# SUFFICIENT SECOND ORDER OPTIMALITY CONDITIONS FOR A STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM OF A WEAKLY SINGULAR INTEGRAL EQUATION

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**Abstract**

An optimal control problem for a nonlinear weakly singular Volterra integral equation is investigated, where pointwise constraints are given on the control and the state. The state constraints are of bottleneck type, hence associated Lagrange multipliers can assumed to be bounded and measurable functions. Based on this property a second order sufficient optimality condition is established, which considers strongly active constraints.

**Keywords:** Optimal control, nonlinear integral equation, sufficient second order optimality condition, pointwise mixed control-state constraints, bottleneck constraints

**AMS subject classification:** 49K22, 49K20, 90C48

# 1 Introduction

In this paper, we discuss second order sufficient optimality conditions for the following optimal control problem

$$(P)\begin{cases} \min & J(u) = F(y,u) = \displaystyle\int_0^T f(t, y(t), u(t))\, dt \\ \text{subject to} & \\ & y(t) = y_0(t) + \displaystyle\int_0^t k(t,s) b(s, y(s), u(s))\, ds \qquad (1.1) \\ \text{and} & \\ & 0 \le u(t) \le c(t) + \gamma(t) y(t), \quad 0 \le t \le T. \qquad (1.2) \end{cases}$$

In this setting, $u \in L^\infty[0,T]$ is the control while $y \in C[0,T]$ the associated state. The following fixed quantities are given : $T > 0$, $y_0 \in C[0,T]$, $f, b : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, $k \in C(D)$, where $D = [0,T]^2 \setminus \{(t,s) : \ s \ge t\}$, $c \in C[0,T]$, $\gamma \in C[0,T]$.

The main difficulty in our problem is the presence of the pointwise state constraint

$$u(t) \le c(t) + \gamma(t) y(t) \tag{1.3}$$

in (1.2). Moreover, the kernel function $k(t,s)$ is allowed to be weakly singular so that equation (1.1) covers nonlinear initial boundary value problems for 1-D parabolic equations.

The theory of second order sufficient optimality conditions for optimal control problems governed by partial differential equations with pointwise state constraints is faced with specific difficulties, which are still far from being solved. We refer to Casas/Tröltzsch/Unger [4] and Raymond/Tröltzsch [9]. The source of difficulties is that the Lagrange multipliers associated with the pointwise state constraints are measures.

In our problem (P), the situation is slightly simpler, since the constraint (1.3) is a mixed control-state constraint of bottleneck type. In this case, the Lagrange multipliers are more regular, they are functions of $L^\infty(0,T)$, see Bergounioux/Tröltzsch [2], [3], Tröltzsch [11] or Arada/Raymond [1].

The higher regularity of the multipliers is the main prerequisite enabling us to work out second order conditions. However, the arguments are still quite complicated.

We need tedious estimations to obtain conditions being close to the associated necessary second order conditions. All this shows that second order conditions for the optimal control of PDEs with more general pointwise state-constraints are a delicate issue, and we hope to contribute some new and meaningful techniques to this field.

The following assumptions are required:

(A1) The functions $b$, $f$ satisfy the following Carathéodory conditions: $b, f$ are of class $C^2$ with respect to $(y,u)$. Moreover, for all $(y,u) \in \mathbb{R}^2$, they are measurable with respect to $t$. By $b'(t,y,u)$ and $b''(t,y,u)$ we denote the gradient and the Hessian matrix of $b$ with respect to $(y,u)$:

$$b'(t,y,u) = \begin{pmatrix} b_y(t,y,u) \\ b_u(t,y,u) \end{pmatrix}, \qquad b''(t,y,u) = \begin{pmatrix} b_{yy}(t,y,u) & b_{yu}(t,y,u) \\ b_{uy}(t,y,u) & b_{uu}(t,y,u) \end{pmatrix},$$

$|b'|$, $|b''|$ are defined by adding the absolute values of all entries.

For all $M > 0$, there are constants $C_M > 0$ and a continuous, monotone increasing function $\eta \in C(\mathbb{R}^+ \cup \{0\})$ with $\eta(0) = 0$ such that $b(.,0,0) \in L^\infty(0,T)$,

$$\begin{aligned} |b'(t,y,u)| + |b''(t,y,u)| &\le C_M, \\ |b''(t,y_1,u_1) - b''(t,y_2,u_2)| &\le \eta(|y_1 - y_2| + |u_1 - u_2|) \end{aligned}$$

for almost all $t$ and all $|y|, |u|, |y_1|, |y_2|, |u_1|, |u_2| \le M$.

In addition, on $[0,T] \times \mathbb{R}^2$ we suppose

$$b_u(t,y,u) \ge 0, \quad b_y(t,y,u) \ge 0, \quad |b(t,y_1,u) - b(t,y_2,u)| \le L|y_1 - y_2|.$$

The non-negativity of $b_u$ and $b_y$ is not supposed to have existence and uniqueness for the state equation. It is needed to show the regularity of Lagrange multipliers. Moreover, this is the main assumption for the proof of Lemma 3, which is also related to the regularity of multipliers and a basic tool for the second order conditions.

For the function $f$, we require $f(\cdot,0,0) \in L^\infty(0,T)$, $f'(\cdot,0,0) \in L^\infty(0,T;\mathbb{R}^2)$, $f''(\cdot,0,0) \in L^\infty(0,T;\mathbb{R}^{2\times 2})$,

$$|f''(t,y_1,u_1) - f''(t,y_2,u_2)| \le \eta(|y_1 - y_2| + |u_1 - u_2|)$$

for almost all $t \in [0,T]$ and $|y_1| \le M$, $|y_2| \le M$, $|u_1| \le M$, $|u_2| \le M$.

(A2) The kernel $k$ is nonnegative in $D$ and satisfies $k(t,s) \le c_k(t-s)^{-\alpha}$ for all $(t,s) \in D$ with some $\alpha \in (0,1)$ ($k$ is weakly singular).

(A3) It holds $c(t) > 0$, $\gamma(t) \ge 0 \quad \forall t \in [0,T]$.

Various other estimates for $f$, $f'$, $b$, $b'$ can be derived from (A1) by the mean value theorem. For convenience, we assume in (A1) a global Lipschitz continuity with respect to $y$, which is not really a strong assumption. Our aim is to later investigate optimal control problems for parabolic partial differential equations. In this case, the maximum principle for the parabolic equation delivers a-priori-bounds for the solution of the parabolic equation. Therefore, the Lipschitz continuity with respect to $y$ is only needed on a bounded set, which is defined by the data.

