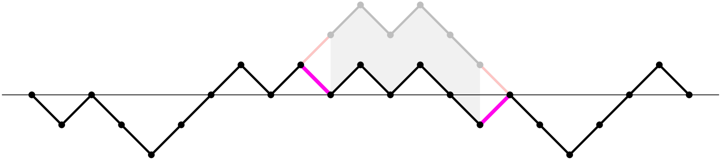
A minimum-change version of the Chung-Feller theorem for Dyck paths

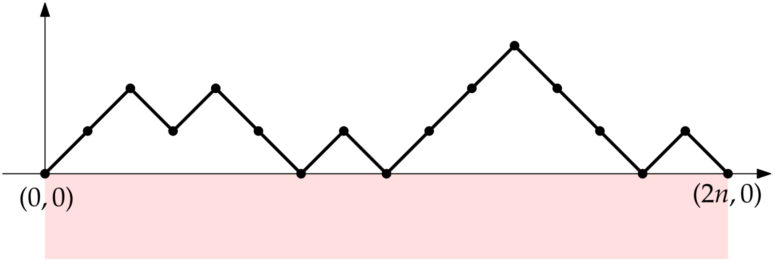
Torsten Mütze, Christoph Standke, **Veit Wiechert**



7ECM 2016, Berlin

Dyck Paths

A *Dyck path* is a lattice path from $(0, 0)$ to $(2n, 0)$ that consists of diagonal up- and downsteps, and that never goes below the line $y = 0$.



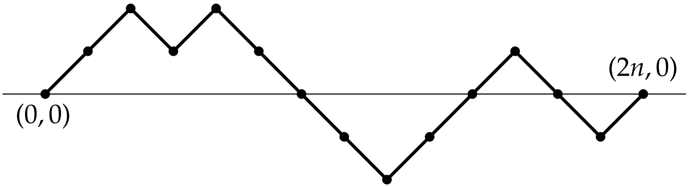
Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

C_n counts

- Dyck paths of order n,
- binary trees with $n+1$ leaves,
- triangulations of the regular $(n+2)$-gon.

$\binom{2n}{n}$ is the number of lattice paths from $(0,0)$ to $(2n,0)$.



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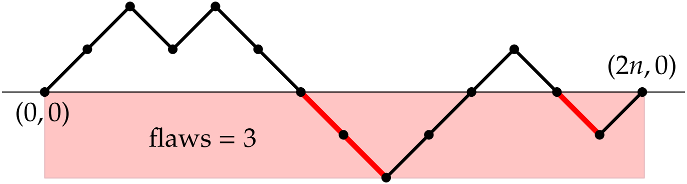
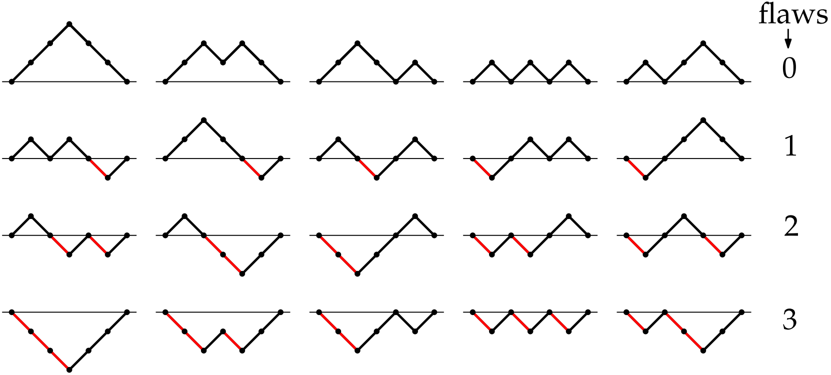


Lattice paths of length 6



Chung-Feller Theorem

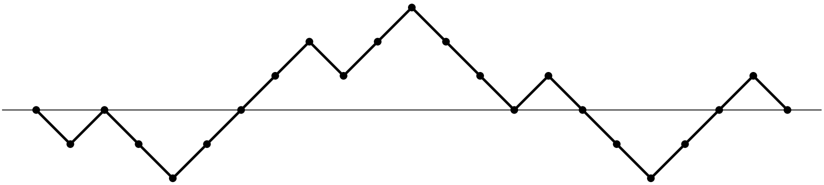
D_n^k := set of Dyck paths of length $2n$ with k flaws.

Theorem [Chung-Feller '49]
For each $n \geq 1$, we have

$$\left|D_n^0\right| = \left|D_n^1\right| = \cdots = \left|D_n^n\right| = \frac{1}{n+1}\binom{2n}{n} = C_n.$$

Proof of Chung-Feller Thm [Hodges 55], [Chen 08]

Establish bijection $f\colon D_n^k \to D_n^{k+1}$.



