Lipschitz stability of optimal controls for the steady-state Navier-Stokes equations

by

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Abstract: An optimal control problem with quadratic cost functional for the steady-state Navier-Stokes equations with no-slip boundary condition is considered. Lipschitz stability of locally optimal controls with respect to certain perturbations of both the cost functional and the equation is proved provided a second-order sufficient optimality condition holds. For a sufficiently small Reynolds number, even global Lipschitz stability of the unique optimal control is shown.

Key Words: Incompressible viscous fluids, flow control, first- and second-order optimality conditions, Lipschitz stability.

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0 Introduction

Optimization of incompressible viscous Newtonean fluid flows governed by the Navier-Stokes system enjoys recently significant attention within the mathematical community and has important engineering applications. In this paper, we confine ourselves to steady-state problems. Optimal control problem of this sort was already studied in Bilić (1985), Bubák (2002), Burkardt and Peterson (1995), Casas 1995, Desai and Ito (1994), Gunzburger (1995), Gunzburger, Hou and Svobodny (1991,1992), Lions (1983), Málek and Roubíček (1999) and Roubíček (2002), but the relevant literature is much more extensive. Moreover, the optimal control of the evolutionary Navier-Stokes system was treated, e.g., in Chebotarev (1993), Fattorini (1995,1999), Fattorini and Sritharan (1994), Fursikov (2000), Gunzburger and Manservisi (1999), Hinze (1999), Hinze and Kunisch (1998), Sritharan (1992,2000), Temam (1995) and also in Lions (1983).

Our main goal is to investigate the stability of the optimal control problem under perturbations of both the cost functional and the state equation. In nonconvex smooth optimization, this stability can only be expected if the solution satisfies a second-order sufficient optimality condition. Roughly speaking, second-order sufficient conditions are necessarily satisfied at the optimal solution, if stability holds. For optimal control of ordinary differential equations, this was addressed by Dontchev and Malanowski (1995), while the case of semilinear elliptic and parabolic PDEs has been discussed by Malanowski and Tröltzsch (1999,2000). In a more general setting, the problem of sensitivity analysis is extensively studied in the book by Bonnans and Shapiro (2000), where second order sufficient conditions are important as well.

To solve our problem, we perform a second-order analysis in two different ways. In the first part of the paper, the solution is assumed to satisfy the standard second-order sufficient optimality conditions (2.17)-(2.18). In the context of flow problems without constraints on the controls, conditions of this type have already been used by several authors. We only mention Desai and Ito (1994), who used second-order conditions to investigate convergence of the augmented-Lagrangean method, and Hinze (1999,2001) who assumes second-order conditions to prove the convergence of Newton- and SQP-methods. We should mention that second-order sufficient optimality conditions are also natural assumptions to prove convergence of numerical algorithms and to derive error estimates for numerical approximations of control problems.

In a second approach, following Málek and Roubíček (1999), we invoke the increment formula (2.7) to obtain global stability. This increment formula is equivalent to a second-order expansion of the objective functional. Known regularity results for the Navier-Stokes system as well as for the linearized Navier-Stokes system and for the adjoint system will systematically be exploited. Essentially, to guarantee the above outlined global stability, we have to assume a sufficiently viscous flow, i.e. a small Reynolds number, cf. the assumptions (1.3) and (4.1) below. Applications of flows with low Reynolds numbers are polymer manufacturing processes or nanotechnology.

The scheme of the paper is the following. In Section 1, we specify the optimal-control problem $(\mathfrak{P})$ we will deal with and recall some of its basic properties already known. In Section 3, we address the local Lipschitz stability of locally optimal controls, states, and adjoint states with respect to certain perturbations of both the cost functional and the equation. Here, we assume standard second-order sufficient optimality conditions which are formulated, together with first-order conditions, in Section 2. Finally, in Section 4 even the global stability of the unique optimal control is shown provided that the Reynolds number is sufficiently small.

1 Problem formulation

Assuming Ω a bounded domain in $\mathbb{R}^n$, $n \leq 3$, with C^2-boundary Γ, we will deal with the following "velocity tracking" optimal control problem for flows governed by the steady-state incompressible Navier-Stokes system:

$$(\mathfrak{P}) \left\{ \begin{array}{lll} \text{Minimize } \ J(u,f) := \displaystyle\int_\Omega \frac{1}{2}|u-u_\mathrm{d}|^2 + \frac{\gamma}{2}|f|^2 \,\mathrm{d}x & & \text{(cost functional)} \\ \text{subject to } (u\cdot\nabla)u - \nu\Delta u + \nabla p = f & \text{on } \Omega, & \text{(state system)} \\ \qquad\qquad \operatorname{div} u = 0 & \text{on } \Omega, & \text{(incompressibility)} \\ \qquad\qquad f(x) \in S(x) & \text{for a.a. } x \in \Omega, & \text{(control constraints)} \\ \qquad\qquad u \in W_0^{1,2}(\Omega;\mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad f \in L^2(\Omega;\mathbb{R}^n), & & \end{array} \right.$$

where $L_0^2(\Omega) := \{p \in L^2(\Omega);\ \int_\Omega p \,\mathrm{d}x = 0\}$.

Here, the distributed force f represents the *control* and (u,p) is the *state* response, where u is the velocity field, p is the pressure, while u_d stands for a given *desired velocity* profile. By $\nu > 0$ we denote the fluid viscosity which is indirectly proportional to the Reynolds number.

The quadratic velocity-tracking cost functional J we use in $(\mathfrak{P})$ is a standard option in flow control, see Gunzburger (1995) or Bilić (1985), Gunzburger and Manservisi (1999). It has reasonable applicability and simplifies the analysis considerably. Anyhow, $(\mathfrak{P})$ is obviously not a linear-quadratic problem because of the bilinear convective term $(u\cdot\nabla)u$ in the state equation.

As to the parameter γ, the desired velocity profile u_d, and the set-valued mapping $S : \Omega \rightrightarrows \mathbb{R}^n$, we assume

$$\gamma \geq 0, \quad u_\mathrm{d} \in L^q(\Omega;\mathbb{R}^n), \tag{1.1}$$

$$S \text{ measurable, closed- and convex-valued,} \tag{1.2}$$

$$\sup|S(x)| \leq \rho(x), \quad \rho \in L^r(\Omega), \quad \frac{N_2 N_4^2}{\nu^2}\|\rho\|_{L^2(\Omega)} < 1\,, \tag{1.3}$$

with $q, r \geq 2$ to be specified later and with N_p, $p < 2n/(n-2)$, denoting the norm of the embedding $W_0^{1,2}(\Omega;\mathbb{R}^n) \subset L^p(\Omega)$. The adjective "measurable" in (1.2) has a standard meaning: for any open $A \subset \mathbb{R}^n$, the set $S^{-1}(A) := \{x \in \Omega;\ S(x) \cap A \neq \emptyset\}$ is Lebesgue measurable. Examples for mappings S satisfying (1.2) are

$$S(x) = \big\{s \in \mathbb{R}^n; |s| \leq \rho(x)\big\} \tag{1.4}$$

or

$$S(x) = \big\{s \in \mathbb{R}^n;\ a_i(x) \leq s_i \leq b_i(x),\ i = 1,...,n\big\} \tag{1.5}$$

with measurable radius $\rho : \Omega \to \mathbb{R}$ or measurable functions a_i, $b_i : \Omega \to \mathbb{R}$. Of course, (1.4) satisfies $\sup|S(x)| = \rho(x)$, cf. (1.3), while in case (1.5) one has to assume $\max(|a(x)|, |b(x)|) \leq \rho(x)$ to meet (1.3).

