Balancing Pairs for Perfect Elimination Orderings

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Kolkom 2013, Ilmenau
Let $\pi$ be a hidden linear order on the set $\{1, \ldots, n\}$.

Task: Find $\pi$ by asking questions like

Is $i < j$ in $\pi$?

$f(n) := \# \text{ needed questions.}$
Sorting with Comparisons

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**Task:** Find $\pi$ by asking questions like

$$
\text{Is } i < j \text{ in } \pi? 
$$

$f(n) := \# \text{ needed questions.}$

**Fact:**

$$
f(n) = \Theta(\log(n!))
$$

lower bound: Information theoretic argument
upper bound: Binary Insertion Sort
Sorting under Partial Information

Assume we are given some partial information on $\pi$:

- $2 < 4$
- $2 < 3$
- $4 < 1$
- $5 < 4$
- $5 < 3$

possible linear orders:

- $2 < 5 < 4 < 1 < 3$
- $2 < 5 < 4 < 3 < 1$
- $2 < 5 < 3 < 4 < 1$
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\begin{align*}
2 < 5 < 4 < 1 < 3 \\
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2 < 5 < 3 < 4 < 1 \\
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5 < 2 < 3 < 4 < 1
\end{align*}

Poset

Linear Extensions
Query Complexity for Posets

\( e(P) := \# \text{ linear extensions of } P \).
\( f(P) := \# \text{ questions needed.} \)

\[ f(P) = \Theta(\log(e(P)))? \]
Balancing Pairs

Definition
\[ \mathbb{P}(x \prec y) := \frac{\text{#lin. extensions with } x \prec y}{\text{#lin. extensions}} \]

Definition
A poset \( P \) is called \( \delta \)-balanced for \( 0 < \delta \leq 0.5 \), if there are elements \( x, y \) in \( P \) with
\[ \delta \leq \mathbb{P}(x \prec y) \leq 1 - \delta \]
Balancing Pairs

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Observation

*If every poset (not a chain) is \( \delta \)-balanced, then*

\[ f(P) = O(\log(e(P))) \]
The Balancing Pair Conjecture

Conjecture (Kislitsyn '68)

Every finite poset $P$ (not a chain) is $\frac{1}{3}$-balanced, i.e. $\exists x, y \in P$ such that

$$\frac{1}{3} \leq P(x \prec y) \leq \frac{2}{3}$$

Best possible:
Every poset is

- $\frac{1}{3}$-balanced (Conjecture)
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- $\frac{3}{11}$-balanced (Kahn, Saks '84)
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- $\frac{3}{11}$-balanced (Kahn, Saks ’84)
- $\frac{5-\sqrt{5}}{10}$-balanced (Brightwell, Felsner, Trotter ’95)

→ both using Aleksandrov-Fenchel Inequalities
Every poset is

\[ \frac{1}{3} \]-balanced (Conjecture)
\[ \frac{3}{11} \]-balanced (Kahn, Saks ’84)
\[ \frac{5-\sqrt{5}}{10} \]-balanced (Brightwell, Felsner, Trotter ’95)
\[ \frac{1}{2e} \]-balanced (Kahn, Linial ’91)

\[ \rightarrow \] both using Aleksandrov-Fenchel Inequalities
\[ \rightarrow \] using Brunn-Minkowski
The Conjecture is true for...

width 2 posets
(Linial '84)
The Conjecture is true for...

semiorders
(Brightwell ’89)
Antimatroids

A finite setsystem $\mathcal{A}$ of finite sets is called an **antimatroid** if

- $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
- $\emptyset \neq A \in \mathcal{A} \implies \exists x \in A : A \setminus \{x\} \in \mathcal{A}$
Downsets of a poset form an antimatroid

Definition
A subset $D$ of $P$ is called a *downset* if for all $x \in D$

$$y \leq_{P} x \implies y \in D$$
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Basic Words

