Balancing Pairs for Perfect Elimination Orderings

Veit Wiechert

Kolkom 2013, Ilmenau

Sorting with Comparisons

Let π be a hidden linear order on the set $\{1, \ldots, n\}$.
Task: Find π by asking questions like

$$\text{Is } i < j \text{ in } \pi?$$

$f(n) := \#$ needed questions.

Sorting with Comparisons

Let π be a hidden linear order on the set $\{1, \ldots, n\}$.
Task: Find π by asking questions like

$$\text{Is } i < j \text{ in } \pi?$$

$f(n) := \#$ needed questions.

Fact:

$$f(n) = \Theta(\log(n!))$$

lower bound: Information theoretic argument
upper bound: Binary Insertion Sort

Sorting under Partial Information

Assume we are given some partial information on π:

$2 < 4$

$2 < 3$

$4 < 1$

$5 < 4$

$5 < 3$

$2 < 5 < 4 < 1 < 3$

$2 < 5 < 4 < 3 < 1$

$2 < 5 < 3 < 4 < 1$

$5 < 2 < 4 < 1 < 3$

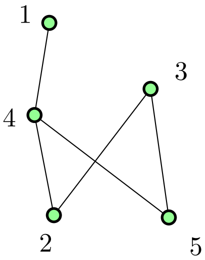
$5 < 2 < 4 < 3 < 1$

$5 < 2 < 3 < 4 < 1$

possible linear orders

Sorting under Partial Information

Assume we are given some partial information on π:



Poset

$2 < 5 < 4 < 1 < 3$

$2 < 5 < 4 < 3 < 1$

$2 < 5 < 3 < 4 < 1$

$5 < 2 < 4 < 1 < 3$

$5 < 2 < 4 < 3 < 1$

$5 < 2 < 3 < 4 < 1$

Linear Extensions

Query Complexity for Posets

$e(\mathbf{P}) :=$ # linear extensions of $\mathbf{P}$.
$f(\mathbf{P}) :=$ # questions needed.

$$f(\mathbf{P}) = \Theta(\log(e(\mathbf{P})))?$$

Balancing Pairs

Definition

$$\mathbb{P}(x \prec y) := \frac{\#\text{lin. extensions with } x < y}{\#\text{lin. extensions}}$$

Definition

A poset **P** is called *δ-balanced* for $0 < \delta \leq 0.5$, if there are elements x, y in **P** with

$$\delta \leqslant \mathbb{P}(x \prec y) \leqslant 1 - \delta$$

Balancing Pairs

Definition

$$\mathbb{P}(x \prec y) := \frac{\#\text{lin. extensions with } x < y}{\#\text{lin. extensions}}$$

Definition

A poset $\mathbf{P}$ is called *δ-balanced* for $0 < \delta \leq 0.5$, if there are elements x, y in $\mathbf{P}$ with

$$\delta \leqslant \mathbb{P}(x \prec y) \leqslant 1 - \delta$$

Observation

If every poset (not a chain) is δ-balanced, then

$$f(\mathbf{P}) = \mathcal{O}(\log(e(\mathbf{P})))$$

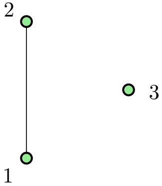
The Balancing Pair Conjecture

Conjecture (Kislitsyn '68)

Every finite poset **P** *(not a chain) is* $\frac{1}{3}$*-balanced, i.e.* $\exists x, y \in$ **P** *such that*

$$\frac{1}{3} \leqslant \mathbb{P}(x \prec y) \leqslant \frac{2}{3}$$

Best possible:



$1 < 2 < 3$

$1 < 3 < 2$

$3 < 1 < 2$

Every poset is

- $\frac{1}{3}$-balanced (Conjecture)

Every poset is

- $\frac{1}{3}$-balanced (Conjecture)
- $\frac{3}{11}$-balanced (Kahn, Saks '84)