In what follows, we denote the set of *admissible controls* by

$$\mathcal{F}_{\rm ad} := \big\{ f \in L^2(\Omega;\mathbb{R}^n);\ f(x) \in S(x) \text{ for a.a. } x \in \Omega \big\}.$$

In $L^2(\Omega,\mathbb{R}^n)$ we introduce the scalar product $(u,v) := \int_\Omega \sum_{i=1}^n u_i v_i \mathrm{d}x$, while $(\xi : \zeta) := \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \xi_{ij}\zeta_{ij}\mathrm{d}x$ is the associated one in $L^2(\Omega,\mathbb{R}^{n\times n})$. For convenience we recall the frequently used notation $(u\cdot\nabla)u := \sum_{k=1}^n u_k \frac{\partial u}{\partial x_k}$. Moreover, for $u \in W_0^{1,2}(\Omega;\mathbb{R}^n)$, $(\nabla u)^\top$ is the matrix having the column vectors $\nabla u_1, ..., \nabla u_n$. It is common to use the trilinear form $b : W^{1,2}(\Omega;\mathbb{R}^n)^3 \to \mathbb{R}$,

$$b(w,u,v) := ((w\cdot\nabla)u, v). \tag{1.6}$$

It is known that $b(w,u,v) = -b(w,v,u)$ if $\operatorname{div} w = 0$ and the normal component of w on Γ vanishes; here we will always have even $w|_\Gamma = 0$. This property immediately implies that $b(w,v,v) = 0$ holds under the same assumption.

The solution (u,p) to the Navier-Stokes system in $(\mathfrak{P})$ is understood in the weak sense. As p does not occur in J, we can advantageously use divergence-free test functions to remove p from the weak formulation. For $k = 1,2$, let us introduce the state space

$$W_{0,\mathrm{DIV}}^{k,2}(\Omega;\mathbb{R}^n) := \big\{ v \in W_0^{k,2}(\Omega;\mathbb{R}^n);\ \operatorname{div} v = 0 \big\}. \tag{1.7}$$

Definition 1.1 *We call $u \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n)$ a weak solution to the no-slip boundary-value problem for the steady-state Navier-Stokes system in $(\mathfrak{P})$ if the variational equation*

$$((u\cdot\nabla)u, v) + \nu(\nabla u : \nabla v) = (f,v) \quad \forall v \in W_{0,\mathrm{DIV}}^{1,2}(\Omega;\mathbb{R}^n) \tag{1.8}$$

is satisfied.

Testing (1.8) by $v := u$, the basic a-priori estimate

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \le \frac{N_2}{\nu}\|\rho\|_{L^2(\Omega)} \tag{1.9}$$

is easily obtained. Thanks to this, the following existence theorem can standardly be proved:

Theorem 1.2 (Galdi (1994)) *Let the assumptions (1.2)-(1.3) be satisfied. Then, for each $f \in \mathcal{F}_{\rm ad}$, there exists a unique weak solution $u =: u(f)$ of the state equations according Definition 1.1.*

Moreover, the (nowadays standard) regularity result

$$\|u\|_{L^\infty(\Omega;\mathbb{R}^n)} \le c\,\|u\|_{W^{2,2}(\Omega;\mathbb{R}^n)} \le C \equiv C(\Omega, \|\rho\|_{L^2(\Omega)}) \tag{1.10}$$

is known, see e.g. Constantin and Foias (1989) or Galdi (1994), Chpt.VIII, Thm.5.2. The dependence of the solution u on f is Lipschitzian:

Theorem 1.3 *Let the assumptions (1.2)-(1.3) be satisfied and $f_i \in \mathcal{F}_{\rm ad}$ be given, $i = 1, 2$. There exists a constant C_0 being independent of f_1, f_2 such that*

$$\|u_1 - u_2\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \le C_0 \, \|f^1 - f^2\|_{L^2(\Omega;\mathbb{R}^n)} \tag{1.11}$$

holds for the associated solutions u_i of (1.8).

Proof. We test the variational equality (1.8) for u_1 and u_2 by $v := u_1 - u_2$ and subtract the associated identities. Then, abbreviating shortly $U \equiv u_1 - u_2$, we get

$$((u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2, U) + \nu(\nabla U : \nabla U) = (f_1 - f_2, U).$$

Write

$$((u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2, U) = ((u_1 \cdot \nabla)U, U) + ((U \cdot \nabla)u_2, U). \tag{1.12}$$

Thanks to $\operatorname{div} u_1 = 0$ and $u_1|_\Gamma = 0$, the identity $b(u_1, U, U) = 0$ holds and we can estimate the nonlinear term by

$$\begin{aligned}
&((u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2, U) = -((U \cdot \nabla)u_2, U) \\
&\le \|\nabla u_2\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \|U\|^2_{L^4(\Omega;\mathbb{R}^n)} \le \frac{N_2 N_4^2}{\nu} \|\rho\|_{L^2(\Omega)} \|\nabla U\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} \\
&< \nu \|\nabla U\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})}
\end{aligned} \tag{1.13}$$

Now, the Lipschitz estimate is easy to derive since the nonlinear term can be absorbed by $\nu\,(\nabla U : \nabla U)$. ∎

Moreover, besides the (norm,norm)-continuity of the mapping $f \mapsto u(f) : L^2(\Omega;\mathbb{R}^n) \to W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ implied by Theorem 1.3, it is a standard exercise to show also its (weak,norm)-continuity. Under our assumptions (1.2)–(1.3), $\mathcal{F}_{\rm ad}$ is weakly compact in $L^2(\Omega;\mathbb{R}^n)$. Therefore, the existence of at least one globally optimal pair $(\bar u, \bar f)$ for $(\mathfrak{P})$ follows by standard weak compactness arguments. The uniqueness of $\bar u$ will be investigated later in Section 4. However, for several reasons we do not confine ourselves to globally optimal controls. Optimal control theory essentially relies on first-order necessary optimality conditions forming the so-called optimality system. The majority of optimization algorithms computes solutions of that system. Due to the nonconvexity of our problem, not all of these solutions are optimal, and second-order sufficient conditions are usually verified to guarantee *local* optimality. For instance, second-order conditions can be checked numerically.

Only in exceptional cases one is able to verify global optimality. Therefore, it is natural to investigate the stability of single local solutions with respect to perturbations rather than to restrict the analysis to global solutions. Since any global solution is also a local one, this discussion is even more general. In view of this, we will just *assume* that a locally optimal *reference control* $\bar f$ is given with the associated state $\bar u$.

Remark 1.4 Without (1.3), we could get existence of the globally optimal pair too, provided $\gamma > 0$ and provided we give up the natural requirement of a unique response to a particular control. In this case, the argument is that J is a coercive (with respect to the control) weakly lower-semicontinuous functional on a closed graph of the pairs (u, f) satisfying (1.8).

