A Variational Structure for Integrable Hierarchies

Mats Vermeeren

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Integrable systems

An integrable system is a (system of) differential equation(s) with too much structure.

Lax Pairs,

Bi-Hamiltonian structure,

Commuting flows $\rightarrow$ hierarchy,

$\ldots$

"An integrable system is a system that I can solve but you cannot."
Lagrangian PDEs

Lagrangian density $L(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots)$

Action $S = \int L \, dx \, dt$

Look for a function $v$ that is a critical point of the action, i.e. for arbitrary infinitesimal variations $\delta v$:

$$0 = \delta S = \int \delta L \, dx \, dt = \int \sum_I \frac{\partial L}{\partial v_I} \delta v_I \, dx \, dt$$

$$= \int \sum_I (-1)^{|I|} D_I \left( \frac{\partial L}{\partial v_I} \right) \delta v \, dx \, dt$$

Euler-Lagrange equation:

$$\frac{\delta L}{\delta v} := \sum_I (-1)^{|I|} D_I \left( \frac{\partial L}{\partial v_I} \right) = 0$$
Example of a Lagrangian PDE

Lagrangian density \( L = \frac{1}{2} \nu_x \nu_t - \nu_x^3 - \frac{1}{2} \nu_x \nu_{xxx} \)

Euler-Lagrange Equation:

\[
0 = \frac{\delta L}{\delta \nu} = \sum_{I} (-1)^{|I|} D_I \left( \frac{\partial L}{\partial \nu_I} \right)
= -\frac{1}{2} D_t(\nu_x) - \frac{1}{2} D_x(\nu_t) + 3 D_x(\nu_x^2) + \frac{1}{2} D_x(\nu_{xxx}) + \frac{1}{2} D_{xxx}(\nu_x)
= -\nu_{xt} + 6\nu_x \nu_{xx} + \nu_{xxxx}
\]

\( \Rightarrow \nu_{xt} = 6\nu_x \nu_{xx} + \nu_{xxxx} \)

Substitute \( u = \nu_x \) to find the Korteweg-de Vries equation

\[ u_t = 6uu_x + u_{xxx}. \]
Example of a Lagrangian PDE

Lagrangian density \( L = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx} \)

Euler-Lagrange Equation:

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= -\frac{1}{2} D_t(v_x) - \frac{1}{2} D_x(v_t) + 3 D_x(v_x^2) + \frac{1}{2} D_x(v_{xxx}) + \frac{1}{2} D_{xxx}(v_x) \\
= -v_{xt} + 6v_x v_{xx} + v_{xxxx} \\
\Rightarrow v_{xt} = 6v_x v_{xx} + v_{xxxx}
\]

Substitute \( u = v_x \) to find the Korteweg-de Vries equation

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\]

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= -v_{xt} + 6v_x v_{xx} + v_{xxxx}
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\[\Rightarrow v_{xt} = 6v_x v_{xx} + v_{xxxx}\]

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Example of a Lagrangian PDE

Lagrangian density $L = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx}$

Euler-Lagrange Equation:

$$0 = \delta L = \sum_l (\sum_I (-1)^{|I|} D_I \left( \frac{\partial L}{\partial v_I} \right)$$

$$= -\frac{1}{2} D_t(v_x) - \frac{1}{2} D_x(v_t) + 3 D_x(v_x^2) + \frac{1}{2} D_x(v_{xxx}) + \frac{1}{2} D_{xxx}(v_x)$$

$$= -v_{xt} + 6 v_x v_{xx} + v_{xxxx}$$

$\Rightarrow v_{xt} = 6 v_x v_{xx} + v_{xxxx}$

Substitute $u = v_x$ to find the Korteweg-de Vries equation

$$u_t = 6uu_x + u_{xxx}.$$
Pluri-Lagrangian systems

Multi-time $\mathbb{R}^N$, coordinates $(t_1, \ldots, t_N) = (x_1, \ldots x_{d-1}, t_d, \ldots, t_N)$

Field $u : \mathbb{R}^N \to \mathbb{R}$

Lagrangian $d$-form $\mathcal{L}(u, u_{t_1}, \ldots, u_{t_n}, u_{t_1t_1}, u_{t_1t_2}, \ldots, u_{t_nt_n})$

Definition

A field $u$ solves the pluri-Lagrangian problem for $\mathcal{L}$ if

- $u$ is a critical point of the action $\int_S \mathcal{L}$
  for all $d$-dimensional surfaces $S$ in $\mathbb{R}^N$ simultaneously.

The differential equations describing this condition are called the multi-time Euler-Lagrange equations.
Stepped Surfaces

Definition

A d-dimensional coordinate surface is a surface $S$ such that for distinct $i_1, \ldots, i_d$ and for all $x \in S$ we have

$$T_x S = \text{span} \left( \frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_d}} \right).$$

A stepped surface is a finite union of coordinate surfaces.