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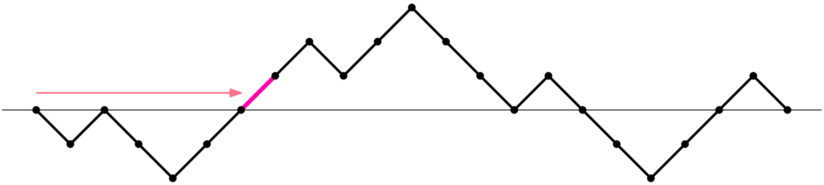

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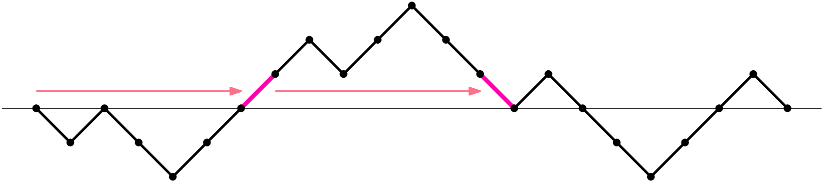

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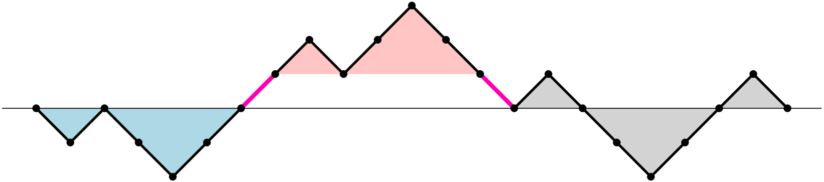

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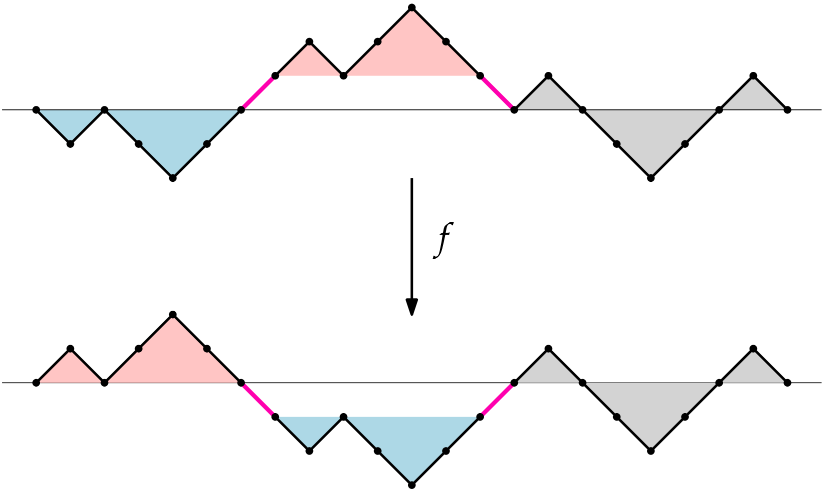
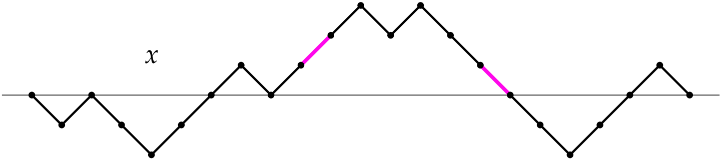


Our Strengthening

Theorem [Mütze, Standke, W. 16]
There is a bijection $f\colon D_n^k \to D_n^{k+1}$, such that x and $f(x)$ differ only in two positions.



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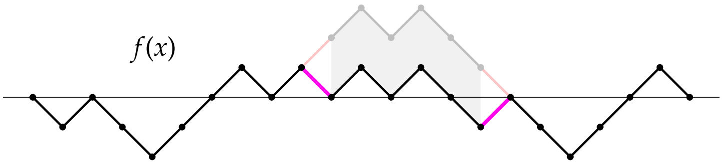

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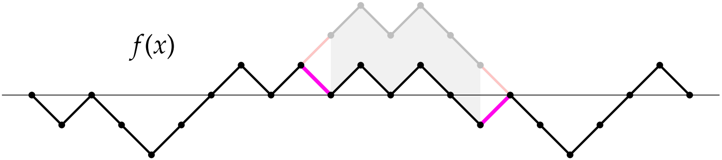


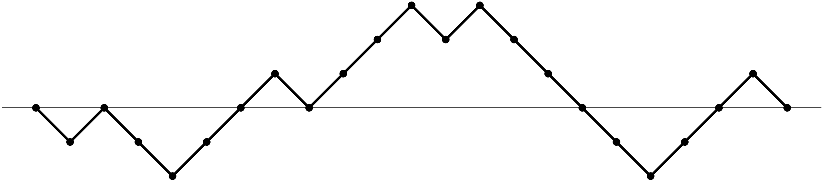
Theorem [Mütze, Standke, W. 16]
There is an algorithm which for given $x \in D_n^0$ computes each path in $f(x), f^2(x), \ldots, f^n(x)$ in $\mathcal{O}(1)$.

