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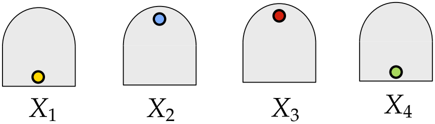

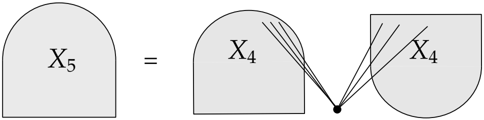

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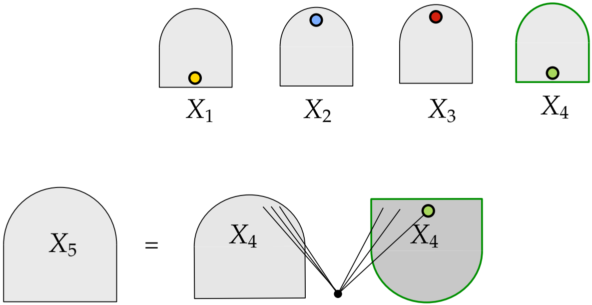

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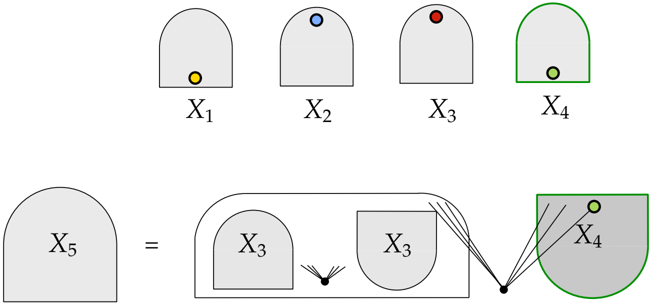

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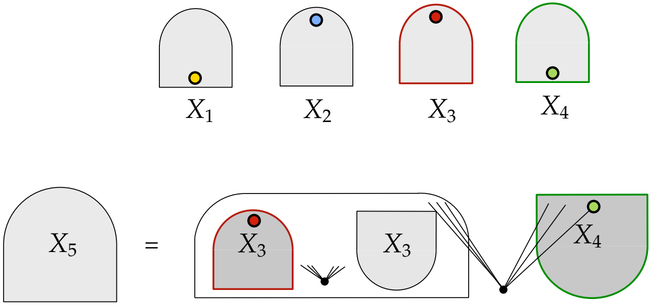

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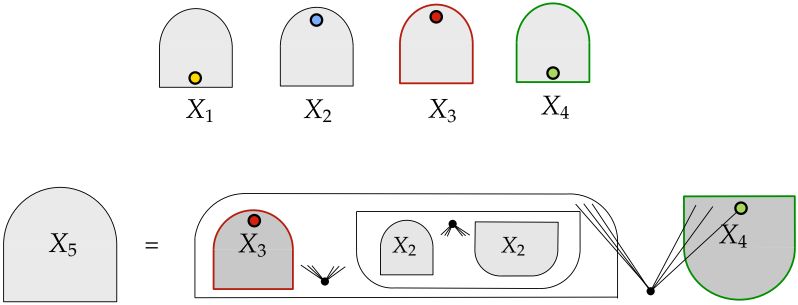

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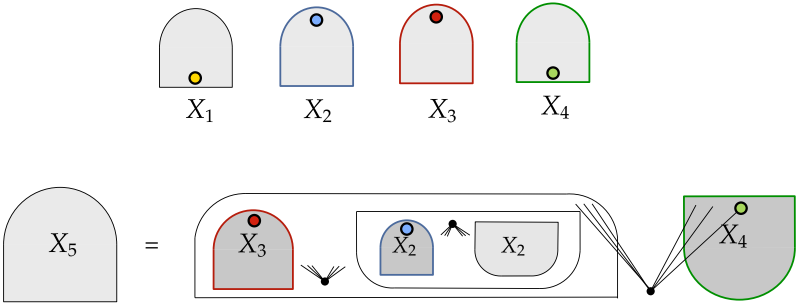