2 First- and second-order optimality conditions

Consider a given locally optimal reference pair $(\bar{f}, \bar{u})$. We begin with recalling the first-order necessary optimality conditions. Formally, they can be found by applying the well-known Lagrange principle, where the state-equations are eliminated by the *Lagrange function*

$$L(u, f, w) = J(u, f) - (f - (u \cdot \nabla u), w) - \nu(\nabla u : \nabla w), \tag{2.1}$$

cf. (1.8). Obviously, for a fixed multiplier $w \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$, the function $L(\cdot, \cdot, w) : W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \to \mathbb{R}$ is quadratic and continuous, hence it is a C^2-function. According to the Lagrange principle, $(\bar{f}, \bar{u})$ should satisfy the necessary optimality conditions for minimizers of L with respect to $f \in \mathcal{F}_{\mathrm{ad}}$, i.e. $L'_u(\bar{u}, \bar{f}, w)(u) = 0$ for all $u \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ and $L'_f(\bar{u}, \bar{f}, w)(f - \bar{f}) \geq 0$ for all $f \in \mathcal{F}_{\mathrm{ad}}$. The first relation leads to the *adjoint system* to the Navier-Stokes equations linearized at $u = \bar{u}$,

$$\begin{aligned} -\nu\Delta w + (\nabla u)^\top w - (u \cdot \nabla)w + \nabla\pi &= u_{\mathrm{d}} - u, \\ \operatorname{div} w &= 0\,, \end{aligned} \tag{2.2}$$

for the so-called *adjoint state* $w = w(u)$, which is associated with a given state u. Notice that $(\nabla u)^\top w - (u \cdot \nabla)w$ means $\big(\sum_{k=1}^n (\frac{\partial u_k}{\partial x_i} w_k - u_k \frac{\partial w_i}{\partial x_k})\big)_{i=1,\dots,n}$.

Definition 2.1 *Under a weak solution to the adjoint system (2.2) we understand any $w \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ satisfying the integral identity*

$$\nu(\nabla w : \nabla v) - ((u \cdot \nabla)w, v) + (w, (v \cdot \nabla)u) = (u_{\mathrm{d}} - u, v) \tag{2.3}$$

for all $v \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$.

Now we formulate the standard *first-order necessary optimality conditions.* They were proven for the case without control constraints by Desai and Ito (1994), for instance. This proof extends to control constraints by obvious modifications.

Theorem 2.2 *Let (1.1)-(1.3) hold, and let $\bar{f}$ be a locally optimal control for $(\mathfrak{P})$ with associated state $\bar{u} = u(\bar{f})$. Then the variational inequality*

$$(\gamma\bar{f} - \bar{w}, f - \bar{f}) \geq 0 \qquad \forall f \in \mathcal{F}_{\mathrm{ad}} \tag{2.4}$$

is satisfied for $\bar{w} = w(\bar{u}) \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ being the unique weak solution to the adjoint equation (2.2) according to Definition 2.1.

Let us only briefly sketch existence and uniqueness of the adjoint state $w(u)$. Consider the adjoint variational equation (2.3) for a given u. Testing (2.3) by $v := w$, we get

$$\begin{aligned}\nu\|\nabla w\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &= -((w\cdot\nabla)u, w) + (u_\mathrm{d} - u, w)\\ &\le \|w\|^2_{L^4(\Omega;\mathbb{R}^n)}\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\\ &\quad + N_{4,2}\|u_\mathrm{d} - u\|_{L^2(\Omega;\mathbb{R}^n)}\|w\|_{L^4(\Omega;\mathbb{R}^n)},\end{aligned} \tag{2.5}$$

where $N_{4,2}$ is the norm of the embedding $L^4(\Omega) \subset L^2(\Omega)$, from which we get easily existence and uniqueness of w solving (2.3) provided that assumption (1.3) holds. We should mention that even $W^{1,\infty}$-regularity of the adjoint state,

$$\|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} \le C_1\|u - u_\mathrm{d}\|_{L^q(\Omega;\mathbb{R}^n)} \tag{2.6}$$

holds with $C_1 = C_1(q)$ for all $q > n$. This result has been proved in Málek and Roubíček (1999) provided that (1.1) is satisfied. For other (but weaker) results concerning regularity of w see also Theorem 3.2 in Gunzburger, Hou and Svobodny (1991).

Let us denote $\Phi(f) = J(f, u(f))$. Recall that $u = u(f)$ is unique under the assumption (1.3). The following *increment formula* has been derived in a slightly modified, relaxed form in Málek and Roubíček (1999):

$$\begin{aligned}\Phi(\tilde f) - \Phi(f) + \big(w - \gamma f, \tilde f - f\big) &= \frac{1}{2}\|\tilde u - u\|^2_{L^2(\Omega;\mathbb{R}^n)}\\ &\quad + \frac{\gamma}{2}\|\tilde f - f\|^2_{L^2(\Omega;\mathbb{R}^n)} - \Big(((\tilde u - u)\cdot\nabla)w, \tilde u - u\Big),\end{aligned} \tag{2.7}$$

where $\tilde u = u(\tilde f)$, $u = u(f)$, $w = w(u)$. Up to the second-order terms on the right-hand side, (2.7) gives immediately the directional derivative of Φ, namely $\mathrm{D}\Phi(f, h) = -(w - \gamma f, h)$ hence the Gâteaux derivative of Φ, denoted by $\Phi'(f) \in L^2(\Omega;\mathbb{R}^n)^* \cong L^2(\Omega;\mathbb{R}^n)$, is given by $\Phi'(f) = \gamma f - w$.

It is more convenient, however, to consider the variational inequality (2.4) in a formally different way: We know that $\mathcal{F}_\mathrm{ad} \subset L^2(\Omega;\mathbb{R}^n)$ and $\bar w \in L^2(\Omega;\mathbb{R}^n)$. Therefore, $\gamma f - w \in L^2(\Omega;\mathbb{R}^n)$. Define the set

$$N_{\mathcal{F}_\mathrm{ad}}(\bar f) := \begin{cases}\big\{z \in L^2(\Omega;\mathbb{R}^n);\ (z, f - \bar f) \le 0\ \forall f \in \mathcal{F}_\mathrm{ad}\big\} & \text{if } \bar f \in \mathcal{F}_\mathrm{ad}\\ \emptyset & \text{if } \bar f \notin \mathcal{F}_\mathrm{ad},\end{cases} \tag{2.8}$$

which is the standard *normal cone* to $\mathcal{F}_\mathrm{ad}$ at $\bar f$. Then the variational inequality (2.4) says that $-\Phi'(\bar f) \in N_{\mathcal{F}_\mathrm{ad}}(\bar f)$, i.e. the negative Gâteaux derivative of Φ at $\bar f$, identified with an L^2-function, belongs to $N_{\mathcal{F}_\mathrm{ad}}$. In other words,

$$\gamma\bar f - \bar w + N_{\mathcal{F}_\mathrm{ad}}(\bar f) \ni 0.$$

The variational inequality (2.4) can also be written as $(\bar w - \gamma\bar f, \bar f) = \max_{f\in\mathcal{F}_\mathrm{ad}}(\bar w - \gamma\bar f, f)$ being equivalent to the pointwise condition

$$(\bar w(x) - \gamma\bar f(x))\cdot\bar f(x) = \max_{s\in S(x)}(\bar w(x) - \gamma\bar f(x))\cdot s$$

a.e. on Ω. In view of the convexity of $S(x)$ and concavity of the *Hamiltonian* $\mathcal{H}_w(x,s) := w(x)\cdot s - \frac{\gamma}{2}|s|^2$ in the s-variable, this can be rewritten as the pointwise *maximum principle*

$$\mathcal{H}_{\bar{w}}(x,\bar{f}(x)) = \max_{s\in S(x)} \mathcal{H}_{\bar{w}}(x,s) \tag{2.9}$$

for a.e. $x\in\Omega$. Expressing the same fact in terms of minimization, $\bar{f}(x)$ is the unique solution of

$$\min_{s\in S(x)} \frac{\gamma}{2}|s|^2 - s\cdot\bar{w}(x) = \min_{s\in S(x)} \frac{\gamma}{2}|s-\gamma^{-1}\bar{w}(x)|^2 + c.$$