Every poset is

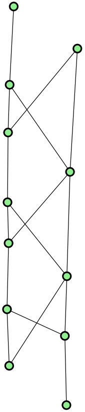
- $\frac{1}{3}$-balanced (Conjecture)
- $\frac{3}{11}$-balanced (Kahn, Saks '84)
- $\frac{5-\sqrt{5}}{10}$-balanced (Brightwell, Felsner, Trotter '95)
  $\longrightarrow$ both using Aleksandrov-Fenchel Inequalities

Every poset is

- $\frac{1}{3}$-balanced (Conjecture)
- $\frac{3}{11}$-balanced (Kahn, Saks '84)
- $\frac{5-\sqrt{5}}{10}$-balanced (Brightwell, Felsner, Trotter '95)
  $\longrightarrow$ both using Aleksandrov-Fenchel Inequalities
- $\frac{1}{2e}$-balanced (Kahn, Linial '91)
  $\longrightarrow$ using Brunn-Minkowski

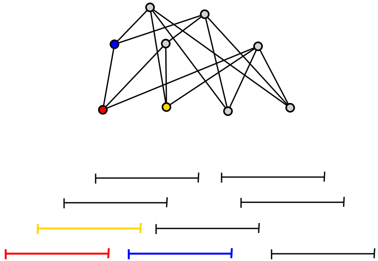
The Conjecture is true for...

width 2 posets
(Linial '84)



The Conjecture is true for...

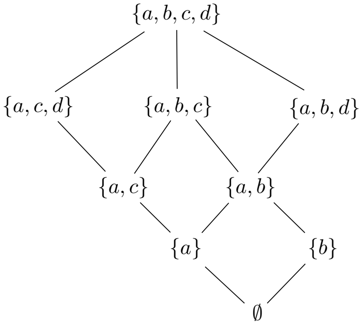
semiorders
(Brightwell '89)



Antimatroids

A finite setsystem $\mathscr{A}$ of finite sets is called an *antimatroid* if

- $A, B \in \mathscr{A} \implies A \cup B \in \mathscr{A}$
- $\emptyset \neq A \in \mathscr{A} \implies \exists x \in A : A \setminus \{x\} \in \mathscr{A}$