# 2 Preliminary results

**Lemma 1** *For each $u \in L^\infty(0,T)$, the integral equation (1.1) admits an unique solution $y \in C[0,T]$.*

**Proof:** Let $u$ be given fixed. We define an operator $T : C[0,T] \to C[0,T]$ by

$$(Ty)(t) := y_0(t) + \int_0^t k(t,s) b(s, y(s), u(s))\, ds.$$

Let us introduce the norm $\|f\|_\lambda := \max_{t\in[0,T]} |e^{-\lambda t} f|$, which is equivalent to the usual maximum norm. We show that $T$ is a contraction in this norm for a suitable $\lambda > 0$. For this purpose, we estimate $T(y_1) - T(y_2)$. We define $d(t) := e^{-\lambda t}(Ty_1)(t) - e^{-\lambda t}(Ty_2)(t)$ and obtain

$$\begin{aligned}
|d(t)| &\le \int_0^t |k(t,s) e^{-\lambda t}(b(s, y_1(s), u(s)) - b(s, y_2(s), u(s)))|\, ds \\
&\le \int_0^t c_k\, (t-s)^{-\alpha} e^{-\lambda t} L |y_1(s) - y_2(s)|\, ds \\
&\le c_k L \cdot \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} e^{-\lambda s} |y_1(s) - y_2(s)|\, ds
\end{aligned}$$

by (A1), (A2). Taking advantage of our equivalent norm and applying Hölder's inequality, we continue by

$$\begin{aligned}
|d(t)| &\le c_k L \|y_1 - y_2\|_\lambda \cdot \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)}\, ds \\
&\le c_k L \|y_1 - y_2\|_\lambda \cdot \left(\int_0^t (t-s)^{-\alpha p}\, ds\right)^{\frac{1}{p}} \cdot \left(\int_0^t e^{-\lambda q(t-s)}\, ds\right)^{\frac{1}{q}}
\end{aligned}$$

with $1 < p < \frac{1}{\alpha}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Evaluating these two integrals, we find

$$\begin{aligned}
|d(t)| &\le c_k L\, \|y_1 - y_2\|_\lambda \cdot \left(\frac{t^{1-\alpha p}}{1-\alpha p}\right)^{\frac{1}{p}} \cdot \left(\frac{1 - e^{-\lambda q t}}{\lambda q}\right)^{\frac{1}{q}}, \\
&\le c_k L \|y_1 - y_2\|_\lambda \cdot \left(\frac{T^{1-\alpha p}}{1-\alpha p}\right)^{\frac{1}{p}} \cdot \left(\frac{1}{\lambda q}\right)^{\frac{1}{q}}.
\end{aligned}$$

For sufficiently large $\lambda$, the following holds

$$c_k L \cdot \left(\frac{T^{1-\alpha p}}{1-\alpha p}\right)^{\frac{1}{p}} \cdot \left(\frac{1}{\lambda q}\right)^{\frac{1}{q}} \leq \varrho < 1,$$

hence

$$\|Ty_1 - Ty_2\|_\lambda = \max_{t\in[0,T]} |e^{-\lambda t}(Ty_1)(t) - e^{-\lambda t}(Ty_2)(t)| \leq \varrho\|y_1 - y_2\|_\lambda.$$

Thus, $T$ is a contraction and Banach's fixed point principle implies the assertion. □

By Lemma 1, a solution mapping $S : L^\infty(0,T) \to C[0,T]$ is defined, which assigns to $u \in L^\infty(0,T)$ the solution $y$ of (1.1). We shall prove that $S$ is Fréchet differentiable on $L^\infty(0,T)$. To this aim, we define operators $K : L^\infty(0,T) \to C[0,T]$ or $L^2(0,T) \to L^2(0,T)$, respectively, and $B : C[0,T] \times L^\infty(0,T) \to L^\infty(0,T)$ by

$$\begin{aligned} (Kz)(t) &:= \int_0^t k(t,s)z(s)\,ds \\ B(y,u)(t) &:= b(t,y(t),u(t)). \end{aligned}$$

Let $(\bar{y},\bar{u}) \in C[0,T] \times L^\infty(0,T)$ be a fixed reference pair. Later, this couple will stand for a local minimum of (P). At $(\bar{y},\bar{u})$ we define multiplication operators $\bar{B}_y$ and $\bar{B}_u$,

$$(\bar{B}_y y)(t) = b_y(t,\bar{y},\bar{u}) \cdot y(t), \quad (\bar{B}_u u)(t) = b_u(t,\bar{y},\bar{u}) \cdot u(t).$$

The operators $\bar{B}_y$ and $\bar{B}_u$ are linear and continuous in $L^\infty(0,T)$ as well as in $L^2(0,T)$, hence they belong to $\mathcal{L}(L^\infty(0,T))$ and $\mathcal{L}(L^2(0,T))$, respectively.

By means of these definitions, the integral equation (1.1) admits the form

$$y = y_0 + KB(y,u).$$

In all what follows, $(\bar{y},\bar{u})$ is an *admissible pair* of our problem (P). This means that (1.1) is satisfied together with (1.2). In particular, $\bar{y} = S(\bar{u})$ holds, i.e. $\bar{y} = y_0 + KB(\bar{y},\bar{u})$.

**Lemma 2** *The nonlinear mapping $S : L^\infty(0,T) \to C[0,T]$ is of class $C^1$. Its Fréchet derivative $S'(\bar{u})$ at $\bar{u}$ has the representation*

$$S'(\bar{u})u = (I - K\bar{B}_y)^{-1} K\bar{B}_u u. \tag{2.1}$$

**Proof:** The mapping $y = S(u)$ is defined implicitely by $\Phi(y,u) = y - y_0 - KB(y,u) = 0$. Thanks to our assumptions, $\Phi: C[0,T] \times L^\infty(0,T) \to C[0,T]$ is continuously Fréchet–differentiable. Moreover, the partial derivative $\Phi_y(\bar{y},\bar{u})$ is nonsingular: We find

$$\Phi_y(\bar{y},\bar{u})\, y = y - K\bar{B}_y\, y.$$

The equation

$$y - K\bar{B}_y\, y = z$$

is a weakly singular linear Volterra equation, which is uniquely solvable for all $z \in C[0,T]$, see Michlin [8]. The continuity of $\Phi_y^{-1} = (I - K\bar{B}_y)^{-1}$ in $C[0,T]$ follows from Banach's theorem on the inverse. According to the implicit function theorem, the mapping $S$ is of class $C^1$. The formula for $S'(\bar{u})$ follows immediately by the chain rule for the differentiation of $\Phi(S(u),u) = 0$ with respect to $u$. □

**Remark 1** *The operator $S'$ can be represented as a Volterra integral operator in the form*

$$\left(S'(\bar{u})u\right)(t) = \int_0^t \sigma(t,s)u(s)\, ds,$$

*where $\sigma$ is a weakly singular kernel, see [5]. Therefore, the operator $I - \gamma \cdot S'(\bar{u})$ is invertible in $C[0,T]$ as well.*

**Remark 2** *Let $[0,T]$ be the union of two disjoint sets $M_1$, $M_2$, and let $f_1 \in L^\infty(M_1)$, $f_2 \in L^\infty(M_2)$ be given. Define $\tilde{f} \in L^\infty(0,T)$ by*

$$\tilde{f} = \begin{cases} f_1(t) & \text{on } M_1, \\ f_2(t) & \text{on } M_2. \end{cases}$$