Therefore, we have the important projection formula

$$\bar{f}(x) = \operatorname{Proj}_{S(x)}\{\gamma^{-1}\bar{w}(x)\}, \tag{2.10}$$

where $\operatorname{Proj}_{S(x)}$ denotes the projection operator from $\mathbb{R}^n$ to $S(x)$.

For further purposes, we equivalently re-formulate the first-order optimality conditions (1.8), (2.3), (2.4) as the abstract inclusion (generalized equation)

$$\mathcal{F}(\bar{u},\bar{w},\bar{f}) + (0,0,N_{\mathcal{F}_{\rm ad}}(\bar{f})) \ni 0, \tag{2.11}$$

where $N_{\mathcal{F}_{\rm ad}}(\bar{f})$ is from (2.8) and the mapping $\mathcal{F}$ is defined by

$$[\mathcal{F}(u,w,f)]_1(v) := ((u\cdot\nabla)u - f, v) + \nu(\nabla u : \nabla v), \tag{2.12a}$$
$$[\mathcal{F}(u,w,f)]_2(v) := \nu(\nabla w : \nabla v) + (w,(v\cdot\nabla)u) - ((u\cdot\nabla)w - u + u_{\rm d}, v) \tag{2.12b}$$
$$[\mathcal{F}(u,w,f)]_3 := \gamma f - w. \tag{2.12c}$$

The inclusion (2.11) is a condensed form of the first-order necessary optimality conditions, i.e. of the *optimality system*.

Let us first discuss the right spaces between which $\mathcal{F}$ should be defined to finally obtain the best stability results. Take $(\bar{u},\bar{w},\bar{f})$ satisfying (1.8), (2.3), and (2.4). By the definition of $\mathcal{F}_{\rm ad}$, we have $\bar{f}\in L^2(\Omega;\mathbb{R}^n)$. The fact that $\bar{u}\in W^{1,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)$ solves (1.8) implies in particular $\bar{u}\in W^{2,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)$ due to a well-known regularity result by Galdi (1994) Chpt. VIII, Thm.5.2, provided Ω has a C^2-boundary, as indeed assumed.

The adjoint equation (2.3) can equally be viewed as the Stokes system with the right-hand side $(\bar{u}\cdot\nabla)\bar{w} - (\nabla\bar{u})^\top\bar{w} + u_{\rm d} - \bar{u}$, which certainly belongs to $L^2(\Omega;\mathbb{R}^n)$. So, $\bar{w}\in W^{1,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)$ solving (2.3) must belong to $W^{2,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)$ due to the well-known regularity for Stokes systems, see Galdi (1994), Chpt. IV, Thm.6.1. This justifies to define $W^{2,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)^2\times L^2(\Omega;\mathbb{R}^n)$ as the domain of $\mathcal{F}$.

Let us now consider the range of $\mathcal{F}$. The first two components of $\mathcal{F}$ define elements of $W^{1,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)^*$, hence functionals. On $W^{2,2}_{0,\rm DIV}(\Omega;\mathbb{R}^n)^2$ we get for the first component

$$[\mathcal{F}(u,w,f)]_1(v) = ((u\cdot\nabla)u - \nu\Delta u - f, v),$$

where $(u\cdot\nabla)u-\nu\Delta u-f\in L^2(\Omega;\mathbb{R}^n)$, hence $[\mathcal{F}(u,w,f)]_1$ can be identified with an L^2-function. The same holds true for $[\mathcal{F}(u,w,f)]_2$. Notice that, despite of their simple structure, both functionals are only applied to divergence free test functions. Therefore, we consider $\mathcal{F}(u,w,f)$ as follows:

$$\mathcal{F}: W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)^2\times L^2(\Omega;\mathbb{R}^n)\to L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)^2\times L^2(\Omega;\mathbb{R}^n), \tag{2.13}$$

where

$$\begin{aligned} L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n) &:= \big\{f\in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)^*;\ \exists\hat f\in L^2(\Omega;\mathbb{R}^n) \\ &\qquad \forall v\in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n):\ \langle f,v\rangle=(\hat f,v)\,\big\} && (2.14)\\ W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n) &:= \big\{v\in W^{2,2}(\Omega;\mathbb{R}^n);\ v|_\Gamma=0,\ \operatorname{div}v=0\big\}. && (2.15)\end{aligned}$$

The space $L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ is the space of equivalence classes in $L^2(\Omega;\mathbb{R}^n)$ of functions having the same rotation (in the distributional sense) and is naturally normed by $\|f\|_{L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}:=\inf_{\hat f}\|\hat f\|_{L^2(\Omega;\mathbb{R}^n)}$, where the infimum is taken over all $\hat f$ occurring in (2.14) for f under consideration.

Lemma 2.3 *The mapping $\mathcal{F}$ is of class C^1.*

Proof. On $W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$, we know

$$\langle[\mathcal{F}(u,w,f)]_1,v\rangle=((u\cdot\nabla)u-\nu\Delta u-f,v)$$

for all $v\in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$, and $(u\cdot\nabla)u-\nu\Delta u-f\in L^2(\Omega;\mathbb{R}^n)$. The mapping $(u,f)\mapsto-\nu\Delta u-f$ is linear and continuous. The same holds true for the embedding of $L^2(\Omega;\mathbb{R}^n)$ into $W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)^*$. Therefore, the linear part of $[\mathcal{F}]_1$ is trivially of class C^1. Its nonlinear part can be identified with the convective term $B(u):=b'(u,u,\cdot)$, i.e. in the classical formulation $B(u):=(u\cdot\nabla)u$, and we find

$$B(u+\tilde u)-B(u)=(\tilde u\cdot\nabla)u+(u\cdot\nabla)\tilde u+(\tilde u\cdot\nabla)\tilde u=:B(u)'\tilde u+r_2(\tilde u),$$

where the second-order remainder term $\|r_2(\tilde u)\|_{L^2(\Omega;\mathbb{R}^n)}=o\big(\|\tilde u\|_{W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}\big)$. The Fréchet-differentiability of B is shown. By injection into $W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)^*$, this yields the differentiability of the nonlinear part of $[\mathcal{F}_r]_1$.