Lemma

If the action is stationary on every stepped surface, then it is stationary on every smooth surface.
Multi-time Euler-Lagrange equations for curves

Theorem

The multi-time Euler-Lagrange equations for the Lagrangian one-form $\mathcal{L} = \sum_i L_i \, dt_i$ are

\[
\frac{\delta_i L_i}{\delta u_I} = 0 \quad \forall I \not\ni t_i, \\
\frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \quad \forall I,
\]

where $i$ and $j$ are distinct, and

\[
\frac{\delta_i L_i}{\delta u_I} := \sum_{\alpha \in \mathbb{N}} (-1)^{\alpha} D_i^{\alpha} \frac{\partial L_i}{\partial u_{It_i}^{\alpha}} = \frac{\partial L_i}{\partial u_I} - D_i \frac{\partial L_i}{\partial u_{It_i}} + D_i^2 \frac{\partial L_i}{\partial u_{It_i}^2} - \ldots.
\]
Multi-time Euler-Lagrange equations for curves

It is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

The variation of the action on $S_i$ is

$$\int_{S_i} (\delta L_i) \, dt_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \, dt_i + \sum_{I} \left( \frac{\delta_i L_i}{\delta u_{lt_i}} \right) \delta u_I \bigg|_{p}$$
Multi-time Euler-Lagrange equations for curves

We have

\[
\int_{S_i} (\delta L_i) \, dt_i = \int_{S_i} \sum_{l \not\in t_i} \frac{\delta_i L_i}{\delta u_l} \delta u_l \, dt_i + \sum_l \left( \frac{\delta_i L_i}{\delta u_{lt_i}} \delta u_l \right) \bigg|_p.
\]

The other piece, \( S_j \), contributes

\[
\int_{S_j} (\delta L_j) \, dt_j = \int_{S_j} \sum_{l \not\in t_j} \frac{\delta_j L_j}{\delta u_l} \delta u_l \, dt_j - \sum_l \left( \frac{\delta_j L_j}{\delta u_{lt_j}} \delta u_l \right) \bigg|_p.
\]
Multi-time Euler-Lagrange equations for curves

We have

\[ \int_{S_i} (\delta L_i) \, dt_i = \int_{S_i} \sum_{l \not\ni t_i} \frac{\delta_i L_i}{\delta u_l} \delta u_l \, dt_i + \sum_l \left( \frac{\delta_i L_i}{\delta u_{lt_i}} \delta u_l \right) \bigg|_p. \]

The other piece, \( S_j \), contributes

\[ \int_{S_j} (\delta L_j) \, dt_j = \int_{S_j} \sum_{l \not\ni t_j} \frac{\delta_j L_j}{\delta u_l} \delta u_l \, dt_j - \sum_l \left( \frac{\delta_j L_j}{\delta u_{lt_j}} \delta u_l \right) \bigg|_p. \]

Summing the two contributions we find

\[ \int_S \delta \mathcal{L} = \int_{S_i} \sum_{l \not\ni t_i} \frac{\delta_i L_i}{\delta u_l} \delta u_l \, dt_i + \int_{S_j} \sum_{l \not\ni t_j} \frac{\delta_j L_j}{\delta u_l} \delta u_l \, dt_j \]

\[ + \sum_l \left( \frac{\delta_i L_i}{\delta u_{lt_i}} - \frac{\delta_j L_j}{\delta u_{lt_j}} \right) \delta u_l \bigg|_p. \]
Multi-time Euler-Lagrange equations for curves

Theorem

The multi-time Euler-Lagrange equations for the Lagrangian one-form $\mathcal{L} = \sum_i L_i \, dt_i$ are

$$\frac{\delta_i L_i}{\delta u_I} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_i L_i}{\delta u_{lt_i}} = \frac{\delta_j L_j}{\delta u_{lt_j}} \quad \forall I,$$

where $i$ and $j$ are distinct, and

$$\frac{\delta_i L_i}{\delta u_I} := \sum_{\alpha \in \mathbb{N}} (-1)^\alpha D_i^\alpha \frac{\partial L_i}{\partial u_{lt_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - D_i \frac{\partial L_i}{\partial u_{lt_i}} + D_i^2 \frac{\partial L_i}{\partial u_{lt_i^2}} - \ldots.$$
Multi-time Euler-Lagrange equations for 2d surfaces