*Clearly, there exists a unique solution $u \in L^\infty(0,T)$ of the Volterra integral equation*

$$u(t) = \tilde{f}(t) + \int_0^t \gamma(t)\chi_{M_2}(t)\sigma(t,s)u(s)\, ds. \tag{2.2}$$

*This solution satisfies*

$$u(t) = \begin{cases} f_1(t) & \text{on } M_1, \\ f_2(t) + \gamma(t)\int_0^t \sigma(t,s)u(s)\, ds & \text{on } M_2. \end{cases} \tag{2.3}$$

*On the other hand, any solution of (2.3) satisfies (2.2) and is therefore unique. In other words, there exists a unique $u \in L^\infty(0,T)$ satisfying $u = f_1$ on $M_1$ and $u = \gamma S'(\bar{u})u + f_2$ on $M_2$. Moreover, one easily verifies the following estimates*

$$\|u\|_{L^\infty(0,T)} \leq c_1 \|\tilde{f}\|_{L^\infty(0,T)}, \tag{2.4}$$

$$\|u\|_{L^2(0,T)} \leq c_2 \|\tilde{f}\|_{L^2(0,T)}. \tag{2.5}$$

Let us introduce the first order remainder term $r(\bar{u}, h) := S(\bar{u}+h) - S(\bar{u}) - S'(\bar{u})h$. It satisfies

$$\frac{\|r(\bar{u}, h)\|_{C[0,T]}}{\|h\|_{L^\infty(0,T)}} \to 0 \quad \text{as } \|h\|_{L^\infty(0,T)} \to 0,$$

since $S$ is Fréchet differentiable at $\bar{u}$. We extend the linear operator $S'(\bar{u})$ to an operator belonging to $\mathcal{L}(L^2(0,T))$ defined by (2.1). To deal with the well-known two-norm discrepancy (see Maurer [7]), we have to estimate the $L^2$-norm of the remainder. We cannot in general substitute above the $L^\infty$-norm by the $L^2$-norm, since, in $L^2(0,T)$, $S$ is not in general of class $C^1$ . However, it is possible to prove the property

$$\frac{\|r(\bar{u}, h)\|_{L^2(0,T)}}{\|h\|_{L^2(0,T)}} \to 0 \quad \text{if } \|h\|_{L^\infty(0,T)} \to 0.$$

We refer to Maurer [7].

**Definition 1** *An operator $A \in \mathcal{L}(L^2(0,T))$ is said to be nonnegative, if $u \geq 0$ a.e. on $(0,T)$ implies $Au \geq 0$ a.e. on $(0,T)$. In this case we write $A \geq 0$.*

**Lemma 3** *(Comparison principle) Under assumptions (A1)-(A3), the following relations of nonnegativity hold*

$$S'(\bar{u}) \geq 0, \tag{2.6}$$

$$(I - \gamma \cdot S'(\bar{u}))^{-1} \geq 0. \tag{2.7}$$

**Proof:** From (A1) we get $\bar{B}_y \geq 0$, while (A2) ensures $K \geq 0$, hence $K\bar{B}_y \geq 0$. Analogously, (A1) and (A2) imply $K\bar{B}_u \geq 0$.

Let $u \geq 0$. Then the Neumann series expansion

$$\begin{aligned} S'(\bar{u})u &= (I - K\bar{B}_y)^{-1} K\bar{B}_u u \\ &= K\bar{B}_u u + K\bar{B}_y K\bar{B}_u u + (K\bar{B}_y)^2 K\bar{B}_u u + ... \end{aligned}$$

shows $S'(\bar{u})u \geq 0$, as $K\bar{B}_u \geq 0$, $(K\bar{B}_y)^n \geq 0 \; \forall n \in \mathbb{N}$. Therefore, (2.6) is true. (2.7) is obtained by the same argument, since

$$(I - \gamma S'(\bar{u}))^{-1} u = u + \gamma S'(\bar{u})u + (\gamma S'(\bar{u}))^2 u + ...$$

and $(\gamma S'(\bar{u}))^n \geq 0 \; \forall n \in \mathbb{N}$. □

**Remark 3** *The relations (2.6) and (2.7) extend to the associated $L^2$-adjoint operators. Therefore, we have especially*

$$S'(\bar{u})^* \geq 0. \tag{2.8}$$

*Moreover, the nonnegativity property extends to the constructions in Remark 2. In this case, it is possible to prove that*

$$\tilde{f} \geq 0 \quad \textit{implies} \quad u \geq 0 \tag{2.9}$$

*for the solution $u$ of (2.2).*

# 3 Optimality system and auxiliary estimates

Let $(\bar{y}, \bar{u})$ be a locally optimal solution of (P). In this section, we set up the associated first order necessary optimality conditions in form of a Kuhn Tucker type theorem. To this aim, we introduce the Lagrange functional $L : C[0,T] \times L^\infty(0,T) \times C[0,T] \times L^\infty(0,T)^2 \to \mathbb{R}$,

$$\begin{aligned} L(y, u, p, \mu_1, \mu_2) \;=\; & F(y,u) \\ & + \int_0^T (y(t) - y_0(t) - \int_0^t k(t,s) b(s, y(s), u(s))\, ds) p(t)\, dt \\ & - \int_0^T \mu_1(t) u(t)\, dt + \int_0^T (u(t) - c(t) - \gamma(t) y(t)) \mu_2(t)\, dt. \end{aligned}$$

Let us comment this choice for $L$. The equation (1.1) is considered in $C[0,T]$, while the inequality constraints (1.2) are given in $L^\infty(0,T)$. Knowing the general Kuhn Tucker theory in Banach spaces, one expects associated Lagrange multipliers $p \in (C[0,T])^*$ and $\mu_i \in (L^\infty(0,T))^*$ together with a related quite complicated Lagrange functional. In contrast to this, special techniques for optimal control problems of bottleneck type have shown that, under natural assumptions, the Lagrange multipliers can be expressed by regular functions, i.e. $p \in C[0,T]$ and $\mu_i \in L^\infty(0,T)$, see Bergounioux/Tröltzsch [2],[3]. This well known advantage of bottleneck type problems is our key idea to establish special second order sufficient optimality conditions, which are hardly to expect for $\mu_i \in (L^\infty(0,T))^*$. The existence of such regular multipliers can be shown under a Slater type condition and the assumption $\gamma(t) \geq 0$. Here, the nonnegativity of $\gamma$ plays a crucial role.

Therefore, we are justified to *assume* that Lagrange multipliers $\bar{p} \in C[0,T]$ and $\bar{\mu}_i \in L^\infty(0,T)$ exist such that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ satisfies the following first order necessary optimality system (FON),

$$(FON) \begin{cases} D_y L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) & = 0 \\ D_u L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) & = 0 \\ \text{and for almost all } t \in [0,T] & \\ \bar{\mu}_1(t) & \geq 0 \\ \bar{\mu}_2(t) & \geq 0 \\ \bar{u}(t)\bar{\mu}_1(t) & = 0 \\ (\bar{u}(t) - c(t) - \gamma(t)\bar{y}(t))\bar{\mu}_2(t) & = 0. \end{cases}$$

In the next section, we discuss a second order sufficient optimality condition (SSC). For this purpose, we define here *strongly active sets* and the associated *critical subspace*. Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfills (FON).