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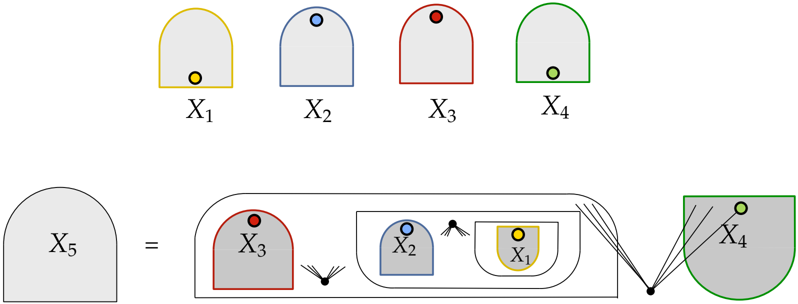

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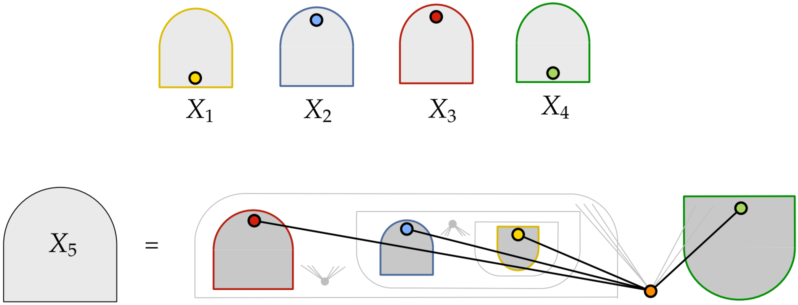
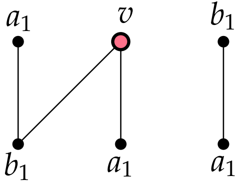


BicolorMax algorithm (Broersma et al.)

- Two palettes of colors: $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$.
- Let C be a connected induced subgraph of G. Color a_i is mixed in C if a_i is used on both sides of C.
- $G(v)$ = graph spanned by v and vertices presented before v.
- $C(v)$ = connected component of $G(v)$ that contains v



BicolorMax algorithm (Broersma et al.)

Algorithm: BicolorMax

Vertex v is introduced
$m \leftarrow \max\{\, i \geq 1 \mid a_i \text{ is mixed in } C(v)\} + 1$ // $\max\{\} := 0$
let I_1, I_2 be the sides of $C(v)$ such that $v \in I_1$
if $a_m \in \mathrm{col}(I_2)$ **then** $\mathrm{col}(v) = b_m$
else $\mathrm{col}(v) = a_m$

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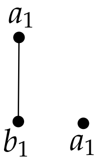

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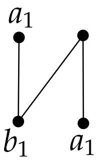

BicolorMax algorithm (Broersma et al.)

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Vertex v is introduced
$m \leftarrow \max\{\, i \geq 1 \mid a_i \text{ is mixed in } C(v)\} + 1$ // $\max\{\} := 0$
let I_1, I_2 be the sides of $C(v)$ such that $v \in I_1$
if $a_m \in \mathrm{col}(I_2)$ **then** $\mathrm{col}(v) = b_m$
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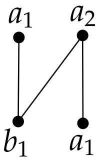

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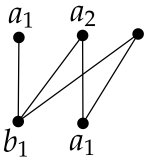

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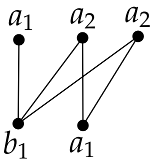

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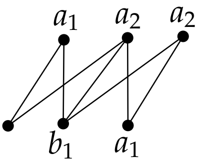

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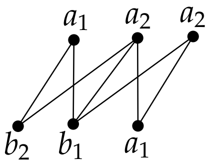

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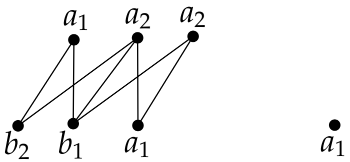

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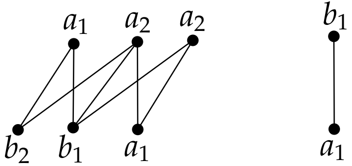

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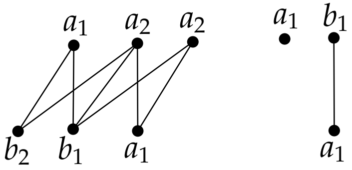

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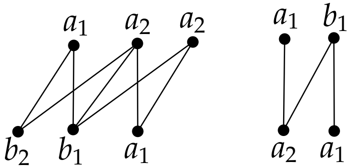

