$x = y \qquad x, y \in \mathbb{Z}$ dann ist klar

$A = B \qquad A, B$ Mengen

$\mathbb{Z}$ (a) $x \sim y \iff x \geq y$ (RE), (TRA)

$\boxed{\text{(b) } x \sim y \iff x = y}$ (RE), (SY), (TRA)

(c) $x \sim y \iff x + y = 0$ (SY)

$\boxed{\text{(d) } x \sim y \iff 2 \mid (x - y)}$ (RE), (SY), (TRA)

2 teilt

Äquivalenz Relation $\iff$ (RE), (SY), (TRA)

(RE): $\forall x \in M: \; x \sim x$

(SY): $\forall x, y \in M: \; x \sim y \Rightarrow y \sim x$

(TRA): $\forall x, y, z \in M:$

$x \sim y, \; y \sim z \Rightarrow x \sim z$

$m \mid n$ m teilt n

$n = m \cdot k$ für $k \in \mathbb{Z}$

$2 \mid x - y \Rightarrow x - y = 2k_1$

$2 \mid y - z \Rightarrow y - z = 2k_2$

dann

$x - z = (x - y) + (y - z) = 2(k_1 + k_2) \iff 2 \mid (x - z)$

Frage $x \sim y \iff 3 \mid (x-y)$
auch Äquivalenzrel. ?

JA auch alle Rel.
$x \sim y \iff m \mid (x-y), m \neq 0$

Beispiele

(e) $M = \{\text{Menschen}\}$

$x \sim y \iff$ x und y sind am gleichen Wochentag geboren

$M, \sim$ Äquiv.

$x \in M$ $[x]_\sim := \{y \in M : x \sim y\} \subseteq M$

(b) $x \sim y \iff x = y$ $(M = \mathbb{Z})$

$\forall x \in \mathbb{Z}, [x]_\sim = \{y \in \mathbb{Z} : x = y\} = \{x\}$

(d) $x \sim y \iff 2 \mid (x-y)$

$[0]_\sim = \{y \in \mathbb{Z} : 2 \mid (-y)\} = \{0, 2, -2, 4, -4, \ldots\}$

$= \{2k, k \in \mathbb{Z}\}$ alle gerade Zahlen

$[1]_\sim = \{y \in \mathbb{Z} : 2 \mid (1-y)\} = \{1, -1, 3, -3, \ldots\} = \{2k+1, k \in \mathbb{Z}\}$