**Definition 2** *Let $\delta_1, \delta_2 > 0$ be real numbers and $\bar{\mu}_1, \bar{\mu}_2 \in L^\infty(0,T)$ be the Lagrange multipliers introduced in (FON). The sets*

$$A_1(\delta_1) := \{t \in [0,T] : \bar{\mu}_1(t) \geq \delta_1\}, \tag{3.1}$$
$$A_2(\delta_2) := \{t \in [0,T] : \bar{\mu}_2(t) - (S'(\bar{u})^* \gamma\, \bar{\mu}_2)(t) \geq \delta_2\} \tag{3.2}$$

*are called strongly active sets. We say that $(y,u) \in C[0,T] \times L^\infty(0,T)$ belongs to the critical subspace, if*

*(a) $u = 0$ on $A_1$ and*

*(b) $u = \gamma y$ on $A_2$,*

*where $y = S'(\bar{u})u$.*

To prove a theorem on second order sufficiency, we later have to compare the reference pair $(\bar{y}, \bar{u})$ with another admissible pair $(u,y)$

$$y - \bar{y} = S(u) - S(\bar{u}) = S'(\bar{u})(u - \bar{u}) + r_1(\bar{u}, u - \bar{u}).$$

Let us denote for short the remainder and the derivative by $r_1$ and $S'$, respectively.

Before continuing our analysis, we briefly discuss the main difficulties and our main ideas to resolve them. We start with the case of pure control constraints, i.e. $\gamma(t) \equiv 0$. Then the constraints are

$$0 \leq u(t) \leq c(t).$$

On $A_1$, we have $\bar{u}(t) \equiv 0$, hence $u - \bar{u} \geq 0$ on $A_1$, while $\bar{u}(t) = c(t)$ holds on $A_2$, thus $u - \bar{u} \leq 0$ on $A_2$. The associated terms in the Lagrange functional can be estimated by

$$\begin{aligned}\int_{A_1} \bar{\mu}_1(u-\bar{u})\,dt - \int_{A_2} \bar{\mu}_2(u-\bar{u})\,dt &\geq \int_{A_1} \delta_1(u-\bar{u})\,dt + \int_{A_2} \delta_2(\bar{u}-u)\,dt \\ &= \delta_1\|u-\bar{u}\|_{L^1(A_1)} + \delta_2\|u-\bar{u}\|_{L^1(A_2)}.\end{aligned}$$

Now we return to the state constraints, i.e.

$$0 \leq u(t) \leq c(t) + \gamma y(t).$$

Assume that the control-state mapping is linear, which is true for $y_0 = 0$ and a linear function $b$. Then $S' = S$, hence

$$0 \leq u \leq c + \gamma S' u \tag{3.3}$$

holds for any admissible control $u$. On $A_1$, we have again $0 = \bar{u} \leq u$, hence $u - \bar{u} \geq 0$ on $A_1$. In contrast to the case of pure control constraints, the relation $u \leq \bar{u}$ cannot be expected on $A_2$ now. If $u > \bar{u}$ holds somewhere on $[0,T] \setminus A_2$, then $S'u > S'\bar{u}$ can hold on $A_2$. Then the right hand side of (3.3) is greater than $c + \gamma S'\bar{u}$ and $u > \bar{u}$ can happen.

We represent $u$ in the form $u = u_1 + u_2$, such that we can prove $u_1 \leq \bar{u}$ on $A_2$, where $u_2$ stands for the additional margin of freedom, which is caused by $u > \bar{u}$ outside of $A_2$. Hence, we split $u$ in two parts, $u = u_1 + u_2$ on $[0,T]$, where

$$\begin{array}{lll} u_1 = \bar{u}, & u_2 = u - \bar{u} & \text{on } [0,T] \setminus A_2, \\ u_2 = \gamma(S'u_2 + r_1), & u_1 = u - u_2 & \text{on } A_2. \end{array}$$

The functions $u_1$ and $u_2$ are well defined. To see this, we apply Remark 2, where $M_1 = [0,T] \setminus A_2$ and $M_2 = A_2$. Note that $S'u_2 = S'(\chi_{M_1}(u-\bar{u}) + \chi_{M_2} u_2)$. From (2.4) and the properties of the remainder we get easily

$$\|u_2\|_{L^\infty(0,T)} \leq c_3 \|u - \bar{u}\|_{L^\infty(0,T)}.$$

Therefore, we find

$$\begin{aligned}\|u_1 - \bar{u}\|_{L^\infty(A_2)} &\leq \|u - \bar{u}\|_{L^\infty(A_2)} + \|u_2\|_{L^\infty(A_2)} \\ &\leq c_4 \|u - \bar{u}\|_{L^\infty(0,T)}.\end{aligned} \tag{3.4}$$

**Lemma 4** *It holds*

$$\bar{u} - u_1 \geq 0 \quad a.e. \text{ on } [0,T]. \tag{3.5}$$

**Proof:** Notice that, on $A_2$, the inequality $\bar{\mu}_2 \geq \bar{\mu}_2 - S'(\bar{u})^*\gamma\bar{\mu}_2 \geq \delta_2 > 0$ holds. Thus (FON) implies $\bar{u} = c + \gamma\bar{y} = 0$ there. In addition, we know on $A_2$

$$u - \gamma S(u) \leq c = \bar{u} - \gamma S(\bar{u}).$$

In view of this, it holds on $A_2$

$$\begin{aligned}
u - \gamma(S(u) - S(\bar{u})) &\leq \bar{u} \\
u - \gamma(S'(u - \bar{u}) + r_1) &\leq \bar{u} \\
u_1 - \gamma S' u_1 + (u_2 - \gamma(S' u_2 + r_1)) &\leq \bar{u} - \gamma S' \bar{u} \\
u_1 - \gamma S' u_1 &\leq \bar{u} - \gamma S' \bar{u} \\
(I - \gamma S')(u_1 - \bar{u}) &\leq 0,
\end{aligned} \tag{3.6}$$

where we have used the definition of $u_2$. Outside of $A_2$ we get by definition $u_1 = \bar{u}$. We are now in the situation, which was described on the end of Remark 3. Indeed, if we denote $f_2 := (I - \gamma S')(\bar{u} - u_1)$ and $f_1 := 0$, then we have $\tilde{f} \geq 0$. Applying (2.9) to (3.6), we obtain

$$\bar{u} - u_1 \geq 0 \quad \text{a.e. on } [0, T]$$

which is just inequality (3.5). □

**Lemma 5** *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfills the first order optimality system (FON). Then*

$$\int_0^T (u - \bar{u})\bar{\mu}_1 \, dt \geq \frac{\delta_1}{\varepsilon}\|u - \bar{u}\|^2_{L^2(A_1)}, \tag{3.7}$$

$$-\int_0^T (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dt \geq \frac{\delta_2}{c_4\varepsilon}\|u_1 - \bar{u}\|^2_{L^2(A_2)} \tag{3.8}$$

