07.1.14 - $\boxed{\text{Ue } 8}$

$r = \frac{m}{n} \in \mathbb{Q}$

Zeigen : $\exp(r) = \sqrt[n]{e^m}$

— // —

$\exp(n) = e^n$ (bekannt)

$\exp\left(n \cdot \frac{m}{n}\right) = \exp(m) = \boxed{e^m}$

$= \exp\Big(\underbrace{\overbrace{\tfrac{m}{n}}^{x_1} + \overbrace{\tfrac{m}{n} + \dots + \tfrac{m}{n}}^{x_2}}_{n \cdot \text{mal}}\Big)$

$\exp(x_1 + x_2)$
$= \exp(x_1) \cdot \exp(x_2)$

$= \exp\left(\frac{m}{n}\right) \exp\Big(\underbrace{\tfrac{m}{n} + \dots + \tfrac{m}{n}}_{(n-1)\text{ mal}}\Big)$

$= \exp\left(\frac{m}{n}\right) \cdot \exp\left(\frac{m}{n}\right) \cdot \dots \exp\left(\frac{m}{n}\right)$

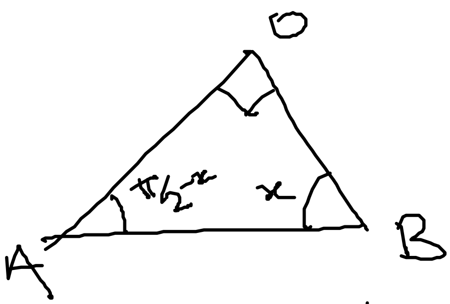
Produkt von n-Zahlen

$= \prod_{k=1}^{n} \exp\left(\frac{m}{n}\right) = \boxed{\left[\exp\left(\frac{m}{n}\right)\right]^n}$

$\Longrightarrow e^m = \left[\exp\left(\frac{m}{n}\right)\right]^n$

$\Longrightarrow \exp\left(\frac{m}{n}\right) = \sqrt[n]{e^m}$

1. Erweiterung von $\sin(x)$ und $\cos(x)$ auf $\mathbb{R}$
2. Polynom
3. Vollständige Induktion



$\sin x$, $\cos x$

$x \in \,]0, \pi/2[$

Wir wissen schon, dass:

$$\sin^{\Delta}(x) = \cos(\pi/2 - x)$$

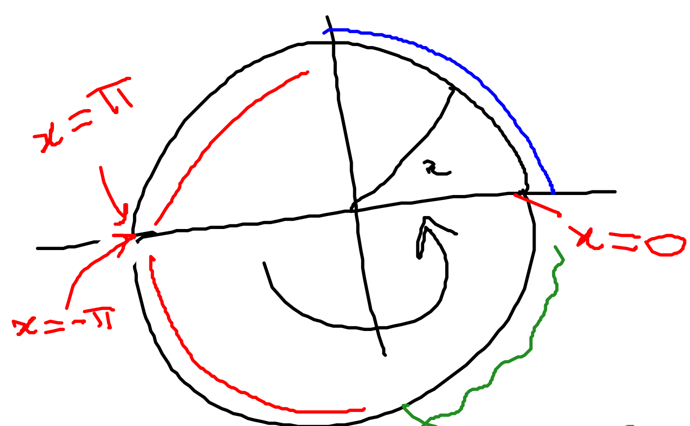
$$\cos^{\Delta}(x) = \sin(\pi/2 - x)$$

$$\sin(0) = 0 \qquad \text{(VL 19.12)}$$

$$\cos(0) = 1$$

A – Erweiterung auf Einheitskreis

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

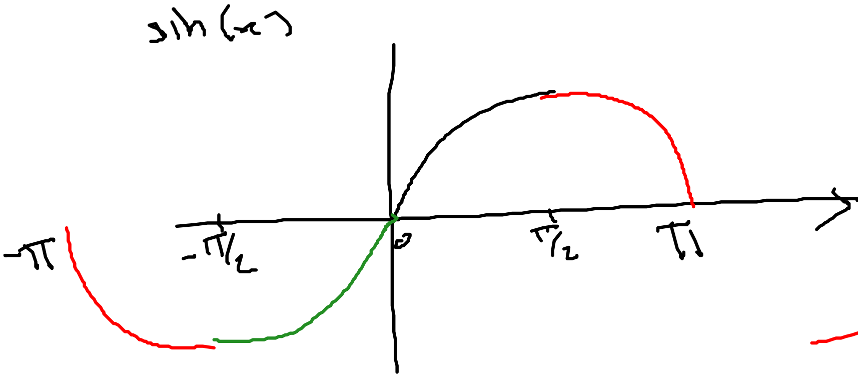


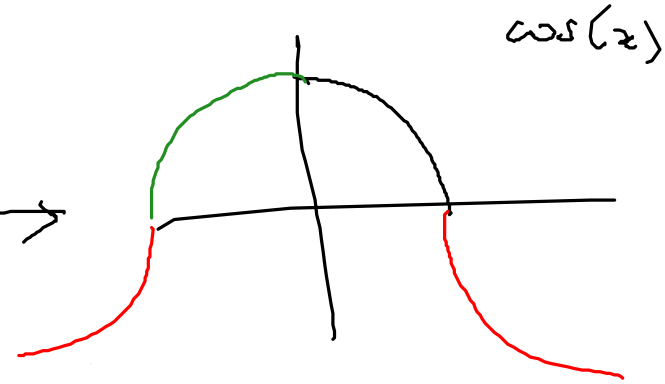
$$\sin(x) = \begin{cases} \sin^{\Delta}(x) & x \in [0, \pi/2] \\ -\sin^{\Delta}(-x) & x \in [-\pi/2, 0] \end{cases}$$

ungerade Fortsetzung

$$\cos(x) = \begin{cases} \cos^{\Delta}(x) & x \in [0, \pi/2] \\ \cos^{\Delta}(-x) & x \in [-\pi/2, 0] \end{cases}$$

gerade Fortsetzung





Im Intervall $]-\pi, -\frac{\pi}{2}]$ und $[\frac{\pi}{2}, \pi]$

Idee: wir definieren $\sin(x)$ und $\cos(x)$
so dass Additionstheorem noch gilt

$u \in [\pi/2, \pi] \iff u = x + \pi/2$, mit $x \in [0, \pi/2]$

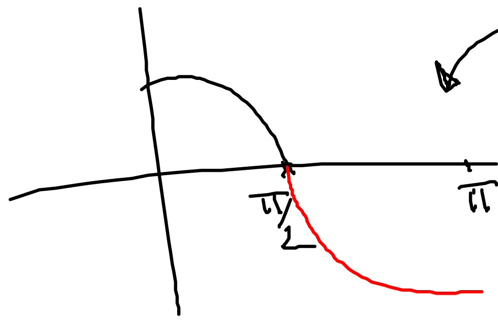
$v \in [-\pi, -\pi/2] \qquad v = x - \pi/2$, mit $x \in [-\pi/2, 0]$

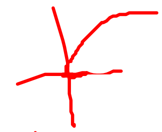
dann $\sin(u) := \sin(x + \pi/2) = \sin(x)\cos(\pi/2) + \cos(x)\sin(\pi/2)$
$= \cos(x)$

$\sin(v) = \sin(x - \pi/2) = -\cos(x)$

$$\cos(u) = \cos(x + \pi/2) = -\sin(x) \qquad (\text{Analog})$$
$$\cos(v) = \cos(x - \pi/2) = \sin(x)$$