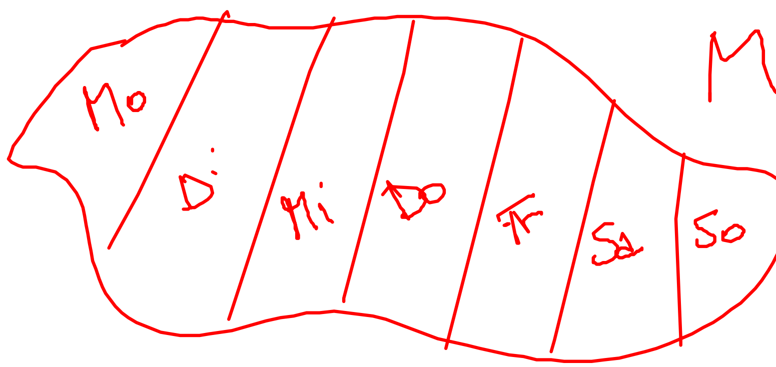
$[2]_\sim = [0]_\sim$

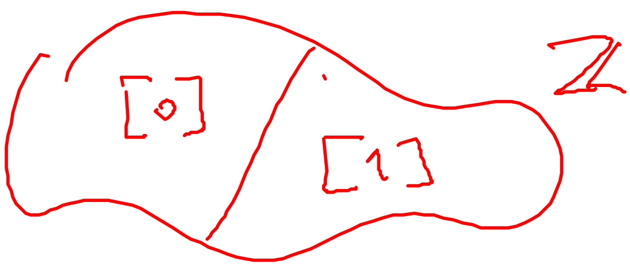
1e) $M = \{\text{Menschen}\}$

$x \in M$

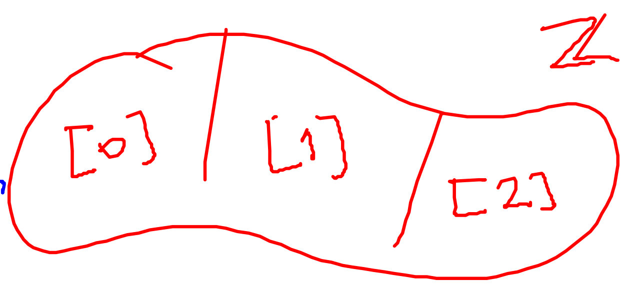
$[x]_\sim = \{ y \in M : y \text{ und } x \text{ sind am gleichen Wochentag geboren} \}$

Es gibt nur 7 Klassen





$2 \mid (x-y)$



$3 \mid (x-y)$

$[0] = \{0, 3, 6, 9, \dots\} = \{3k, k \in \mathbb{Z}\}$

$[1] = \{3k+1, k \in \mathbb{Z}\}$

$[2] = \{3k+2, k \in \mathbb{Z}\}$

$[0] + [1] = [1]$ $\quad 0+1 = 1 \in [1]$

$[1] + [2] = [0]$ $\quad 3+10 = 13 \in [1]$

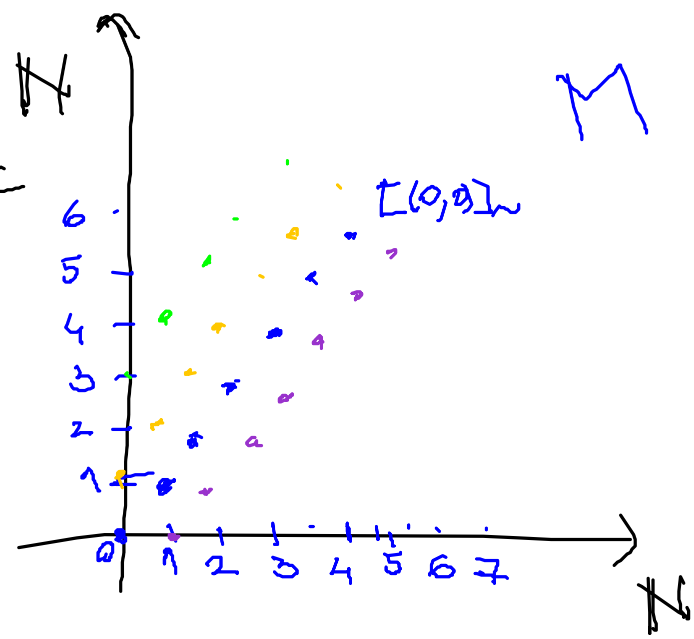
Aufgabe 12

① Wieso brauchen wir $\mathbb{Z}$?

$x + a = b$ Kann man in $\mathbb{N}$ NICHT immer lösen

$M = \mathbb{N} \times \mathbb{N}$

$(a,b) \sim (c,d) \Leftrightarrow a+d = b+c$



$(0,0) \sim (1,1)$

$(1,2) \sim (3,4) \sim (5,6)$

$(2,6) \sim (4,8)$

(RE) $(a,b) \sim (a,b)$?