*hold true for all $\varepsilon > 0$ and all admissible pairs $(u, y)$ satisfying $\|u - \bar{u}\|_{L^\infty(0,T)} < \varepsilon$.*

**Proof: (i)** By (FON) we have $\bar{\mu}_1 > 0$ only if $\bar{u} = 0$. Moreover, $u$ is admissible, hence $u \geq 0$ and we have almost everywhere

$$(u - \bar{u})\bar{\mu}_1 \geq 0.$$

Therefore, (3.1) yields

$$\int_0^T (u - \bar{u})\bar{\mu}_1 \, dt \geq \int_{A_1} (u - \bar{u})\bar{\mu}_1 \, dt \geq \delta_1 \|u - \bar{u}\|_{L^1(A_1)}.$$

By our assumption, we have $\|u-\bar u\|_{L^\infty(0,T)} < \varepsilon$. In particular, this inequality includes $\|u-\bar u\|_{L^\infty(A_1)} < \varepsilon$. Consequently, we obtain

$$\int_0^T (u-\bar u)\bar\mu_1 \, dt \geq \delta_1 \|u-\bar u\|_{L^1(A_1)} \cdot \frac{\|u-\bar u\|_{L^\infty(A_1)}}{\varepsilon} \geq \frac{\delta_1}{\varepsilon}\|u-\bar u\|^2_{L^2(A_1)},$$

and (3.7) is proved.

**(ii)** Next, we discuss the second integral. Because of (FON) we have $\bar\mu_2 > 0$ only if $\bar u - c - \gamma\bar y = 0$. In addition, $(y,u)$ is admissible, that means in particular $u \leq c + \gamma y$. Thus we obtain almost everywhere

$$-(u-\bar u-\gamma(y-\bar y))\bar\mu_2 \geq 0$$

and

$$-\int_0^T (u-\bar u-\gamma(y-\bar y))\bar\mu_2 \, dt \geq -\int_{A_2} (u-\bar u-\gamma(y-\bar y))\bar\mu_2 \, dt.$$

Let us discuss this integral more detailed. Expressing $y-\bar y$ by the controls,

$$-\int_{A_2} (u-\bar u-\gamma(y-\bar y))\bar\mu_2 \, dt = -\int_{A_2} (u-\bar u-\gamma(S'(u-\bar u)+r_1))\bar\mu_2 \, dt. \quad (3.9)$$

The definition of $u_1$ and $u_2$ yields on $A_2$

$$u-\gamma(S'u+r_1) = u_1+u_2-\gamma S'u_1-\gamma S'u_2-\gamma r_1 = u_1-\gamma S'u_1.$$

Inserting the last equation in (3.9), we continue by

$$\begin{aligned} -\int_{A_2} (u-\bar u-\gamma(y-\bar y))\bar\mu_2 \, dt &= -\int_{A_2} (u_1-\bar u-\gamma(S'(u_1-\bar u)))\bar\mu_2 \, dt \\ &= -\int_0^T (u_1-\bar u-\gamma(S'(u_1-\bar u)))\chi_{A_2}\bar\mu_2 \, dt \\ &= -\int_0^T (u_1-\bar u)(I-(\gamma S')^*)(\chi_{A_2}\bar\mu_2) \, dt \\ &= -\int_{A_2} (u_1-\bar u)(I-(\gamma S')^*)(\chi_{A_2}\bar\mu_2) \, dt \quad (3.10) \end{aligned}$$

For the last equation, we used $\bar u - u_1 = 0$ outside of $A_2$. Now we discuss the expression $(I-(\gamma S')^*)(\chi_{A_2}\bar\mu_2)$ in (3.10). On $A_2$ we have

$$(I-(\gamma S')^*)(\chi_{A_2}\bar\mu_2) = \chi_{A_2}\bar\mu_2 - (S')^*(\gamma\chi_{A_2}\bar\mu_2) = \bar\mu_2 - (S')^*(\gamma\chi_{A_2}\bar\mu_2).$$

Using the non-negativity of $S'^*$, following from (2.8) together with $\chi_{A_2}\bar\mu_2 \leq \bar\mu_2$, we obtain

$$(\gamma S')^*(\chi_{A_2}\bar\mu_2) \leq (\gamma S')^*\bar\mu_2.$$

Combining these results, we continue by

$$(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) = \bar{\mu}_2 - (S')^*(\gamma\chi_{A_2}\bar{\mu}_2) \geq (I - (\gamma S')^*)\bar{\mu}_2 \geq \delta_2, \qquad (3.11)$$

where the last inequality follows from the definition (3.2) of $A_2$. Inserting (3.5) and (3.11) in (3.10), we infer

$$\begin{aligned} -\int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dt &= -\int_{A_2} (u_1 - \bar{u})(I - (\gamma S')^*)(\chi_{A_2}\bar{\mu}_2) \, dt \\ &\geq \delta_2 \|\bar{u} - u_1\|_{L^1(A_2)}. \end{aligned}$$

Now we invoke $\|u - \bar{u}\|_{L^\infty(0,T)} < \varepsilon$ and (3.4) to obtain

$$\begin{aligned} -\int_{A_2} (u - \bar{u} - \gamma(y - \bar{y}))\bar{\mu}_2 \, dt &\geq \delta_2 \|u_1 - \bar{u}\|_{L^1(A_2)} \cdot \frac{\|u - \bar{u}\|_{L^\infty(A_2)}}{\varepsilon} \\ &\geq \frac{\delta_2}{c_4 \varepsilon} \|u_1 - \bar{u}\|^2_{L^2(A_2)} \end{aligned}$$

implying inequality (3.8). □

Note that, if $A_1 \cup A_2 = [0, T]$, the critical subspace contains only the function which is identical 0. Therefore, the assumptions of the following Theorem 1 are automatically fulfilled. In this case, (3.7) and (3.8) express the so-called *first order sufficient optimality conditions.*

# 4 Second order sufficient optimality condition

This section includes the proof of a second order sufficient optimality condition (SSC). Let us first state the result.

**Theorem 1** *Assume that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ fulfills the first order optimality system (FON). Suppose that there exists $\delta > 0$ such that the coercivity condition*

$$(SSC) \qquad L''_{(u,y)}(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)[h_y, h_u]^2 \geq \delta \|h_u\|^2_{L^2(0,T)}$$

*is satisfied for all $(h_y, h_u)$ belonging to the critical subspace. Then there exist $\delta_s > 0$ and $\varepsilon > 0$ such that the quadratic growth condition*

$$F(y, u) - F(\bar{y}, \bar{u}) \geq \delta_s \|u - \bar{u}\|^2_{L^2(0,T)} \qquad (4.1)$$

*holds for all admissible pairs $(y, u)$ with $\|u - \bar{u}\|_{L^\infty(0,T)} < \varepsilon$. Therefore, $\bar{u}$ is a locally optimal control in the norm of $L^\infty(0, T)$.*

First, note that the second order derivative in $L''$ is related to the first two variables $(y, u)$. Before we outline the proof of Theorem 1, we formulate and prove some auxiliary results.