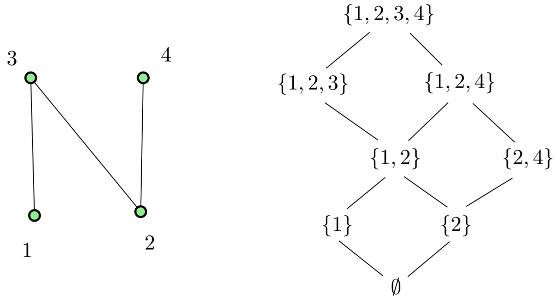


Downsets of a poset form an antimatroid

Definition

A subset D of $\mathbf{P}$ is called a *downset* if for all $x \in D$

$$y \leq_{\mathbf{P}} x \implies y \in D$$

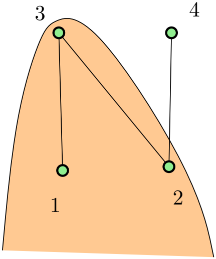


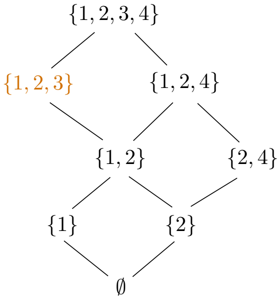
Downsets of a poset form an antimatroid

Definition

A subset D of **P** is called a *downset* if for all $x \in D$

$$y \leq_{\mathbf{P}} x \implies y \in D$$



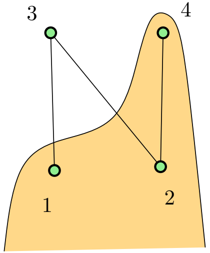


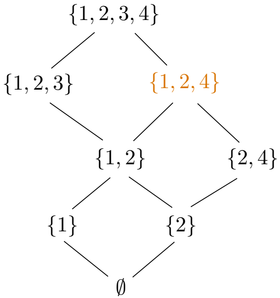
Downsets of a poset form an antimatroid

Definition

A subset D of **P** is called a *downset* if for all $x \in D$

$$y \leq_{\mathbf{P}} x \implies y \in D$$



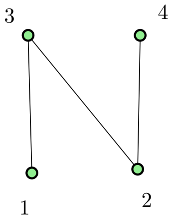


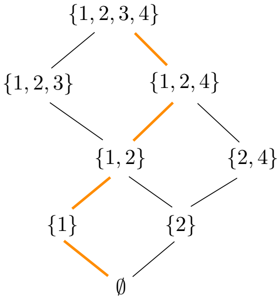
Downsets of a poset form an antimatroid

Definition

A subset D of $\mathbf{P}$ is called a *downset* if for all $x \in D$

$$y \leq_{\mathbf{P}} x \implies y \in D$$





$1 < 2 < 4 < 3$

Basic Words

$M_{\mathscr{A}}$ = groundset of setsystem $\mathscr{A}$
$n = |M_{\mathscr{A}}|$

Definition
$w = (w_1, w_2, \ldots, w_n)$ is called a *basic word* if

$$\{w_1, \ldots, w_k\} \in \mathscr{A} \quad \text{for all } k \geq 1$$

Observation

$$\textit{basic words of } \mathscr{A} \longleftrightarrow \textit{ maximal chains in } (\mathscr{A}, \subseteq)$$
$$\textit{basic words in } D(\mathbf{P}) \longleftrightarrow \textit{ linear extensions of } \mathbf{P}$$

Balancing Pairs in Antimatroids

Conjecture (Eppstein 2013)

Every antimatroid $\mathscr{A}$ (with more than 1 basic word) is $\frac{1}{3}$-balanced, i.e. $\exists x, y \in M_{\mathscr{A}}$ s.t.

$$\frac{1}{3} \leqslant \mathbb{P}(x \prec y) \leqslant \frac{2}{3}$$

Balancing Pairs in Antimatroids

Conjecture (Eppstein 2013)

Every antimatroid $\mathscr{A}$ (with more than 1 basic word) is $\frac{1}{3}$-balanced, i.e. $\exists x, y \in M_{\mathscr{A}}$ s.t.

$$\frac{1}{3} \leqslant \mathbb{P}(x \prec y) \leqslant \frac{2}{3}$$

Eppstein gives proofs for

- convex dimension 2
- height 2

Balancing Pairs in Antimatroids

Conjecture (Eppstein 2013)

Every antimatroid $\mathscr{A}$ (with more than 1 basic word) is $\frac{1}{3}$-balanced, i.e. $\exists x, y \in M_{\mathscr{A}}$ s.t.

$$\frac{1}{3} \leqslant \mathbb{P}(x \prec y) \leqslant \frac{2}{3}$$

Eppstein gives proofs for

- convex dimension 2
- ~~height 2~~ (W.)

Chordal Graphs and Antimatroids

Definition

A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.

Chordal graphs have *perfect elimination orderings (PEO)*:

Chordal Graphs and Antimatroids

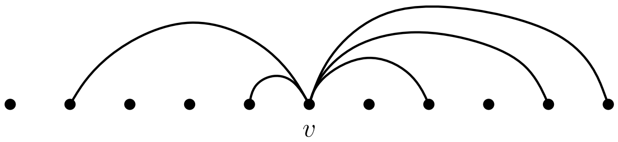
Definition

A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.
Chordal graphs have *perfect elimination orderings (PEO)*:

Chordal Graphs and Antimatroids

Definition

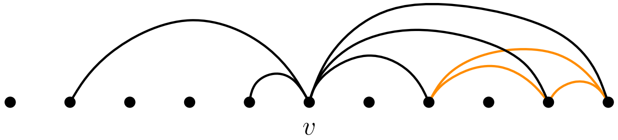
A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.
Chordal graphs have *perfect elimination orderings (PEO)*:



Chordal Graphs and Antimatroids

Definition

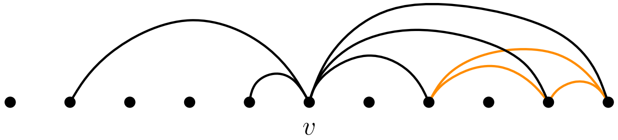
A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.
Chordal graphs have *perfect elimination orderings (PEO)*:



Chordal Graphs and Antimatroids

Definition

A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.
Chordal graphs have *perfect elimination orderings (PEO)*:

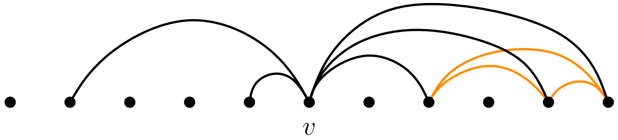


PEOs $\longleftrightarrow$ basic words

Chordal Graphs and Antimatroids

Definition

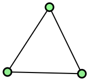
A graph G is *chordal*, if it has no induced cycle C_n for $n \geqslant 4$.
Chordal graphs have *perfect elimination orderings (PEO)*:



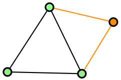
PEOs $\longleftrightarrow$ basic words

$\mathscr{A}_G :=$ PEO-antimatroid of chordal graph G

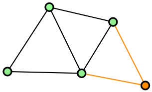
k-trees



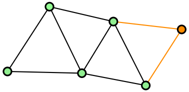
k-trees



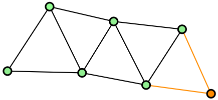
k-trees



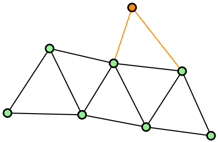
k-trees



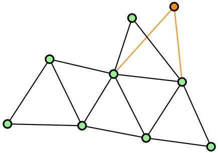
k-trees



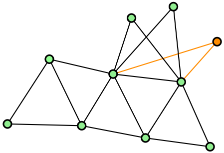
k-trees



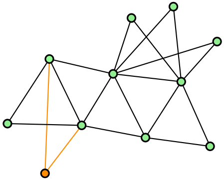
k-trees



k-trees



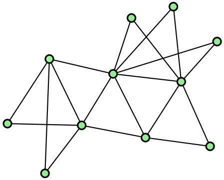
k-trees



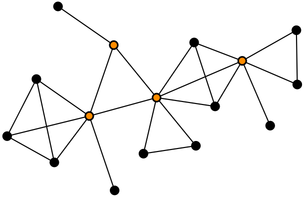
k-trees

Theorem (Eppstein 2013)

If G is a k-tree, then $\mathscr{A}_G$ is $\frac{1}{3}$-balanced.



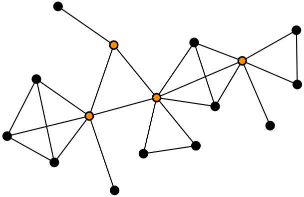
Blockgraphs



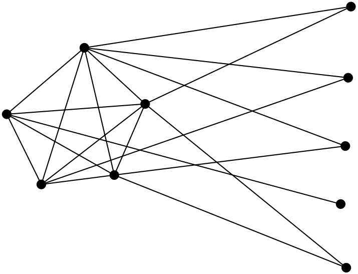
Blockgraphs

Theorem (Eppstein 2013)

If G is a blockgraph, then $\mathscr{A}_G$ is $\frac{1}{3}$-balanced.