$M_{\mathfrak{A}} = \text{groundset of setsystem } \mathfrak{A}$

$n = |M_{\mathfrak{A}}|$

Definition

$w = (w_1, w_2, \ldots, w_n)$ is called a basic word if

$$\{w_1, \ldots, w_k\} \in \mathfrak{A} \quad \text{for all } k \geq 1$$

Observation

basic words of $\mathfrak{A} \leftrightarrow$ maximal chains in $(\mathfrak{A}, \subseteq)$

basic words in $D(P) \leftrightarrow$ linear extensions of $P$
Conjecture (Eppstein 2013)

Every antimatroid $\mathcal{A}$ (with more than 1 basic word) is $\frac{1}{3}$-balanced, i.e. $\exists x, y \in M_\mathcal{A}$ s.t.

$$\frac{1}{3} \leq \mathbb{P}(x \prec y) \leq \frac{2}{3}$$
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Eppstein gives proofs for
- convex dimension 2
- height 2
Balancing Pairs in Antimatroids

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Eppstein gives proofs for

- convex dimension 2
- height 2 (W.)
Chordal Graphs and Antimatroids

Definition
A graph $G$ is *chordal*, if it has no induced cycle $C_n$ for $n \geq 4$. Chordal graphs have *perfect elimination orderings (PEO)*:
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Chordal Graphs and Antimatroids

Definition
A graph \( G \) is chordal, if it has no induced cycle \( C_n \) for \( n \geq 4 \).
Chordal graphs have perfect elimination orderings (PEO):

\[ A_G := \text{PEO-antimatroid of chordal graph } G \]
\textit{k-trees}

Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $A_G$ is $1/3$-balanced.
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If $G$ is a $k$-tree, then $A_G$ is $\frac{13}{3}$-balanced.
Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $A_G$ is 13-balanced.
Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $A_G$ is $1$-$3$-balanced.
Theorem (Eppstein 2013)

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Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $A_G$ is $13$-balanced.
Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $A_G$ is 1-balanced.
Theorem (Eppstein 2013)

If $G$ is a $k$-tree, then $\mathcal{A}_G$ is $\frac{1}{3}$-balanced.
Theorem (Eppstein 2013)

If $G$ is a blockgraph, then $A_G$ is 1-balanced.
Theorem (Eppstein 2013)

If $G$ is a blockgraph, then $\mathcal{A}_G$ is $\frac{1}{3}$-balanced.
Theorem (Eppstein 2013)

If $G$ is a split graph, then $A_G$ is $1$-$3$-balanced.
Theorem (Eppstein 2013)

If $G$ is a splitgraph, then $\mathcal{A}_G$ is $\frac{1}{3}$-balanced.
Theorem (W. 2013+)

If $G$ is a unit interval graph, then $A_G$ is $1\over 3$-balanced.
Theorem (W. 2013+)

If $G$ is a unit interval graph, then $\mathcal{A}_G$ is $\frac{1}{3}$-balanced.
Bonusmaterial
Observation

Semiorders can be quite “unbalanced”. PEO-antimatroids of unit interval graphs seem to be quite “balanced”.

\[ \delta(P) = \frac{1}{3} \]
Semiorders vs. Unit Interval Graphs

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$\delta(P) = \frac{1}{3}$
A useful relation

On $V(G)$ of a chordal graph $G$, the relation $\ll$ is defined by

$$x \ll y \iff \Pr(x < y) > \frac{2}{3}$$

Observation

If $G$ is a counterexample to the Balanced Pair Conjecture, then $\ll$ defines a linear order on $V(G)$. 
Semiorders vs. Unit Interval Graphs

Observation

Given an interval representation $I_P$ for semiorder (and counterexample) $P$, then

$$x \ll y \iff I_x \text{ is more to the left than } I_y$$
Semiorders vs. Unit Interval Graphs

Observation

Given an interval representation $I_P$ for semiorder (and counterexample) $P$, then

$$x \ll y \iff I_x \text{ is more to the left than } I_y$$

1 $\ll$ 2 $\ll$ 3 $\ll$ 4 $\ll$ 5 $\ll$ 6 $\ll$ 7 $\ll$ 8 $\ll$ 9