Let us introduce the increments $\delta u := u - \bar{u}$ and $\delta y := S'\delta u$. Next we split $\delta u = u_0 + u_+$, where

$$\begin{array}{lll} u_0 = 0, & u_+ = \delta u & \text{on } A_1, \\ u_0 = \delta u, & u_+ = 0 & \text{on } [0,T] \setminus (A_1 \cup A_2), \\ u_0 = \gamma S' u_0, & u_+ = \delta u - u_0 & \text{on } A_2. \end{array}$$

Due to Remark 2, the definition of $u_0$ and $u_+$ is correct. The part $u_0$ belongs to the critical subspace, while $u_+$ covers the part of $\delta u$ which exploits the effects of first order sufficiency. Furthermore, we define $y_0 := S'u_0$ and $y_+ := S'u_+$. We have $\delta y = y_0 + y_+$ by the linearity of $S'$.

In the next Lemma we estimate the difference $L(y, u, \bar{p}, \bar{\mu}_1, \bar{\mu}_2) - L(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$. We write for convenience $L(y, u) - L(\bar{y}, \bar{u})$, since $(\bar{p}, \bar{\mu}_1, \bar{\mu}_2)$ remains fixed throughout that paper. We also do not explicitly indicate the point $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_1, \bar{\mu}_2)$, where all derivatives are taken.

**Lemma 6** *Under the assumptions of Theorem 1,*

$$L(y, u) - L(\bar{y}, \bar{u}) \geq \frac{\delta}{4} \|u_0\|^2_{L^2(0,T)} - \frac{c_s}{2} \|u_+\|^2_{L^2(0,T)} + r_2 + r_3 \tag{4.2}$$

*holds, where $r_2$, $r_3$ are remainder terms with*

$$\frac{|r_i|}{\|u - \bar{u}\|^2_{L^2(0,T)}} \to 0 \quad \text{if } \|u - \bar{u}\|_{L^\infty(0,T)} \to 0$$

*for $i = 2, 3$.*

**Proof:** Using a Taylor expansion and (FON), we get

$$\begin{aligned} L(y, u) - L(\bar{y}, \bar{u}) &= L_u[u - \bar{u}] + L_y[y - \bar{y}] + \frac{1}{2}(L_{uu}[u - \bar{u}]^2 \\ &\quad + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2 \\ &= \frac{1}{2}(L_{uu}[u - \bar{u}]^2 + 2L_{uy}[u - \bar{u}, y - \bar{y}] + L_{yy}[y - \bar{y}]^2) + r_2 \end{aligned} \tag{4.3}$$

Similar to the explanation after Remark 2, the following property of the remainder is known

$$\frac{|r_2(\bar{u}, h)|}{\|h\|^2_{L^2(0,T)}} \to 0 \quad \text{if } \|h\|_{L^\infty(0,T)} \to 0.$$

For the proof, we refer to Tröltzsch [12]. According to the notation of Lemma 5, we get $y - \bar{y} = \delta y + r_1$. Replacing $y - \bar{y}$ by $\delta y$ in (4.3), we cause a small error

$$\begin{aligned} r_3 \;:=\; & \frac{1}{2}(L_{uu}[u-\bar{u}]^2 + 2L_{uy}[u-\bar{u}, y-\bar{y}] + L_{yy}[y-\bar{y}]^2) \\ & -\frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2). \end{aligned}$$

It is easy to show that

$$\frac{|r_3|}{\|u-\bar{u}\|^2_{L^2(0,T)}} \to 0 \quad \text{if } \|u-\bar{u}\|_{L^\infty(0,T)} \to 0.$$

By these notations, we can express (4.3) in the form

$$L(y,u) - L(\bar{y},\bar{u}) = \frac{1}{2}(L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2) + r_2 + r_3. \qquad (4.4)$$

We continue by splitting the Lagrange functional,

$$\begin{aligned} L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 \;=\; & L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \\ & + L_{uu}[u_+]^2 + 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ & + 2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ & + 2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]. \end{aligned}$$

As $u_0$ belongs to the critical subspace, (SSC) yields

$$L''[u_0, y_0]^2 = L_{uu}[u_0]^2 + 2L_{uy}[u_0, y_0] + L_{yy}[y_0]^2 \geq \delta \|u_0\|^2_{L^2(0,T)}.$$

The other terms are easily estimated by means of Young's inequality, by $\|y_0\|^2_{L^2(0,T)} \leq \|S'\|^2 \|u_0\|^2_{L^2(0,T)}$, and $\|y_+\|^2_{L^2(0,T)} \leq \|S'\|^2 \|u_+\|^2_{L^2(0,T)}$,

$$\begin{aligned} & |L_{uu}[u_+]^2 + 2L_{uy}[u_+, y_+] + L_{yy}[y_+]^2 \\ & \quad + 2L_{uu}[u_0, u_+] + 2L_{uy}[u_0, y_+] \\ & \quad + 2L_{uy}[u_+, y_0] + 2L_{yy}[y_0, y_+]| \;\leq\; \frac{\delta}{2}\|u_0\|^2_{L^2(0,T)} + c_s \|u_+\|^2_{L^2(0,T)}. \end{aligned}$$

In this setting, $c_s$ is a certain (large) constant. Combining these results, we arrive at

$$L_{uu}[\delta u]^2 + 2L_{uy}[\delta u, \delta y] + L_{yy}[\delta y]^2 \geq \frac{\delta}{2}\|u_0\|^2_{L^2(0,T)} - c_s \|u_+\|^2_{L^2(0,T)}.$$

Returning to (4.3), we end up with

$$L(y,u) - L(\bar{y},\bar{u}) \geq \frac{\delta}{4}\|u_0\|^2_{L^2(0,T)} - \frac{c_s}{2}\|u_+\|^2_{L^2(0,T)} + r_2 + r_3,$$

which is exactly the assertion. □

In the next Lemma we estimate the term $\|u_+\|^2_{L^2(0,T)}$ in (4.2).