Minimum-change bijection

$d_0(x)$:= # down-steps *starting* from the line $y = 0$.

Flip

- $(d_0(x) + 1)$-th down-step **s** *touching* the line $y = 0$,
- first upstep to the left of **s** *touching* the line $y = 1$.



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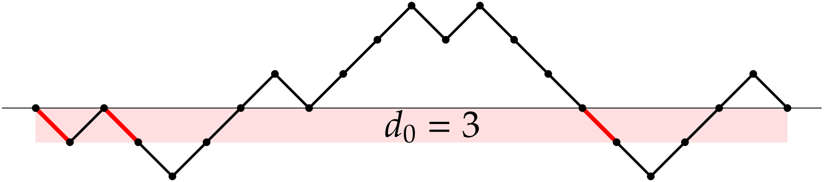

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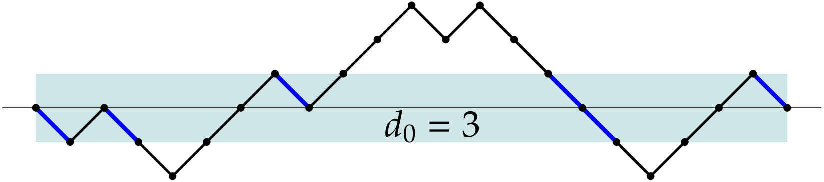

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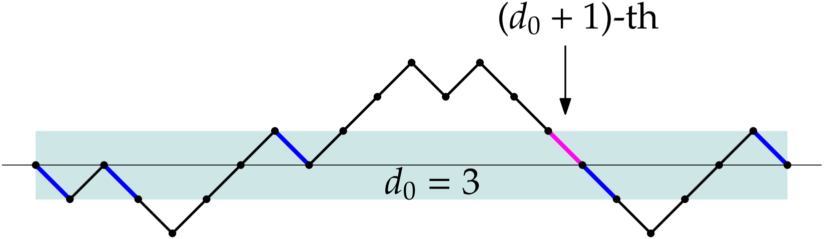

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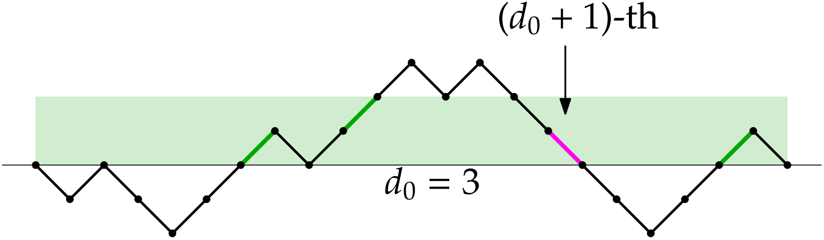

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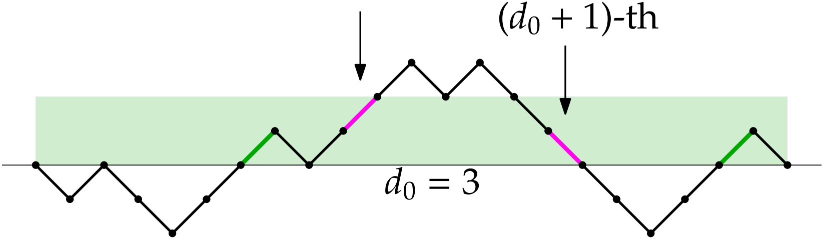

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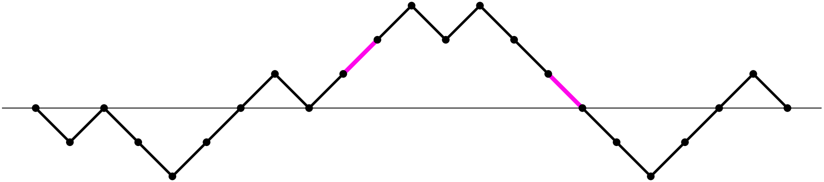

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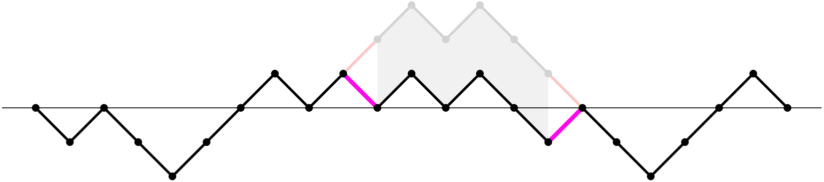

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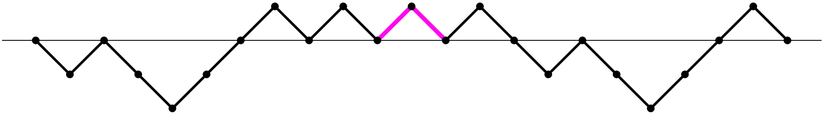