As to the continuity of the differential of $[\mathcal{F}]_1$, it suffices to show the continuity of the mapping $u\mapsto B'(u)$ from $W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ to $\mathcal{L}(W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n),L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n))$. Let u_i be given, $i=1,2$, and abbreviate again $U=u_1-u_2$. Even Lipschitz continuity follows from the estimate

$$\begin{aligned}&\|B'(u_1)-B'(u_2)\|_{\mathcal{L}(W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n),L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n))}\\ &\quad=\sup_{\|\tilde u\|_{W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}\le1}\|(\tilde u\cdot\nabla)U+(U\cdot\nabla)\tilde u\|_{L^{2,*}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}\\ &\quad\le\sup_{\|\tilde u\|_{W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}\le1}\|(\tilde u\cdot\nabla)U+(U\cdot\nabla)\tilde u\|_{L^2(\Omega;\mathbb{R}^n)}\ \le\ C\|\nabla U\|_{L^2(\Omega;\mathbb{R}^n)}\end{aligned}$$

with a suitable $C = C(\Omega)$. The component $[\mathcal{F}]'_2$ is considered analogously, while the continuity of $[\mathcal{F}]'_3$ is obvious by linearity. ∎

In the context of optimization, the definition of $\mathcal{F}$ is justified by the following assertion.

Lemma 2.4 *The optimality system $\bar{f} \in \mathcal{F}_{\mathrm{ad}}$, (1.8), (2.3), (2.4) is equivalent to (2.11).*

Proof: Let $(\bar{u}, \bar{w}, \bar{f})$ satisfy the optimality system (1.8), (2.3), and (2.4). Then $\bar{u}$ and $\bar{w}$ are weak solutions of the state equation and adjoint equation. Moreover, they have the regularity $W^{2,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$. Therefore, $[\mathcal{F}(\bar{u}, \bar{w}, \bar{f})]_i = 0$ holds for $i = 1, 2$. Moreover, the variational inequality implies $[\mathcal{F}(\bar{u}, \bar{w}, \bar{f})]_3 + N_{\mathcal{F}_{\mathrm{ad}}}(\bar{f}) \ni 0$. Altogether, the inclusion (2.11) is fulfilled.

Conversely, if (2.11) is satisfied by $(\bar{u}, \bar{w}, \bar{f})$, then we obtain from its third component $N_{\mathcal{F}_{\mathrm{ad}}}(\bar{f}) \neq \emptyset$, hence $\bar{f} \in \mathcal{F}_{\mathrm{ad}}$. By definition of $N_{\mathcal{F}_{\mathrm{ad}}}(\bar{f})$, the variational inequality follows immediately. Moreover, the first two components are equivalent to the weak formulations of the state- and adjoint equations. Thus $(\bar{u}, \bar{w}, \bar{f})$ solves the optimality system $\bar{f} \in \mathcal{F}_{\mathrm{ad}}$, (1.8), (2.3), and (2.4). ∎

In order to perform a second-order analysis, we need the second order derivative of the Lagrange function. The second differential of $L(\cdot, \cdot, w)$ at a point (u, f), denoted as $L''(u, f, w) : [W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)]^2 \to \mathbb{R}$, is given by

$$\begin{aligned} L''(u, f, w)[(u_1, f_1), (u_2, f_2)] &= (u_1, u_2) + \gamma(f_1, f_2) \\ &\quad + \big((u_1 \cdot \nabla) u_2, w\big) + \big((u_2 \cdot \nabla) u_1, w\big). \end{aligned}$$

It is symmetric and independent of (u, f). We obtain the estimate

$$\begin{aligned} &\big|L''(u, f, w)[(u_1, f_1), (u_2, f_2)]\big| \leq \\ &\quad \leq \big(N_2^2 + 2\, N_4^2 \|\nabla w\|_{L^2(\Omega;\mathbb{R}^n)}\big) \|u_1\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \|u_2\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \\ &\quad + \gamma \|f_1\|_{L^2(\Omega;\mathbb{R}^n)} \|f_2\|_{L^2(\Omega;\mathbb{R}^n)}, \end{aligned}$$

expressing the boundedness of the quadratic form $L''(u, f, w)$, which is even uniform with respect to all w under consideration. If $L''(u, f, w)$ is restricted to the diagonal of $[W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)]^2$, which is what we will need, we simply write $L''(u, f, w)(\tilde{u}, \tilde{f})^2 := L''(u, f, w)[(\tilde{u}, \tilde{f}), (\tilde{u}, \tilde{f})]$. By $b(\tilde{u}, \tilde{u}, w) = -b(\tilde{u}, w, \tilde{u})$, this restricted second differential takes the form

$$L''(u, f, w)(\tilde{u}, \tilde{f})^2 = \|\tilde{u}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \gamma \|\tilde{f}\|^2_{L^2(\Omega;\mathbb{R}^n)} - 2((\tilde{u} \cdot \nabla) w, \tilde{u}). \quad (2.16)$$

This complies with the increment formula (2.7). The standard second-order sufficient optimality condition is:

Definition 2.5 *We say that a second-order sufficient optimality condition, briefly (SSC), is satisfied at* $(\bar{u}, \bar{f}, \bar{w})$ *if there is* $\delta > 0$ *such that the coercivity condition*

$$L''(\bar{u}, \bar{f}, \bar{w})(u, f)^2 \geq \delta \|f\|^2_{L^2(\Omega;\mathbb{R}^n)} \tag{2.17}$$

holds for all (u, f) *solving the Navier-Stokes system linearized at* $(\bar{u}, \bar{f})$*:*

$$((u \cdot \nabla)\bar{u}, v) + ((\bar{u} \cdot \nabla)u, v) + \nu(\nabla u : \nabla v) = (f, v) \tag{2.18}$$

for all $v \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ *.*

Proposition 2.6 *Let (1.3) hold, and let* $(\bar{u}, \bar{f}, \bar{w})$ *satisfy the first-order necessary conditions* $\bar{f} \in \mathcal{F}_{\mathrm{ad}}$*, (1.8) and (2.3) with* $\bar{u}$ *and* $\bar{f}$ *substituted for* u *and* f*, together with the second-order sufficient condition (SSC). Then* $(\bar{u}, \bar{f})$ *is locally optimal with respect to the topology of* $W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)$*.*

Sketch of the proof. By Casas and Tröltzsch (2002), eqs. (4.11)-(4.12), the condition (2.17)-(2.18) yields $\Phi''(\bar{f})(f, f) \geq \delta_1 \|f\|^2_{L^2(\Omega;\mathbb{R}^n)}$ for some $\delta_1 > 0$ and for all $f \in L^2(\Omega; \mathbb{R}^n)$. Moreover, the mapping $(u, f, w) \mapsto L''(u, f, w)$ is continuous. This follows from the estimate

$$\begin{aligned}
&\Big|[L''(u_1, f_1, w_1) - L''(u_2, f_2, w_2)][(\tilde{u}_1, \tilde{f}_1), (\tilde{u}_2, \tilde{f}_2]\Big| \\
&\qquad = \big|(\tilde{u}_1 \cdot \nabla \tilde{u}_2), W) + (\tilde{u}_2 \cdot \nabla \tilde{u}_1), W)\big| \\
&\qquad \leq 2\|\nabla W\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \|\tilde{u}_1\|_{L^4(\Omega;\mathbb{R}^n)} \|\tilde{u}_2\|_{L^4(\Omega;\mathbb{R}^n)} \\
&\qquad \leq 2N_4^2 \|W\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \|\tilde{u}_1\|_{W^{1,2}(\Omega;\mathbb{R}^n)} \|\tilde{u}_2\|_{W^{1,2}(\Omega;\mathbb{R}^n)}
\end{aligned}$$

where we abbreviated $W := w_1 - w_2$. This continuity is inherited also by $\Phi''(\cdot)(f, f)$, so that we can conclude that $\bar{f}$ is locally optimal for Φ with respect to the norm of $L^2(\Omega; \mathbb{R}^n)$. The continuity of the state mapping $f \mapsto u : L^2(\Omega; \mathbb{R}^n) \to W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$ yields the claimed local optimality of $(\bar{u}, \bar{f})$. ∎

In this paper, we will not apply Proposition 2.6. Instead, we shall directly use the condition (SSC) to obtain our result on Lipschitz stability. Therefore, we have only briefly sketched the proof.