**Lemma 7** *Under the assumptions of Theorem 1,*

$$\left(\frac{c_s}{2}+\frac{\delta}{4}\right)\|u_+\|^2_{L^2(0,T)} \le c_5\|u_1-\bar u\|^2_{L^2(A_2)}+c_6\|r_1\|^2_{L^2(0,T)}+c_7\|u-\bar u\|^2_{L^2(A_1)} \quad (4.5)$$

*holds with certain positive constants $c_5$, $c_6$, and $c_7$.*

**Proof:** First we get on $A_1$

$$\|u_+\|_{L^2(A_1)} = \|\delta u\|_{L^2(A_1)} = \|u-\bar u\|_{L^2(A_1)}. \quad (4.6)$$

On the whole interval $[0,T]$ we have

$$u_+ + u_0 = \delta u = u - \bar u.$$

Apply the operator $I-\gamma S'$ to this equation and consider the image only on the set $A_2$. Using $u_0 = \gamma S' u_0$ on $A_2$, we find

$$u_+ - \gamma S' u_+ = u - \gamma S' u - (\bar u - \gamma S' \bar u) \text{ on } A_2.$$

Now, $u$ is again replaced by $u_1+u_2$ to obtain on $A_2$

$$u_+ - \gamma S' u_+ = u_1 - \gamma S' u_1 + u_2 - \gamma S' u_2 - (\bar u - \gamma S' \bar u).$$

On $A_2$, by definition, the equation $u_2 - \gamma S' u_2 = r_1$ is satisfied. Therefore, here we are able to continue by

$$u_+ - \gamma S' u_+ = u_1 - \bar u - (\gamma S'(u_1-\bar u)) + r_1.$$

Due to our definitions, $u_+ = \delta u = u - \bar u$ holds on $A_1$. In addition, $u_+$ vanishes on $[0,T]\setminus(A_1\cup A_2)$. Therefore we find

$$u_+ = \begin{cases} u_1 - \bar u + \gamma S'(u_+ - u_1 + \bar u) + r_1 & \text{on } A_2 \\ u - \bar u & \text{on } A_1 \\ 0 & \text{on } [0,T]\setminus(A_1\cup A_2). \end{cases}$$

Setting $M_2 = A_2,\ M_1 = [0,T]\setminus A_2$ and applying (2.5), we get the inequality

$$\|u_+\|_{L^2(0,T)} \le c_2\|\tilde f\|_{L^2(0,T)},$$

where $\tilde f$ is defined by

$$\tilde f = \begin{cases} r_1 + (u_1-\bar u) - \gamma S'(u_1-\bar u) & \text{on } A_2 \\ u - \bar u & \text{on } A_1 \\ 0 & \text{on } [0,T]\setminus(A_1\cup A_2). \end{cases}$$

Therefore, we obtain

$$\|u_+\|_{L^2(0,T)} \leq c_2(\|u-\bar{u}\|_{L^2(A_1)} + c_8\|u_1-\bar{u}\|_{L^2(0,T)} + \|r_1\|_{L^2(A_2)}),$$

where the positive constant $c_8$ is connected with $\|S'\|$. Using $\|u_1-\bar{u}\|_{L^2(0,T)} = \|u_1-\bar{u}\|_{L^2(A_2)}$,

$$\|u_+\|_{L^2(0,T)} \leq c_9\|u_1-\bar{u}\|_{L^2(A_2)} + c_2\|r_1\|_{L^2(A_2)} + c_2\|u-\bar{u}\|_{L^2(A_1)}$$

is found. Young's inequality yields

$$\|u_+\|^2_{L^2(0,T)} \leq 3c_9\|u_1-\bar{u}\|^2_{L^2(A_2)} + 3c_2\|r_1\|^2_{L^2(0,T)} + 3c_2\|u-\bar{u}\|^2_{L^2(A_1)}.$$

Multiplying by $(c_s/2+\delta/4)$ we find

$$\left(\frac{c_s}{2}+\frac{\delta}{4}\right)\|u_+\|^2_{L^2(0,T)} \leq c_5\|u_1-\bar{u}\|^2_{L^2(A_2)} + c_6\|r_1\|^2_{L^2(0,T)} + c_7\|u-\bar{u}\|^2_{L^2(A_1)},$$

which concludes the proof of the Lemma. □

Now we are able to prove our main Theorem 1.

**Proof:** Inserting (4.5) in (4.2)

$$\begin{aligned} L(y,u)-L(\bar{y},\bar{u}) \ \geq\ & \frac{\delta}{4}(\|u_0\|^2_{L^2(0,T)} + \|u_+\|^2_{L^2(0,T)}) + r_2 + r_3 \\ & -c_7\|u-\bar{u}\|^2_{L^2(A_1)} - c_5\|u_1-\bar{u}\|^2_{L^2(A_2)} - c_6\|r_1\|^2_{L^2(0,T)} \end{aligned}$$

is obtained. Returning to the objective $F$ and using Lemma 5, we find

$$\begin{aligned} F(y,u)-F(\bar{y},\bar{u}) \ \geq\ & \frac{\delta}{4}(\|u_0\|^2_{L^2(0,T)} + \|u_+\|^2_{L^2(0,T)}) + r_2 + r_3 \\ & +(\frac{\delta_1}{\varepsilon} - c_7)\|u-\bar{u}\|^2_{L^2(A_1)} + (\frac{\delta_2}{c_4\varepsilon} - c_5)\|u_1-\bar{u}\|^2_{L^2(A_2)} \\ & -c_6\|r_1\|^2_{L^2(0,T)}. \end{aligned} \qquad (4.7)$$

Next, $\|\delta u\|_{L^2(0,T)} = \|u_0+u_+\|^2_{L^2(0,T)} \leq 2\|u_0\|^2_{L^2(0,T)} + 2\|u_+\|^2_{L^2(0,T)}$ is applied to continue by

$$\begin{aligned} F(y,u)-F(\bar{y},\bar{u}) \ \geq\ & \frac{\delta}{8}\|\delta u\|^2_{L^2(0,T)} + r_2 + r_3 \\ & +(\frac{\delta_1}{\varepsilon} - c_7)\|u-\bar{u}\|^2_{L^2(A_1)} + (\frac{\delta_2}{c_4\varepsilon} - c_5)\|u_1-\bar{u}\|^2_{L^2(A_2)} \\ & -c_6\|r_1\|^2_{L^2(0,T)}. \end{aligned} \qquad (4.8)$$

Take now $\varepsilon$ sufficiently small, such that

$$\frac{\delta_1}{\varepsilon} - c_7 \geq 0 \quad \text{and} \quad \frac{\delta_2}{c_4\varepsilon} - c_5 \geq 0.$$

So we can omit the associated terms in (4.8),

$$F(y,u) - F(\bar{y},\bar{u}) \geq \frac{\delta}{8}\|\delta u\|^2_{L^2(0,T)} + r_2 + r_3 - c_6\|r_1\|^2_{L^2(0,T)}. \tag{4.9}$$

Due to the discussions during the proof, all terms of the righthand side (except the first one) are small with respect to $\|u-\bar{u}\|^2_{L^2(0,T)}$. Therefore,

$$F(y,u) - F(\bar{y},\bar{u}) \geq \frac{\delta}{16}\|u-\bar{u}\|^2_{L^2(0,T)} \tag{4.10}$$

holds, if $\|u-\bar{u}\|_{L^\infty(0,T)} < \varepsilon$ and $\varepsilon$ is sufficiently small. The growth condition is proved. We can choose $\delta_s = \delta/16$. □