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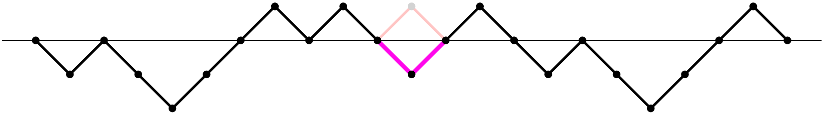

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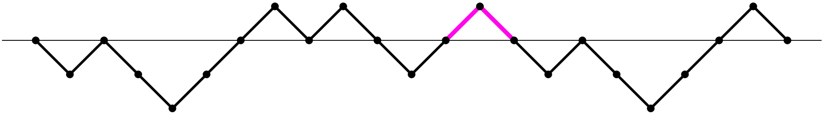

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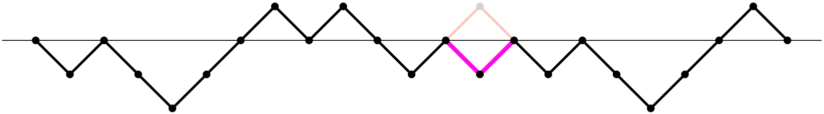

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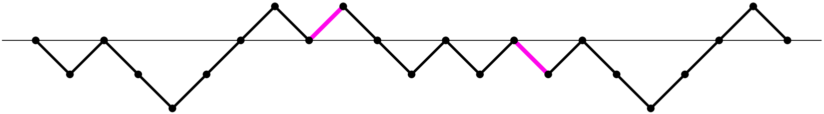

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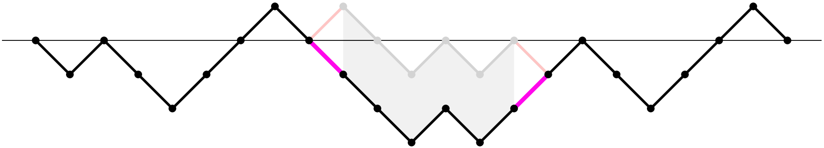


Inverse bijection f^{-1}

$d_0(x)$:= # down-steps *starting* from the line $y = 0$.

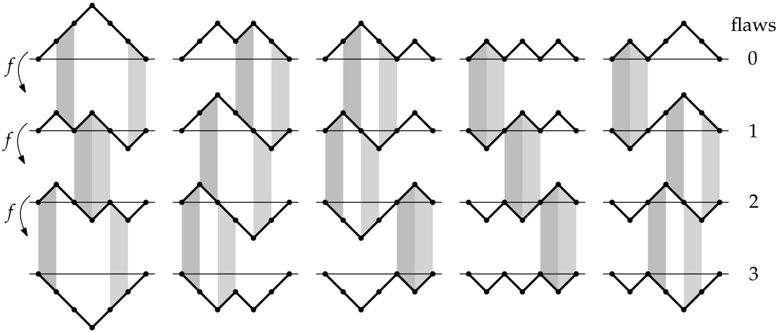
Minimum-change bijection f:

- $(d_0(x) + 1)$-th down-step **s** touching the line $y = 0$,
- first upstep to the left of **s** touching the line $y = 1$.

Inverse bijection f^{-1}:

- $d_0(x)$-th down-step **s** touching the line $y = 0$,
- first upstep to the right of **s** touching the line $y = -1$.

Bijection on paths of length 6



Applications

Combinatorial Gray Codes

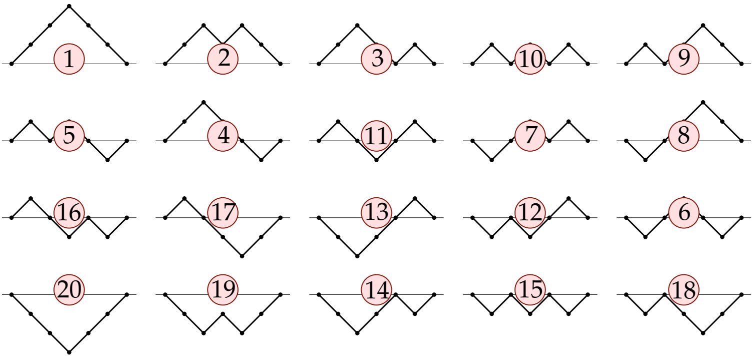
Goal 1: Generate all objects in a combinatorial class s.t. consecutive objects differ only *a little bit*.

Goal 2: Generate each new object in time $\mathcal{O}(1)$.

Examples: Generate all

- permutations of $[n]$ by *single transpositions*.
- subsets of $[n]$ by *adding/removing one element*.
- k-element subsets of $[n]$ by *exchanging one element*.
- Dyck paths in D_n^0 by flipping only *two positions*.