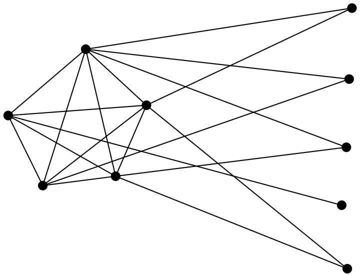
Splitgraphs



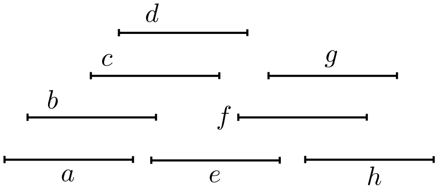
Splitgraphs

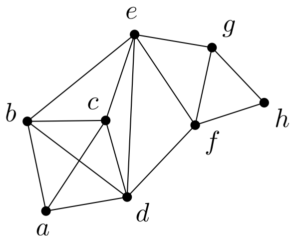
Theorem (Eppstein 2013)

If G is a splitgraph, then $\mathscr{A}_G$ is $\frac{1}{3}$-balanced.



Unit Interval Graphs

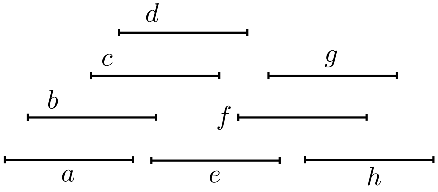


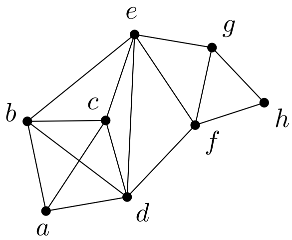


Unit Interval Graphs

Theorem (W. 2013+)

If G is a unit interval graph, then $\mathscr{A}_G$ is $\frac{1}{3}$-balanced.





Bonusmaterial

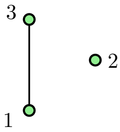
Semiorders vs. Unit Interval Graphs

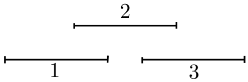
Observation

Semiorders can be quite "unbalanced". PEO-antimatroids of unit interval graphs seem to be quite "balanced".

P

$\delta(\mathbf{P}) = \frac{1}{3}$

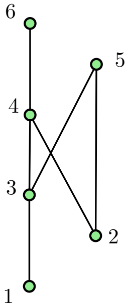




Semiorders vs. Unit Interval Graphs

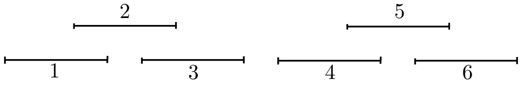
Observation

Semiorders can be quite "unbalanced". PEO-antimatroids of unit interval graphs seem to be quite "balanced".



P

$\delta(\mathbf{P}) = \frac{1}{3}$



A useful relation

On $V(G)$ of a chordal graph G, the relation $\ll$ is defined by

$$x \ll y \iff \mathbb{P}(x \prec y) > \frac{2}{3}$$

Observation

If G is a counterexample to the Balanced Pair Conjecture, then $\ll$ defines a linear order on $V(G)$.

Semiorders vs. Unit Interval Graphs

Observation

Given an interval representation $I_{\mathbf{P}}$ for semiorder (and counterexample) $\mathbf{P}$, then

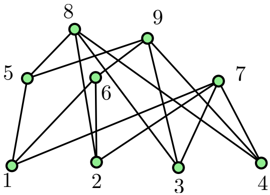
$$x \ll y \iff I_x \text{ is more to the left than } I_y$$

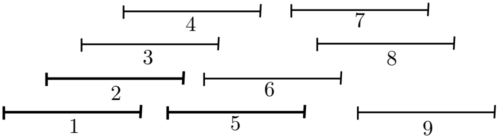
Semiorders vs. Unit Interval Graphs

Observation

Given an interval representation $I_{\mathbf{P}}$ for semiorder (and counterexample) $\mathbf{P}$, then

$$x \ll y \iff I_x \text{ is more to the left than } I_y$$





$$1 \ll 2 \ll 3 \ll 4 \ll 5 \ll 6 \ll 7 \ll 8 \ll 9$$