**Theorem**

The multi-time EL equations for \( \mathcal{L} = \sum_{i<j} L_{ij} \, dt_i \wedge dt_j \) are

\[
\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \quad \forall I \not\ni t_i, t_j,
\]

\[
\frac{\delta_{ij} L_{ij}}{\delta u_{lt_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{lt_k}} \quad \forall I \not\ni t_i,
\]

\[
\frac{\delta_{ij} L_{ij}}{\delta u_{lt_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{lt_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{lt_k t_i}} = 0 \quad \forall I,
\]

where \( i, j, \) and \( k \) are distinct, and

\[
\frac{\delta_{ij} L_{ij}}{\delta u_I} := \sum_{\alpha, \beta \in \mathbb{N}} (-1)^{\alpha+\beta} D_i^\alpha D_j^\beta \frac{\partial L_{ij}}{\partial u_{lt_i^\alpha t_j^\beta}}.
\]
The Korteweg-de Vries hierarchy

KdV hierarchy:

\[ u_{t_k} = D_x(r_k[u]) \]

\[ u_{t_1} = u_x \]
\[ u_{t_2} = u_{xxx} + 6uu_x = D_x(u_{xx} + 3u^2) \]
\[ u_{t_3} = u_x^5 + 20u_xu_{xx} + 10uu_{xxx} + 30u^2u_x \]
\[ = D_x(u_x^4 + 10u_x^2 + 10uu_{xx} + 10u^3) \]
\[ \vdots \]

Motivated by the first equation, we identify space with the first time-coordinate: \( x \equiv t_1 \).
The potential Korteweg-de Vries hierarchy

Potential \( \nu \) such that \( \nu_x = u \), \( g[\nu] := r[\nu_x] \).

KdV equations become:

\[
\nu_{xt_k} = D_x(g_k[\nu])
\]

Lagrangian: \( L_k := \frac{1}{2} \nu_x \nu_t - h_k \), with \( h_k = \frac{1}{4k+2}g_{k+1} \).

\[
\frac{\delta L_k}{\delta \nu} = -\nu_{xt_k} + D_x(g_k[\nu])
\]

PKdV hierarchy:

\[
\nu_{t_k} = g_k[\nu]
\]
Pluri-Lagrangian structure for PKdV hierarchy

Lagrangian two-form

\[ \mathcal{L} = \sum_{i<j} L_{ij} \, dt_i \wedge dt_j, \]

with coefficients:

\[ L_{1i} = \frac{1}{2} \nu_x v_{t_i} - h_i \]

(the classical Lagrangians)

\[ L_{ij} = \frac{1}{2} (v_{t_i} g_j - v_{t_j} g_i) + (a_{ij} - a_{ji}) - \frac{1}{2} (b_{ij} - b_{ji}) \]

(Obtained from the fact that the flows are variational symmetries of each other)
Multi-time Euler-Lagrange equations

- By construction, the equations \( \frac{\delta_{1i}L_{1i}}{\delta v} = 0 \) are

\[
v_{xt_i} = D_x g_i.
\]

- The equation \( \frac{\delta_{1i}L_{1i}}{\delta v_x} = \frac{\delta_{ij}L_{ji}}{\delta v_{t_j}} \) yields

\[
v_{t_i} = g_i.
\]

- All other Euler-Lagrange equations are corollaries of these.
Multi-time Euler-Lagrange equations

- By construction, the equations \( \frac{\delta_1 i L_1 i}{\delta v} = 0 \) are

\[
v_{xt_i} = D_x g_i.
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- The equation \( \frac{\delta_1 i L_1 i}{\delta v_x} = \frac{\delta_{ij} L_{ji}}{\delta v_{t_j}} \) yields

\[
v_{t_i} = g_i.
\]

- All other Euler-Lagrange equations are corollaries of these.
Multi-time Euler-Lagrange equations

The multi-time EL equations for the Lagrangian two-form
\[ \mathcal{L} = \sum_{i<j} L_{ij} \, dt_i \wedge dt_j \] with
\[
L_{1i} = \frac{1}{2} \nu_x \nu_{ti} - h_i
\]
\[
L_{ij} = \frac{1}{2} (\nu_{ti} g_j - \nu_{tj} g_i) + (a_{ij} - a_{ji}) - \frac{1}{2} (b_{ij} - b_{ji})
\]
are the PKdV equations
\[ \nu_{ti} = g_i, \]
and corollaries thereof.
Multi-time Euler-Lagrange equations

The multi-time EL equations are the PKdV equations,

\[ v_{t_i} = g_i, \]

and corollaries thereof.

Surprisingly, our variational method produces first order evolution equations. Discrepancy with:

- Classical Lagrangian formalism
- Discrete pluri-Lagrangian systems on quad graphs
References

Main source.

Discrete Pluri-Lagrangian systems.

Introduction to integrable systems.
Olivier Babelon, Denis Bernard, and Michel Talon. *Introduction to classical integrable systems.* 2003.

Lagrangian methods (and other topics) in classical mechanics.