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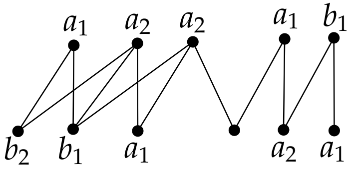

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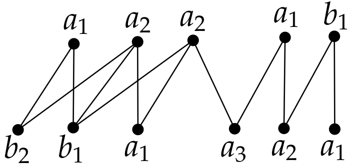
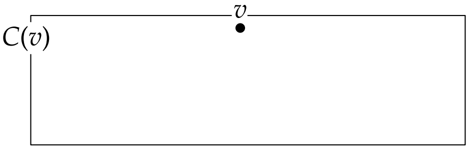


BicolorMax algorithm (Broersma et al.)

- suppose v is the first vertex with $\mathrm{col}(v) = a_m$



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BicolorMax algorithm (Broersma et al.)

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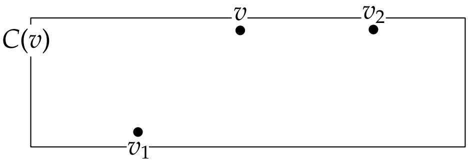

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- suppose v is the first vertex with $\text{col}(v) = a_m$
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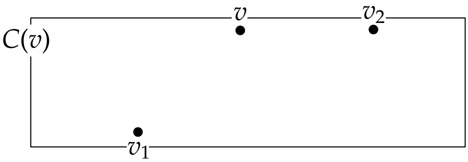
- each v_1-v_2-path in $C(v)$ contains v



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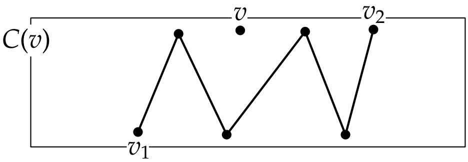

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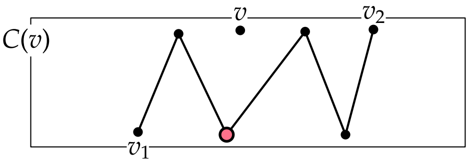

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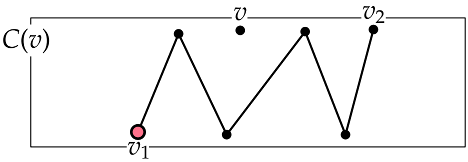

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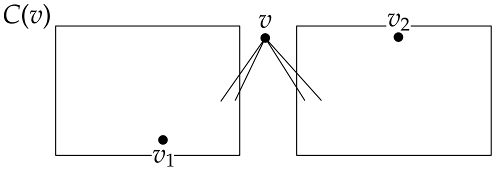
- color a_{m-1} must be mixed in $C(v)$
- each v_1-v_2-path in $C(v)$ contains v
- v is joining connected components



$\text{col}(v) = a_m$

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Claim

Let G be a P_6-free bipartite graph. If $v \in G$ is the first vertex in $C(v)$ with $\mathrm{col}(v) = a_m$, then $C(v)$ contains X_m such that v is the root of X_m.

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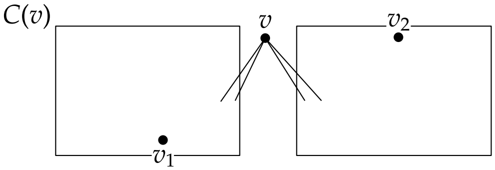

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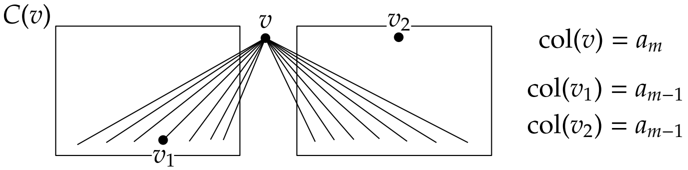

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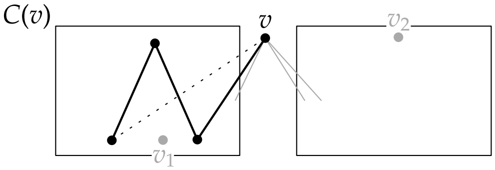