Gray Codes on Lattice Paths



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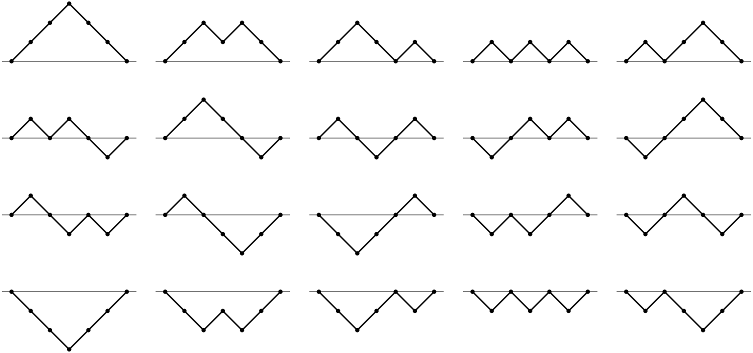

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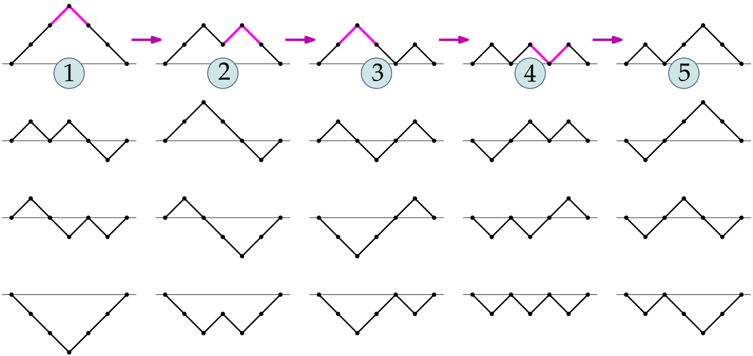

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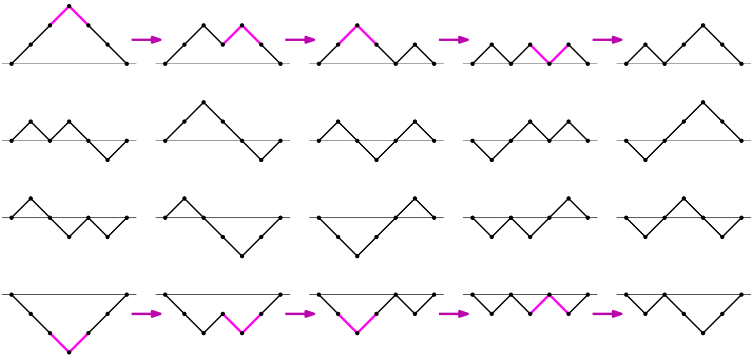

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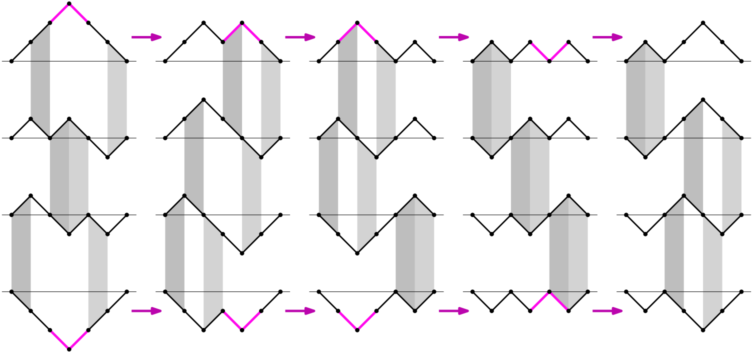

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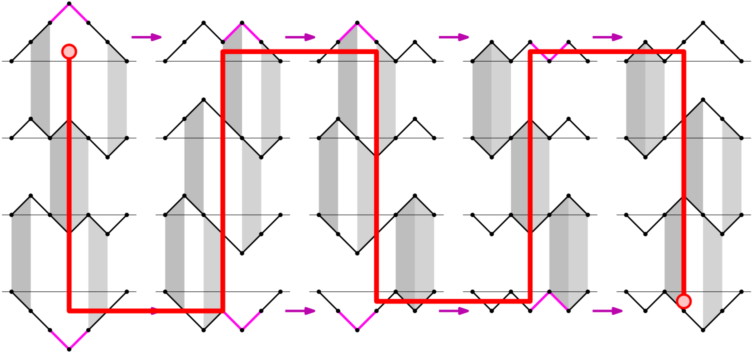
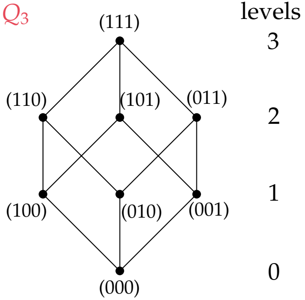