$a+b = b+a$ JA

(SY) $(a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b)$ JA

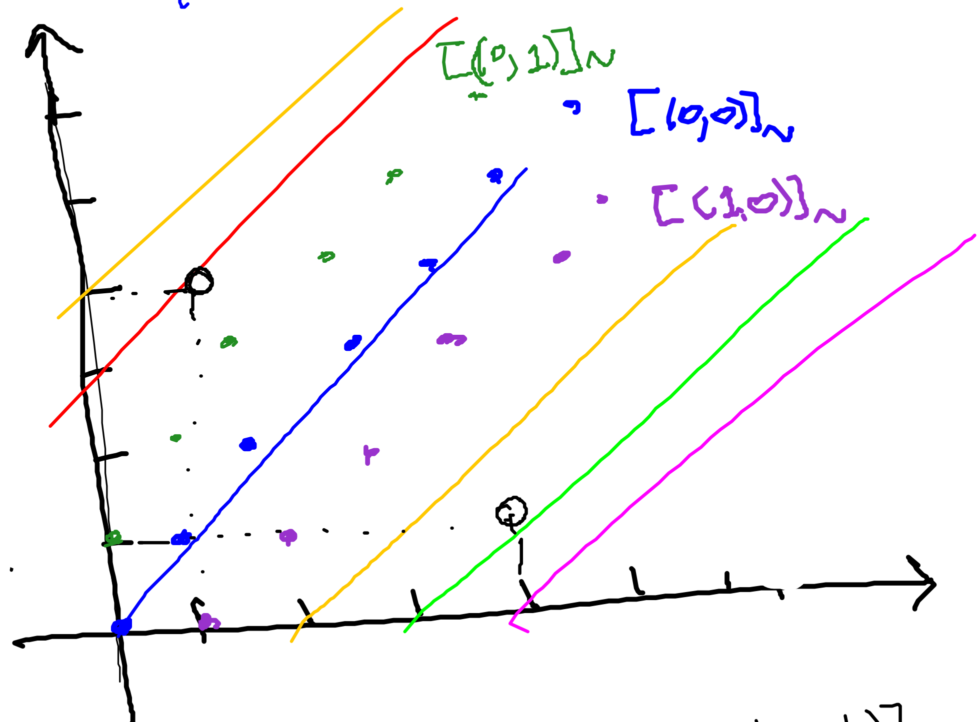
$a+d = b+c \Longrightarrow c+b = d+a$

(TRA) $(a,b) \sim (c,d)$, $(c,d) \sim (e,f) \Longrightarrow (a,b) \sim (e,f)$

Bew. $a+d = b+c$ $c+f = d+e$

$\Rightarrow a+\cancel{d}+\cancel{c}+f = b+\cancel{c}+\cancel{d}+e \Rightarrow a+f=b+e$ ✓

$[(0,0)] = \{(a,b)\in\mathbb{N}\times\mathbb{N} : (a,b)\sim(0,0)\}$



$[(a,b)]_\sim + [(c,d)] = [(a+c,\ b+d)]$

$[(0,0)]_\sim + [(1,4)]_\sim = [(1,4)] = [(0,3)]$

$[(1,4)]_\sim + [(4,1)]_\sim = [(5,5)]_\sim = [(0,0)]_\sim$

$\overset{\shortparallel}{[(0,3)]_\sim} + [(3,0)]_\sim = [(0,0)]_\sim$

$[(7,0)]_\sim + [(0,2)]_\sim = [(7,2)]_\sim = [(5,0)]_\sim$

$\approx\ 7 \quad -2 \quad = \quad 5$

$$\mathbb{N}\times\mathbb{N}/_\sim = \{[(a,b)]_\sim, (a,b)\in\mathbb{N}\times\mathbb{N}\}$$

$$= \mathbb{Z}$$

$$a + x = b$$
$$x = b - a \in \mathbb{Z}$$
$$(b, a) \in \mathbb{N} \times \mathbb{N}$$
$[(b,a)]_\sim \approx x$ dass $a + x = b$ löst

- $M = \{\text{Menschen}\}$

$\sim$ = geboren am gleichen Wochentag

$$M/_\sim = \{[Mo], [Di], [Mi], [Do], [Fr], [Sa], [So]\}$$
$$M/_\sim = \{1, 2, 3, 4, 5, 6, 7\}$$

- $\mathbb{Z}$, $x \sim y \iff 3 \mid (x - y)$

$$\mathbb{Z}/_\sim = \{[0], [1], [2]\} = \{0, 1, 2\}$$

Rechner modulo 3

$$[1]+[1]=[2]$$
$$[2]+[2]=[1]$$
$$[1]\cdot[0]=[0]$$
$$[1]\cdot[2]=[2]$$

$$x\sim y \iff 123 \mid (x-y)$$

$$[20]+[120]$$
$$=[140]$$
$$=[140-123]=[17]$$

OK