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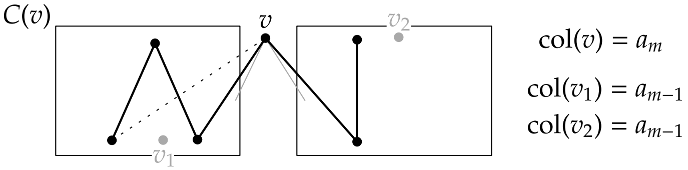

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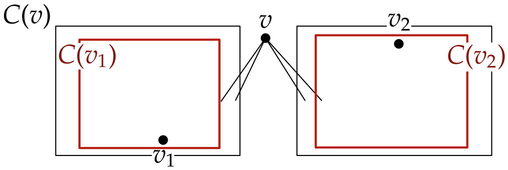

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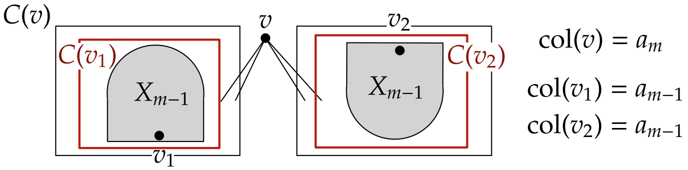
- apply induction to v_1 and v_2, respectively.



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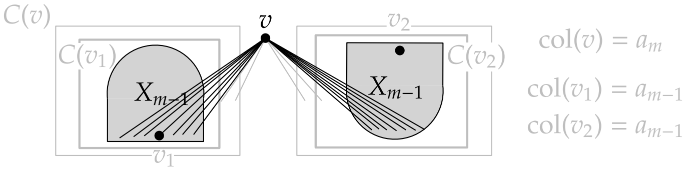

Theorem (Broersma et al.)
BicolorMax uses at most $2\chi^*(G)$ colors on each P_6-free bipartite graph G.

- let $m = \max\{ i \geq 1 \mid \text{there is } v \in G \text{ with } \mathrm{col}(v) = a_i \}$
- so at most $2m$ colors are used
- Claim $\implies$ G contains X_m $\implies$ $\chi^*(G) \geq m$

Claim

Let G be P_7-free bipartite graph. If $v \in G$ is the first vertex in $C(v)$ with $\mathrm{col}(v) = a_{2m}$, then $C(v)$ contains X_{m+1} such that v is the root of X_{m+1}.

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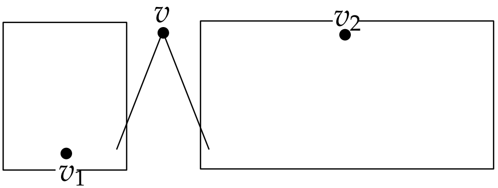

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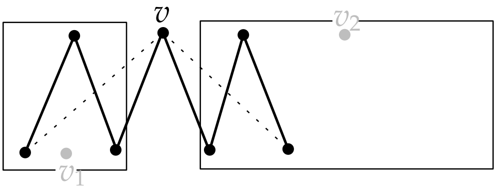

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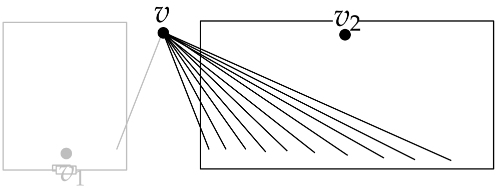

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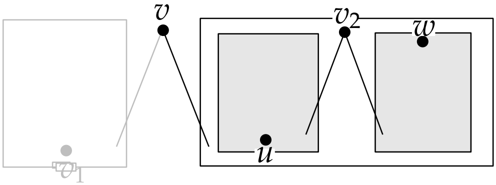

$$\mathrm{col}(v) = a_{2m}$$
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$$\mathrm{col}(w) = a_{2m-2}$$

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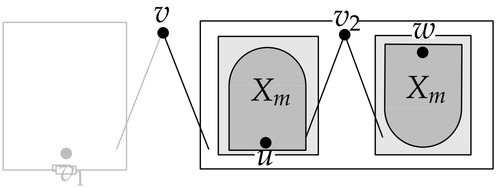