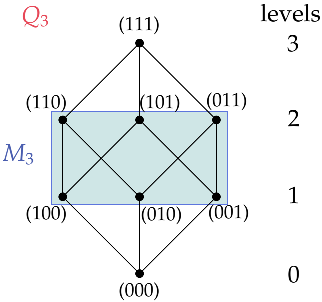
Applications to Graph theory

Q_n n-dimensional cube



Applications to Graph theory

Q_n	n-dimensional cube
M_{2n+1}	subgraph of Q_{2n+1} ind. by levels n and $n+1$



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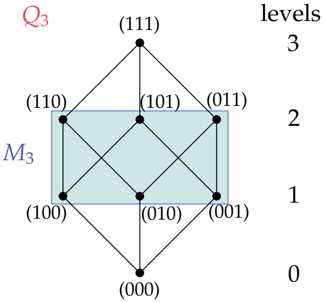
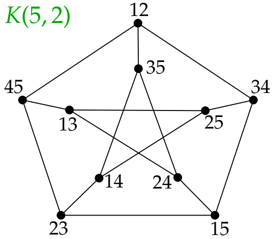
$K(2n+1,n)$	*Odd graph*





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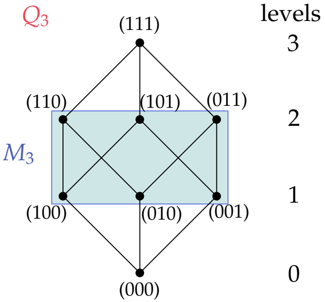
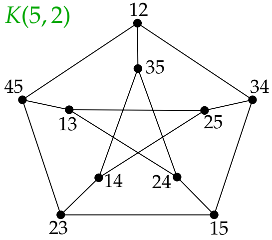
Problem: For which $\ell \geq 1$ do M_{2n+1} or $K(2n+1,n)$ contain a C_ℓ-factor?





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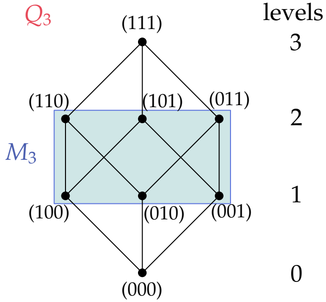

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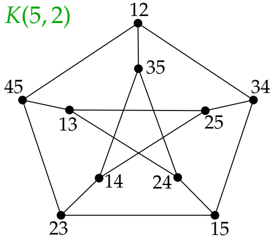


Thm. [Mütze '14]
M_{2n+1} is hamiltonian $\forall n \geq 1$.

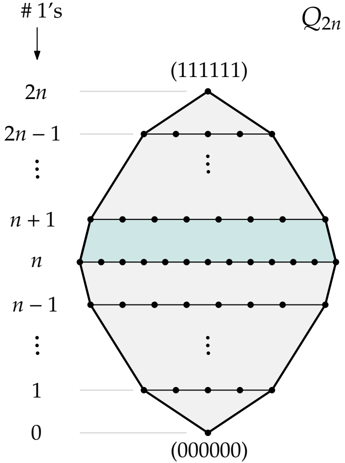
Applications to graph theory

Theorem [Mütze, Standke, W. '16]
For each $n \geq 1$, the middle levels graph M_{2n+1} contains a C_{4n+2}-factor.

Theorem [Mütze, Standke, W. '16]
For each $n \geq 1$, the Odd graph $K(2n+1, n)$ contains a C_{2n+1}-factor.

$\rightarrow$ both cycle-factors contain C_n many cycles.

Cycle-factors in the middle levels graph



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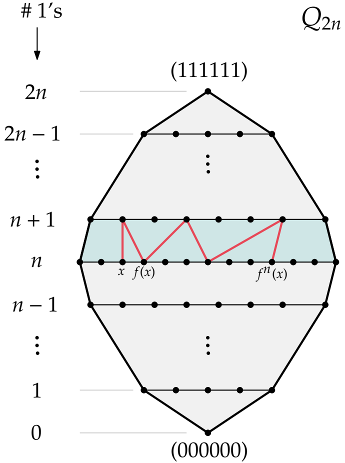

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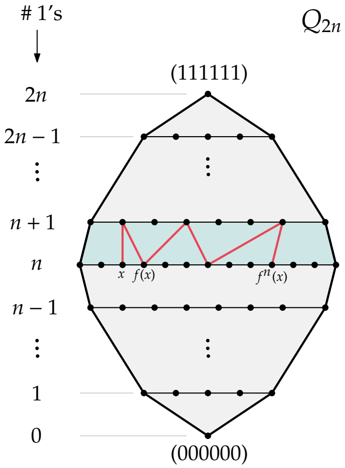
$$x, f(x), \ldots, f^n(x)$$

yields a path P_x.



- $\forall x \in D^0_{2n}$, the sequence

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- then $\bigcup_{x\in D^0_{2n}} P_x$

  covers the levels n and $n+1$.