Remark 2.7 Often, in the literature a seemingly stronger condition is used instead of (2.17), namely

$$L''(\bar{u}, \bar{f}, \bar{w})(u, f)^2 \geq \delta_1 (\|f\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|u\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)}) \tag{2.19}$$

for all (u, f) satisfying (2.18); where $\delta_1 > 0$ is fixed again. Yet, this is equivalent to (2.17) provided the linear mapping $f \mapsto u : L^2(\Omega; \mathbb{R}^n) \to W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n)$, u being the solution to (2.18), is bounded. This can be seen by the following arguments: Let $N = N(\bar{u}, \bar{f})$ denote the norm of this mapping. Then $\delta \|f\|^2_{L^2(\Omega;\mathbb{R}^n)} = \frac{1}{2}\delta \|f\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{1}{2}\delta \|f\|^2_{L^2(\Omega;\mathbb{R}^n)} \geq \frac{1}{2}\delta \|f\|^2 + \frac{1}{2} N^{-1} \delta \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^n)}$.

Hence one can take $\delta_1 = \max(2, 2N)^{-1}\delta$. Here, by putting $v := u$ in (2.18), we can estimate explicitly

$$N \leq \frac{N_2}{\nu - N_4^2 \|\nabla \bar{u}\|_{L^2(\Omega;\mathbb{R}^n)}} \leq \frac{\nu N_2}{\nu^2 - N_4^2 N_2 \|\rho\|_{L^2(\Omega)}} < +\infty \tag{2.20}$$

where the estimate (1.9) for $\bar{u}$ and the assumption (1.3) have been used, too.

Remark 2.8 The condition (2.17) implicitly requires $\gamma > 0$.

3 Local stability analysis of $(\mathfrak{P})$

Let us now address the main focus of the paper, i.e. the *stability* of a locally optimal reference pair $(\bar{u}, \bar{f})$ of the original problem $(\mathfrak{P})$. To be more specific, for a perturbation parameter $\varepsilon \equiv (\varepsilon^q, \varepsilon^u, \varepsilon^f) \in L^2(\Omega;\mathbb{R}^n)^2 \times L^r(\Omega;\mathbb{R}^n)$, we consider the perturbed optimization problem

$$(\mathfrak{P}_\varepsilon) \qquad \begin{cases} \text{Minimize} & J(u,f) := \displaystyle\int_\Omega \frac{1}{2}|u - u_\mathrm{d}|^2 - u \cdot \varepsilon^u + \frac{\gamma}{2}|f|^2 - f \cdot \varepsilon^f \,\mathrm{d}x \\ \text{subject to} & (u \cdot \nabla)u - \nu\Delta u + \nabla p = f + \varepsilon^q \qquad \text{on } \Omega, \\ & \operatorname{div} u = 0 \qquad \text{on } \Omega, \\ & u \in W_0^{1,2}(\Omega;\mathbb{R}^n), \quad p \in L_0^2(\Omega), \quad f \in \mathcal{F}_\mathrm{ad}. \end{cases}$$

As an example, one can think about a perturbation of the desired profile u_d, say $u_\mathrm{d} + e_\mathrm{d}$, which is obviously equivalent to considering the original u_d but taking $\varepsilon^u = e_\mathrm{d}$.

The first-order optimality conditions for $(\mathfrak{P}_\varepsilon)$, written in the condensed form of the inclusion (2.11), now read

$$\mathcal{F}(\bar{u}, \bar{w}, \bar{f}) + (0, 0, N_{\mathcal{F}_\mathrm{ad}}(\bar{f})) \ni (\varepsilon^q, \varepsilon^u, \varepsilon^f). \tag{3.1}$$

To investigate the stability of locally optimal pairs, we rely on a deep stability result by Robinson (1980) formulated for generalized equations covering, in particular, also our inclusion (2.11). Let us briefly recall some definitions that are basic to understand this theorem. We consider the generalized equation

$$0 \in F(z) + N(z), \tag{3.2}$$

where $F : Z \to Y$ is a mapping of class C^1 between two Banach spaces Z and Y, while $N : Z \to 2^Y$ is a set-valued mapping with a closed graph. Let $\bar{z}$ be a solution of (3.2). The generalized equation (3.2) is said to be *strongly regular* at the point $\bar{z}$, if there are open balls $B_Z(\bar{z}, \rho_Z) := \{z \in Z;\ ||z - \bar{z}||_Z \leq \rho_Z\}$ and $B_Y(0, \rho_Y) := \{\varepsilon \in Y;\ ||\varepsilon||_Y \leq \rho_Y\}$ such that, for all $\varepsilon \in B_Y(0, \rho_Y)$, the linearized and perturbed generalized equation

$$\varepsilon \in F(\bar{z}) + F'(\bar{z})(z - \bar{z}) + N(z) \tag{3.3}$$

admits a unique solution $z = z(\varepsilon)$ in $B_Z(\bar{z}, \rho_Z)$ and the mapping $\varepsilon \mapsto z(\varepsilon)$ from $B_Y(0, \rho_Y)$ to $B_Z(\bar{z}, \rho_Z)$ is Lipschitz continuous.

Proposition 3.1 (Robinson (1980), here modified.) *Let $\bar{z}$ be a solution of the generalized equation (3.2), and assume that (3.2) is strongly regular at $\bar{z}$. Then there are open balls $B_Z(\bar{z}, \rho_Z)$ and $B_Y(0, \rho_Y)$ such that, for all $\varepsilon \in B_Y(0, \rho_Y)$, the perturbed generalized equation*

$$\varepsilon \in F(z) + N(z) \tag{3.4}$$

has a unique solution $z = z(\varepsilon)$ in $B_Z(\bar{z}, \rho_Z)$, and the mapping $\varepsilon \mapsto z(\varepsilon)$ from $B_Y(0, \rho_Y)$ to $B_Z(\bar{z}, \rho_Z)$ is Lipschitz continuous.