$-\sin(x)$, mit $x \in [0, \pi/2]$

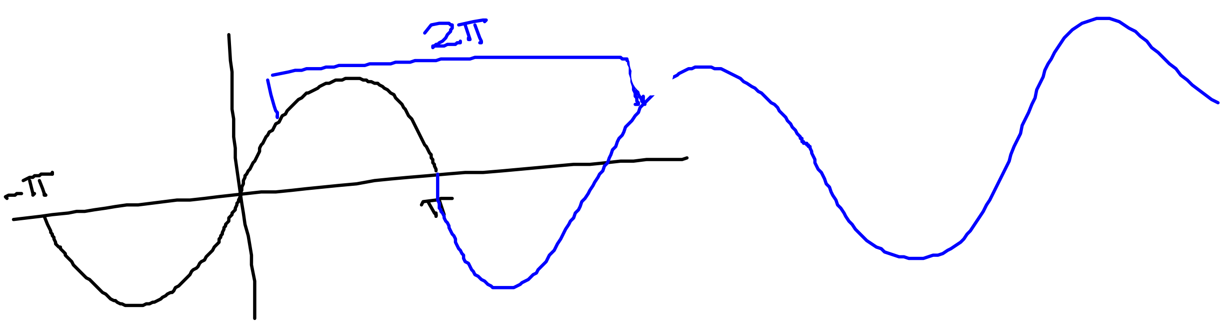




$\sin(x)$

$$\sin(2\pi) = \sin(\pi + \pi) = 2\overbrace{\sin(\pi)}^{0}\,\overbrace{\cos(\pi)}^{-1} = 0$$
$$\cos(2\pi) = \cos(\pi + \pi) = \cos^2(\pi) - \sin^2(\pi) = 1$$

$$\Longrightarrow \quad \sin(x + 2\pi) = \sin(x)\cos(2\pi) + \sin(2\pi)\cos(x) = \sin(x)$$
$$\cos(x + 2\pi) = \cos(x)$$



$$\sin(3\pi) = \sin(2\pi+\pi) = \sin(\pi)$$

$\sin(4\pi) = \sin(2\pi) + \sin(2\pi) = 0$

$$\sin(n\cdot\pi) = \begin{cases} 0 & n \text{ gerade} \\ \sin(\pi) = 0 & n \text{ ung} \end{cases}$$

$$\cos(n\cdot\pi) = \begin{cases} 1 & n \text{ gerade} \\ \cos(\pi) = -1 & n \text{ ungerade} \end{cases}$$

Polynom: eine Funktion $f: \mathbb{R} \to \mathbb{R}$, mit

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$a_0, a_1, \dots, a_n \in \mathbb{R}$$ (f Polynom mit reellen Koeffizienten)

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

mit $a_k = 0$ für $k > n$

Sei $f : \mathbb{R} \longrightarrow \mathbb{R}$ ein Polynom $f(x) = a_0 + a_1 x + \dots + a_n x^n + \dots$

$$n_0 = \max \{ k \, , \, a_k \neq 0 \}$$

$\Longrightarrow$ n_0 : Grad von f $\quad$ $\mathrm{grad}(f)$

Beispiel

$f(x) = 1 + x^2$ $\qquad$ $a_0 = 1, \ a_1 = 0, \ a_2 = 1,$
$a_3 = a_4 = a_5 \dots = 0$

$f(x) = 1 + x^2 - x^{10}$ $\Longrightarrow$ $\mathrm{grad}(f) = 10$

Operationen mit Polynome

$f, g : \mathbb{R} \longrightarrow \mathbb{R}$ Polynome

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \, ; \ g(x) = \sum_{k=0}^{\infty} b_k x^k$$

$$(f+g)(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$f(x) = 1 + x^2 + x^3$
$g(x) = x + x^3 + x^4$
$f+g = 1 + x + x^2 + 2x^3 + x^4$

$$(f \cdot g)(x) = \left(\sum a_k x^k\right)\left(\sum b_k x^k\right) = \sum c_k x^k$$

	a_0	a_1	a_2	a_3	a_4	$\dots$
b_0	$\boxed{a_0 b_0}$	$a_1 b_0$	$a_2 b_0$			
b_1	$a_0 b_1$	$a_1 b_1$				
b_2	$a_0 b_2$					
b_3						

x $\quad x^2$

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

$$(1 - 2x + x^2 + x^3)(2 + x + x^3 + x^4) = c_0 + c_1 x + c_2 x^2 + \dots + c_7 x^7 + \dots$$

$$c_0 = a_0 b_0 = 2$$

$$c_1 = a_0 b_1 + a_1 b_0 = -3$$

$$c_2 = 0$$

$$c_3 =$$

$$\vdots$$

$$c_7 = \sum_{j=0}^{7} a_j b_{7-j} = a_0 \cancel{b_7} + a_1 \cancel{b_6} + a_2 \cancel{b_5} + a_3 b_4 + \cancel{a_4} b_3 + \cancel{a_5} b_2 + \cancel{a_6} b_1 + \cancel{a_7} b_0$$

$$= 1$$

$$c_8 = c_9 = \dots = 0$$