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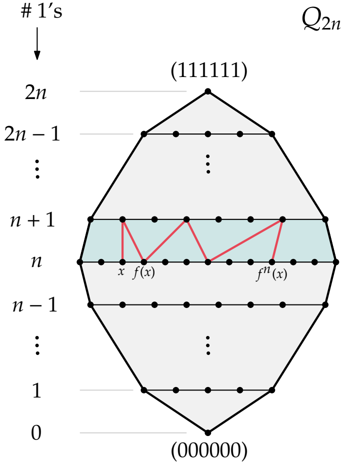

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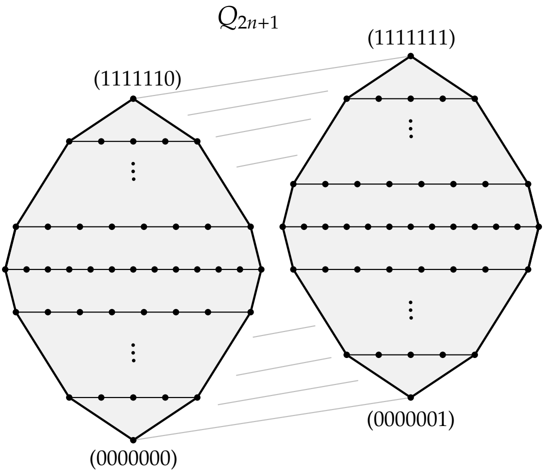
  covers the levels n and $n+1$.
- $\forall x \in D_{2n}^0$ we have $f^n(x) = \overline{x}$.

Cycle-factors in the middle levels graph



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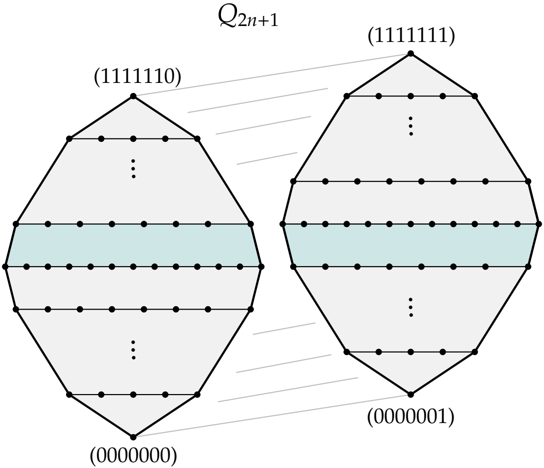

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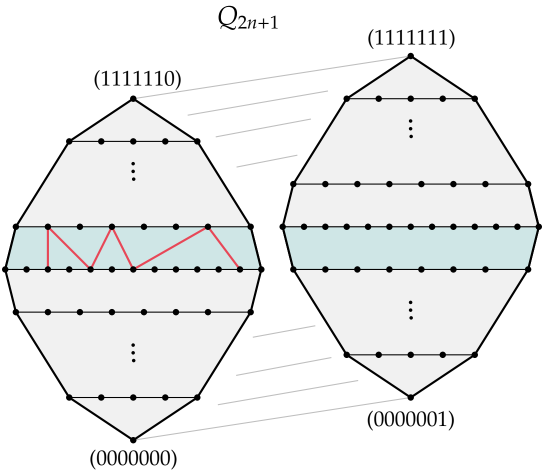

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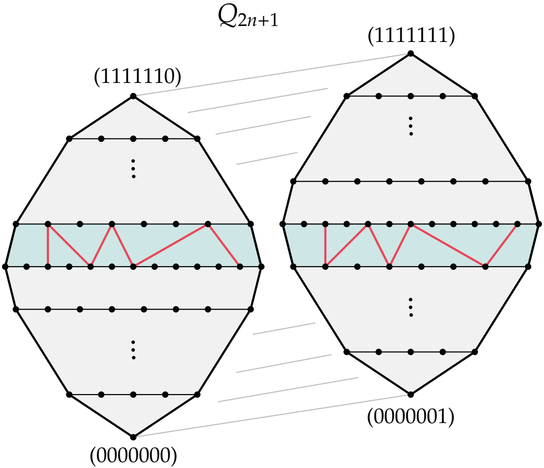

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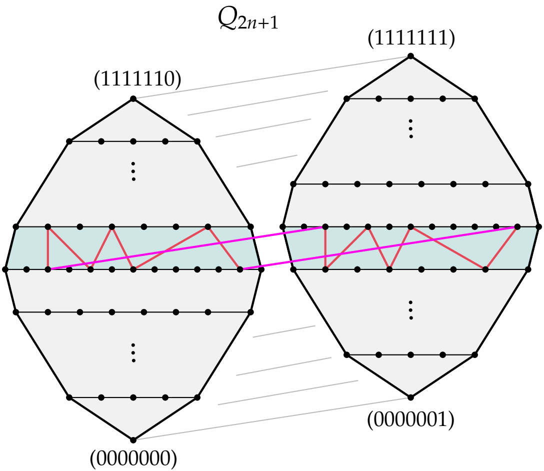
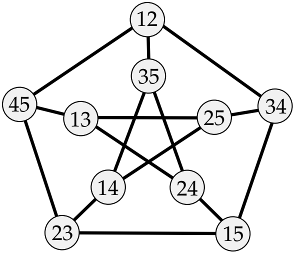
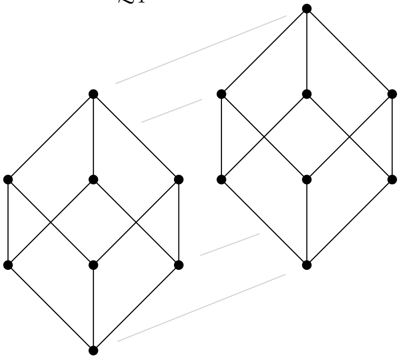