This result enables us to investigate a simpler inclusion arising from (3.1) by linearization of $\mathcal{F}$ around the locally optimal triple $(\bar{u}, \bar{w}, \bar{f})$, i.e. the inclusion

$$\begin{aligned} \mathcal{F}(\bar{u}, \bar{w}, \bar{f}) + \mathcal{F}'(\bar{u}, \bar{w}, \bar{f})(u - \bar{u}, w - \bar{w}, f - \bar{f}) \\ +(0, 0, N_{\mathcal{F}_{\mathrm{ad}}}(f)) \ni (\varepsilon^q, \varepsilon^u, \varepsilon^f). \end{aligned} \tag{3.5}$$

In view of the definition (2.12) of $\mathcal{F}$ and of the fact that $(\bar{u}, \bar{w}, \bar{f})$ satisfies (2.11), in classical formulation it represents the optimality system

$$\begin{aligned} -\nu\Delta u + (u \cdot \nabla)\bar{u} + (\bar{u} \cdot \nabla)u + \nabla p &= f + (\bar{u} \cdot \nabla)\bar{u} + \varepsilon^q, && (3.6a) \\ \operatorname{div} u &= 0 \\ -\nu\Delta w + (\nabla\bar{u})^\top w - (\bar{u} \cdot \nabla)w + \nabla\pi &= u_d - u + ((u - \bar{u}) \cdot \nabla)\bar{w} && (3.6b) \\ &\quad - (\nabla(u - \bar{u}))^\top \bar{w} + \varepsilon^u, \\ \operatorname{div} w &= 0\,, \\ (\gamma f - w - \varepsilon^f, \tilde{f} - f) \geq 0 \quad \forall \tilde{f} &\in \mathcal{F}_{\mathrm{ad}}. && (3.6c) \end{aligned}$$

Lemma 3.2 *Let (1.1)-(1.3) hold and suppose that the triple $(\bar{u}, \bar{w}, \bar{f})$ satisfies the first-order necessary optimality conditions together with the second-order sufficient optimality conditions (SSC). Then, for any $\varepsilon \in L^2(\Omega; \mathbb{R}^n)^3$, the linearized inclusion (3.5) admits a unique solution $(u_\varepsilon, w_\varepsilon, f_\varepsilon)$ and the mapping $\varepsilon \mapsto (u_\varepsilon, w_\varepsilon, f_\varepsilon) : L^2(\Omega; \mathbb{R}^n)^3 \to W^{2,2}(\Omega; \mathbb{R}^n)^2 \times L^2(\Omega; \mathbb{R}^n)$ is Lipschitz continuous.*

Proof. The generalized equation (3.5) represents the first-order optimality conditions for the perturbed linear-quadratic problem

$$(\mathfrak{P}_\varepsilon^{\mathrm{LQ}}) \quad \begin{cases} \text{Minimize} \quad J(u, f) := \displaystyle\int_\Omega \Big(\frac{1}{2}|u - u_d|^2 + \frac{\gamma}{2}|f|^2 \\ \qquad\qquad -((u - \bar{u}) \cdot \nabla)\bar{w} \cdot (u - \bar{u}) - \varepsilon^u \cdot u - \varepsilon^f \cdot f \Big)\,\mathrm{d}x \\ \text{subject to} \quad (u \cdot \nabla)\bar{u} + (\bar{u} \cdot \nabla)u - \nu\Delta u + \nabla p = f + (\bar{u} \cdot \nabla)\bar{u} + \varepsilon^q, \\ \qquad\qquad \operatorname{div} u = 0 \qquad\qquad \text{on } \Omega, \\ \qquad\qquad u \in W^{1,2}_{0,\mathrm{DIV}}(\Omega; \mathbb{R}^n), \quad p \in L^2_0(\Omega), \quad f \in \mathcal{F}_{\mathrm{ad}}. \end{cases}$$

This problem represents a certain linear-quadratic approximation of the problem $(\mathfrak{P}_\varepsilon)$ at the fixed locally optimal pair $(\bar{u}, \bar{f})$ with $\bar{w} = \bar{w}(\bar{u})$ solving the adjoint equation (2.2). Hence, in fact, $(\mathfrak{P}_\varepsilon^{\mathrm{LQ}})$ depends also on $(\bar{u}, \bar{f})$. However, this dependence will not be explicitly indicated, since $(\bar{u}, \bar{f})$ is kept fixed.

The second-order condition (2.17) with (u, f) satisfying (2.18) just says that, disregarding the affine term $(u, f) \mapsto (\bar{u} \cdot \nabla)\bar{w} \cdot u + (u \cdot \nabla)\bar{w} \cdot \bar{u} - (\bar{u} \cdot \nabla)\bar{w} \cdot \bar{u} - u_d \cdot u - \varepsilon^u \cdot u - \varepsilon^f \cdot f$ included in the cost functional and the fixed term $(\bar{u} \cdot \nabla)\bar{u} + \varepsilon^q$ in the right-hand side of the linearized Navier-Stokes equation, the problem $(\mathfrak{P}_\varepsilon^{\mathrm{LQ}})$ has a quadratic and positive definite cost functional. Note that this fixed right-hand-side term, however, cannot change this fact because it only shifts the affine manifold containing all (u, f) satisfying (2.18), and similarly the affine perturbation of the quadratic functional cannot break its positive definiteness. This positive definiteness is even uniform with respect to ε. Therefore, (2.17)–(2.18) ensures the existence and uniqueness of $(u_\varepsilon, w_\varepsilon, f_\varepsilon)$ solving $(\mathfrak{P}_\varepsilon^{\mathrm{LQ}})$ or, equivalently, solving (3.5).

Let us now investigate the Lipschitz continuity of the mapping $\varepsilon \mapsto (u_\varepsilon, f_\varepsilon) : L^2(\Omega; \mathbb{R}^n)^3 \to L^2(\Omega; \mathbb{R}^n)^2$. To this aim, we take two vectors of perturbation parameters $\varepsilon_i \equiv (\varepsilon_i^u, \varepsilon_i^f, \varepsilon_i^q)$, $i = 1, 2$, and write shortly u_i, w_i, and f_i instead of u_{ε_i}, w_{ε_i}, and f_{ε_i}. To shorten the formulas below, we might use the following shorthand notation $U := u_1 - u_2$, $F := f_1 - f_2$, $\mathcal{E}^f := \varepsilon_1^f - \varepsilon_2^f$, $W := w_1 - w_2$, and $\Pi := \pi_1 - \pi_2$.

Now write (3.6c) for $\varepsilon^f = \varepsilon_i^f$ and f_i, $i = 1, 2$,

$$
\begin{aligned}
(\gamma f_1 - w_1 - \varepsilon_1^f, f_2 - f_1) \geq 0, \\
(\gamma f_2 - w_2 - \varepsilon_2^f, f_1 - f_2) \geq 0.
\end{aligned}
$$

Adding these two inequalities, in view of our notation we get

$$
-(\gamma F - W - \mathcal{E}^f, F) \geq 0, \qquad \text{hence} \tag{3.7}
$$

$$
\gamma \|F\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq (W, F) + (\mathcal{E}^f, F) \tag{3.8}
$$

Subtracting the perturbed state equation (3.6a) in the weak formulation for $\varepsilon^q = \varepsilon_1^q$ from that for $\varepsilon^q = \varepsilon_2^q$, and testing it by W, we get

$$
(W, F) = \nu(\nabla U : \nabla W) + ((U \cdot \nabla)\bar{u}, W) + ((\bar{u} \cdot \nabla)U, W) - (\mathcal{E}^q, W). \tag{3.9}
$$

Next we subtract the adjoint equation (3.6b), for $u = u_2$, $\varepsilon^u = \varepsilon_2^u$ from that for $u = u_1$, $\varepsilon^u = \varepsilon_1^u$ and obtain

$$
-\nu \Delta W + (\nabla \bar{u})^\top W - (\bar{u} \cdot \nabla)W + \nabla \Pi = -U + (U \cdot \nabla)\bar{w} - (\nabla U)^\top \bar{w} + \mathcal{E}^u. \tag{3.10}
$$