# 5 Application to the 1-dimensional heat equation

The theory of section 4 can be applied to optimal control problems for semilinear parabolic equations with mixed pointwise control state constraints. We illustrate this by the following optimal control problem

$$\min \int_0^T f(t,\vartheta(t,1),u(t))\,dt \tag{5.1}$$

subject to the parabolic initial-boundary value problem

$$\begin{array}{rcll} \vartheta_t(t,x) &=& \vartheta_{xx}(t,x) & \text{on } (0,\mathrm{T})\times(0,1) \\ \vartheta(0,x) &=& \vartheta^0(x) & \text{on } (0,1) \\ \vartheta_x(t,0) &=& 0 & \text{on } (0,\mathrm{T}) \\ \vartheta_x(t,1) &=& b(t,\vartheta(t,1),u(t)) & \text{on } (0,\mathrm{T}) \end{array} \tag{5.2}$$

and

$$0 \leq u(t) \leq c(t) + \gamma(t)\vartheta(t,1), \quad \text{for } 0 \leq t \leq T. \tag{5.3}$$

This problem may be considered as the optimal heating of an infinite plate of thickness 1, where the control $u$ stands for the outer temperature acting at the side $x=1$. In this case, the mixed control-state constraint (5.3) restricts the difference between the boundary temperature $\vartheta(t,1)$ and the control.

Here we require that the function $b(t,y,u)$ fulfills assumption (A1) except $b_y(t,y,u) \geq 0$. Instead of this, we assume the reversed inequality

$$b_y(t,y,u) \leq 0.$$

Then the boundary condition of (5.2),

$$\vartheta_x - b(t, \vartheta, u) = 0$$

defines a monotone operator, and we are able to show the existence of a unique solution of (5.2) for each control $u$. Nevertheless, the theory of the former sections remains still true, as we point out later.

We should mention that the assumption of global Lipschitz continuity of $b$ with respect to $y$ can be omitted. This follows from maximum principle arguments, see [10]. In addition, $f, c, \gamma$ should fulfill (A1) and (A3). Furthermore $\vartheta^0$ belongs to $C[0,1]$. Let us first discuss the *linear* initial-boundary value problem

$$\begin{aligned}
\vartheta_t(t,x) &= \vartheta_{xx}(t,x) && \text{on } (0,\mathrm{T}) \times (0,1) \\
\vartheta(0,x) &= \vartheta^0(x) && \text{on } (0,1) \\
\vartheta_x(t,0) &= 0 && \text{on } (0,\mathrm{T}) \\
\vartheta_x(t,1) &= g(t) && \text{on } (0,\mathrm{T}),
\end{aligned} \tag{5.4}$$

where $\vartheta^0$ and $g$ are given. We introduce $V = H^1(0,1)$ and $W(0,T) = \{\vartheta \in L^2(0,T;V) : \vartheta_t \in L^2(0,T;V^*)\}$, where $V^*$ is the dual space of $V$. ¿From Lions/Magenes [6] we know that (5.4) has a unique weak solution in $W(0,T)$, i.e. $\vartheta$ solves

$$\begin{aligned}
\int_0^1 \vartheta_t \cdot v \, dx + \int_0^1 \vartheta_x \cdot v_x \, dx &= g \cdot v(1) \\
\vartheta(0) &= \vartheta^0
\end{aligned} \tag{5.5}$$

for almost all $t$ and all $v \in V$. For convenience, we introduce the Green's function

$$G(x,\xi,t) = \sum_{n=0}^{\infty} v_n(x) v_n(\xi) e^{-n^2\pi^2 t}, \tag{5.6}$$

where $v_n(x) = \sqrt{2}\cos n\pi x$ denote the normalized eigenfunctions of a Sturm–Liouville eigenvalue problem associated with the problem (5.4): $v_{xx} = \lambda v,\ v_x = 0$ at $x = 0,\ 1$. By the Fourier method, we get for the solution of (5.4)

$$\vartheta(t,x) = \int_0^t G(x,1,t-s) g(s) \, ds + \int_0^1 G(x,\xi,t) \vartheta^0(\xi) \, d\xi.$$

Returning to the nonlinear problem (5.2), the integral equation

$$\vartheta(t,x) = \int_0^t G(x,1,t-s) b(s,\vartheta(s,1),u(s)) \, ds + \int_0^1 G(x,\xi,t) \vartheta^0(\xi) \, d\xi \tag{5.7}$$

is obtained. It can be shown that the solution of (5.7) is also a solution of (5.2). On $[0, T]$, the Green's function satisfies the estimate

$$|G(x,\xi,t)| < 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < 1 + 2\int_0^{\infty} e^{-x^2\pi^2 t}\, dx < c_k t^{-\frac{1}{2}} \tag{5.8}$$

with a certain constant $c_k > 0$. For our discussion, we need only the values of $\vartheta$ for $x = 1$. Once $\vartheta(t,1)$ is known, $\vartheta(t,x)$ is uniquely determined by (5.7). Denoting $y := \vartheta(.,1)$, we get for the new state $y$ the integral equation

$$y(t) = \int_0^t k(t,s)b(s,y(s),u(s))\, ds + y_0(t), \tag{5.9}$$

where $k(t,s) = G(1,1,t-s)$ and $y_0(t) = \int_0^1 G(1,\xi,t)\vartheta^0(\xi)\, d\xi$. In this way, we have related our state equation to the integral equation in our problem (P). From the representation of the Green's function in (5.6), it is easy to see that the kernel $k(t,s) = G(1,1,t-s)$ is positive. Moreover, we have the inequality (5.8). Therefore, the kernel $k(t,s) = G(1,1,t-s)$ fulfills (A2). In this way, our optimal control problem is converted to

$$(P_{\text{heat}}) \begin{cases} \min & \int_0^T f(t,y(t),u(t))\, dt = J(u) = F(y,u) \\ \text{subject to} & y(t) = y_0(t) + \int_0^t G(1,1,t-s)b(s,y(s),u(s))\, ds \\ \text{and} & 0 \le u(t) \le c(t) + \gamma(t)y(t), \quad 0 \le t \le T. \end{cases}$$

$(P_{\text{heat}})$ fulfills assumptions (A1)-(A3) with $k(t,s) = G(1,1,t-s)$, except $b_y(t,y,u) \ge 0$. However, the non-negativity of $b_y$ was only used to show $S'(\bar{u}) \ge 0$ in Lemma 3. In the case of the parabolic initial-boundary value problem (5.2), this property follows from the maximum principle for the heat equation, which does not rely on the assumption $b_y \ge 0$, see [10]. Therefore we can apply the whole theory to $(P_{\text{heat}})$.

Conversely, if we know the optimal solution $(\bar{y}, \bar{u})$ of $(P_{\text{heat}})$, then we can express the optimal state of the 1-dimensional heat equation $\bar{\vartheta}$ by

$$\bar{\vartheta}(t,x) = \int_0^t G(x,1,t-s)b(t,\bar{y}(s),\bar{u}(s))\, ds + \int_0^1 G(x,\xi,t)\vartheta^0(\xi)\, d\xi. \tag{5.10}$$

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