Cycle-factors in the Odd graph

$K(5,2)$



Q_4



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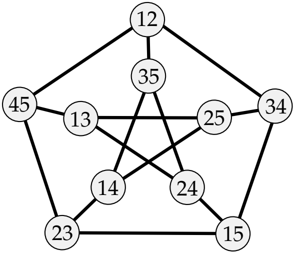

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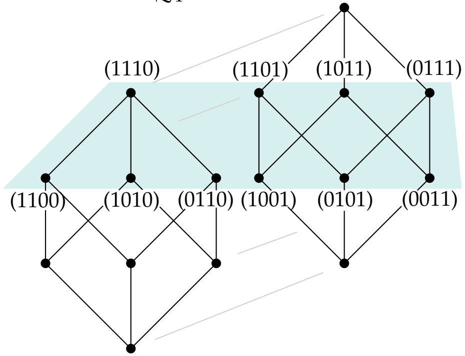

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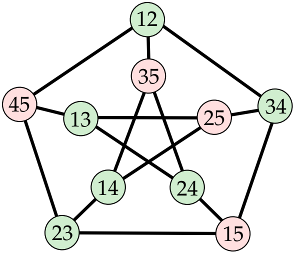

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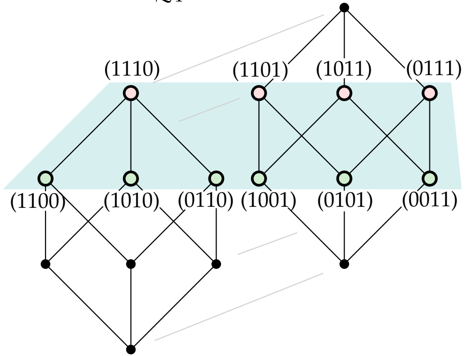

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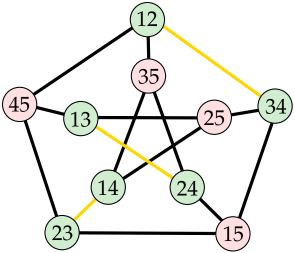

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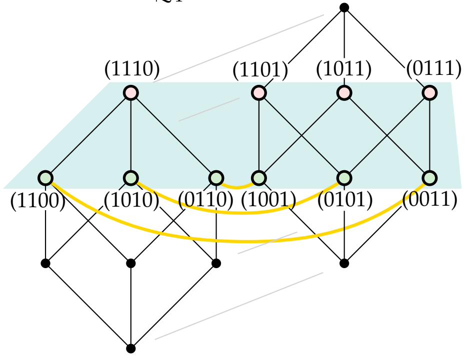

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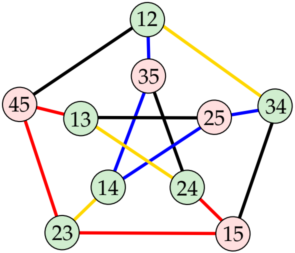

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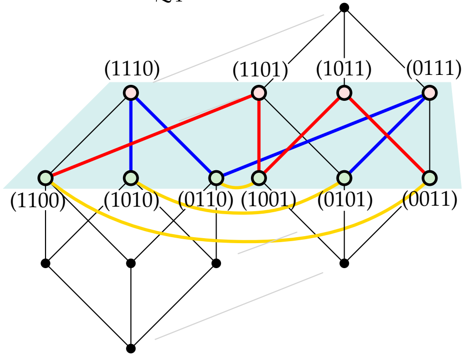

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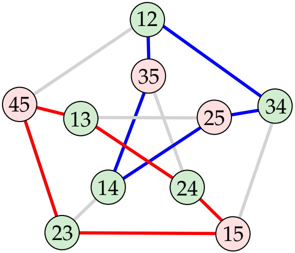

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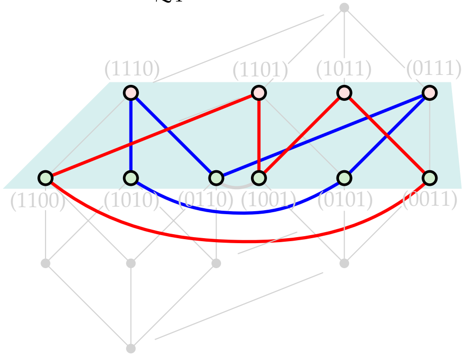

Open Problems

Conjecture
The Odd graph $K(2n+1, n)$ is hamiltonian $\forall n \geq 3$.

Problem
For which ℓ do exist C_ℓ-factors in M_{2n+1} or $K(2n+1, n)$?

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Thank You