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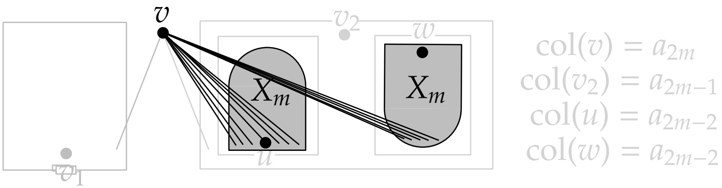
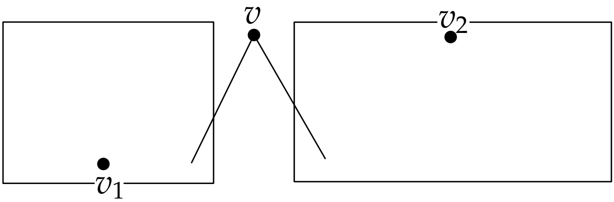


P_7-free bipartite graphs

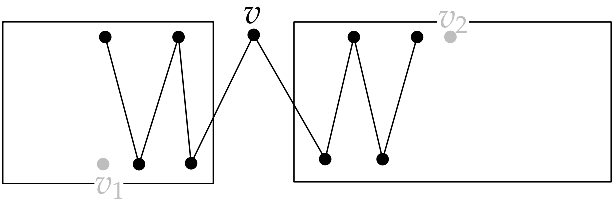
Theorem (Micek, W., 2014)
BicolorMax uses at most $4\chi^*(G) - 2$ colors on each P_7-free bipartite graph G.

- let $m = \max\{ i \geq 1 \mid \text{there is } v \in G \text{ with } \mathrm{col}(v) = a_{2i}\}$
- so at most $4m + 2$ colors are used
- Claim $\implies$ G contains X_{m+1} $\implies$ $\chi^*(G) \geq m + 1$

P_9-free bipartite graphs



P_9-free bipartite graphs



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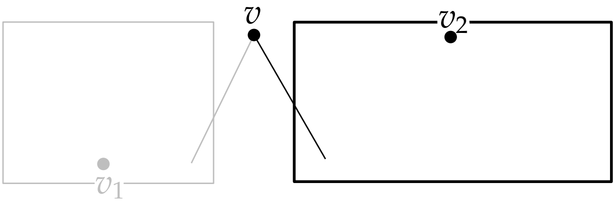
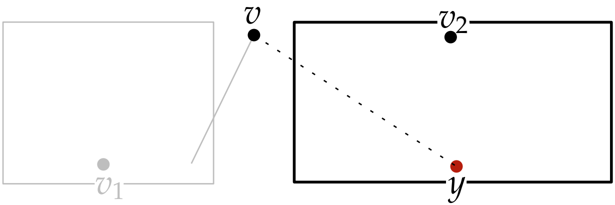
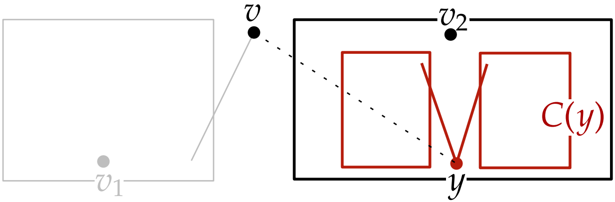
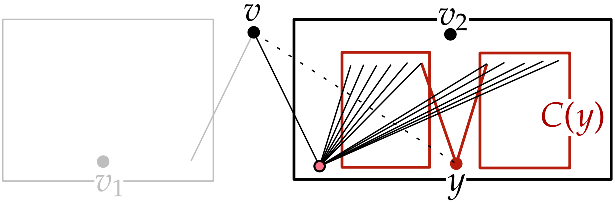


P_9-free bipartite graphs




v
v_2
v_1
y
$C(y)$

P_9-free bipartite graphs



P_8- and P_9-free bipartite graphs

Theorem (Micek, W., 2014)
*BicolorMax** uses at most $3(\chi^*(G) + 1)^2$ colors on each P_8-free bipartite graph G.

Theorem (Micek, W., 2014)
*BicolorMax** uses at most $3(2\chi^*(G) + 1)^2$ colors on each P_9-free bipartite graph G.

Open Problems

Does there exist an on-line competitive algorithm for coloring

- P_k-free bipartite graphs?
- bipartite graphs?
- arbitrary graphs?

Thank You