Testing it by U, in view of the formula

$$
((\nabla u)^\top w, v) = -((v \cdot \nabla)w, u) = -b(v, w, u),
$$

which is valid for all $v, u, w \in W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$, we get

$$\begin{aligned}\nu(\nabla W : \nabla U) = ((\bar{u}\cdot\nabla)W, U) - ((U\cdot\nabla)\bar{u}, W)\\ +(\mathcal{E}^u - U, U) + 2((U\cdot\nabla)\bar{w}, U).\end{aligned} \tag{3.11}$$

Inserting (3.11) into (3.9), we find

$$(W, F) = -\|U\|^2_{L^2(\Omega;\mathbb{R}^n)} + (\mathcal{E}^u, U) - (\mathcal{E}^q, W) + 2((U\cdot\nabla)\bar{w}, U).$$

Thus (3.8) together with the assumed second-order sufficient optimality condition, applied in the form (2.19) with L'' from (2.16), enable us to estimate

$$\begin{aligned}&\delta_1\big(||F||^2_{L^2(\Omega;\mathbb{R}^n)} + ||U||^2_{W^{1,2}(\Omega;\mathbb{R}^n)}\big)\\ &\qquad\le L''(\bar{u}, \bar{f}, \bar{w})(U, F)^2\\ &\qquad= \gamma\|F\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|U\|^2_{L^2(\Omega;\mathbb{R}^n)} - 2((U\cdot\nabla)\bar{w}, U)\\ &\qquad\le (\mathcal{E}^u, U) - (\mathcal{E}^q, W) + (\mathcal{E}^f, F)\\ &\qquad\le \frac{N_2^2}{2\delta_2}\|\mathcal{E}^u\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\delta_2}{2}\|U\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)}\\ &\qquad\quad+\frac{N_2^2}{2\delta_3}\|\mathcal{E}^q\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\delta_3}{2}\|W\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)}\\ &\qquad\quad+\frac{1}{2\delta_2}\|\mathcal{E}^f\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\delta_2}{2}\|F\|^2_{L^2(\Omega;\mathbb{R}^n)}\end{aligned} \tag{3.12}$$

with $\delta_1 > 0$ from (2.19) and with arbitrarily small $\delta_2 > 0$ and $\delta_3 > 0$.

The equation (3.10) is nothing more than an adjoint equation with unknown $W = w_1 - w_2$ and right-hand side $-U + (U\cdot\nabla)\bar{w} - (\nabla U)^\top\bar{w} + \mathcal{E}^u$ depending linearly on $U = u_1 - u_2$ and $\mathcal{E}^u = \varepsilon^u_1 - \varepsilon^u_2$. We know that this solution depends Lipschitz continuously on the right hand side. For instance, this result can be verified by testing (3.10) with W. Therefore, we obtain

$$\|W\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} \le C(\|U\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} + \|\mathcal{E}^u\|^2_{L^2(\Omega;\mathbb{R}^n)}). \tag{3.13}$$

Of course, we now take δ_2 and δ_3 in (3.12) small enough, which enables us to absorb all right-hand-side terms with U and F in the left-hand side of (3.12); eg. we can take $\delta_2 \le \delta_1$ and $\delta_3 < \delta_1/C$. This gives the Lipschitz continuity of $\varepsilon \mapsto (u_\varepsilon, f_\varepsilon) : L^2(\Omega;\mathbb{R}^n)^3 \to W^{1,2}(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{R}^n)$.

Now, the Lipschitz continuity of $\varepsilon \mapsto w_\varepsilon : L^2(\Omega;\mathbb{R}^n)^3 \to W^{1,2}(\Omega;\mathbb{R}^n)$ immediately follows from (3.13). ¿From (3.6a) one can see that

$$-\nu\Delta U + \nabla P = G^{(i)}, \qquad \operatorname{div} U = 0, \tag{3.14}$$

with $G^{(i)} := F + \mathcal{E}^q - (U\cdot\nabla)\bar{u} - (\bar{u}\cdot\nabla)U$. We can estimate

$$\begin{aligned}\|G^{(i)}\|_{L^2(\Omega;\mathbb{R}^n)} \le \|F\|_{L^2(\Omega;\mathbb{R}^n)} + \|\mathcal{E}^q\|_{L^2(\Omega;\mathbb{R}^n)}\\ +\|U\|_{L^6(\Omega;\mathbb{R}^n)}\|\nabla\bar{u}\|_{L^3(\Omega;\mathbb{R}^{n\times n})}\\ +\|\bar{u}\|_{L^\infty(\Omega;\mathbb{R}^n)}\|\nabla U\|_{L^2(\Omega;\mathbb{R}^{n\times n})}.\end{aligned} \tag{3.15}$$

By the regularity (1.10) of $\nabla\bar{u}$, we have the L^2-estimate of $G^{(i)}$ in terms of the assumed L^2-estimate of $\mathcal{E}^q$ and the already proved L^2-estimates of F and ∇U. Notice that $U \in W^{1,2}(\Omega;\mathbb{R}^n) \subset L^6(\Omega;\mathbb{R}^n)$ and $\bar{u} \in W^{2,2}(\Omega;\mathbb{R}^n)$, hence $\nabla\bar{u} \in L^6(\Omega;\mathbb{R}^{n\times n}) \subset L^3(\Omega;\mathbb{R}^{n\times n})$.

Then, using the $W^{2,2}$-regularity for the Stokes system (3.14), see Galdi (1994), Chpt.IV, Thm.6.1, we get the Lipschitz continuity of $\varepsilon \mapsto u_\varepsilon : L^2(\Omega;\mathbb{R}^n)^3 \to W^{2,2}(\Omega;\mathbb{R}^n)$.

Similarly, (3.6b) shows that

$$-\nu\Delta W + \nabla\Pi = G^{(ii)}, \qquad \operatorname{div} W = 0, \tag{3.16}$$

with $G^{(ii)} := -(\nabla\bar{u})^\top W + (\bar{u}\cdot\nabla)W - U + (U\cdot\nabla)\bar{w} - (\nabla U)^\top\bar{w} + \mathcal{E}^u$. Now, by the regularity of both $\bar{u}$ and $\bar{w}$, we have the L^2-estimate of $G^{(ii)}$ in terms of the assumed L^2-estimate of $\mathcal{E}^u$ and the already proved L^2-estimates of ∇U and ∇W. Then, using the $W^{2,2}$-regularity but now for the Stokes system (3.16), we get the Lipschitz continuity of $\varepsilon \mapsto w_\varepsilon : L^2(\Omega;\mathbb{R}^n)^3 \to W^{2,2}(\Omega;\mathbb{R}^n)$. ∎

Proposition 3.3 *Suppose that the assumptions of Lemma 3.2 are fulfilled. Then the generalized equation (2.11) is strongly regular at* $(\bar{u},\bar{w},\bar{f})$.

Proof. In view of the definition of strong regularity, this is a direct conclusion of Lemma 3.2. ∎

Now we apply Robinson's implicit function theorem to the generalized equation (2.11) and obtain the main result of this section, where we write for convenience $Y := L^2(\Omega;\mathbb{R}^n)^3$ and $Z := W^{2,2}(\Omega;\mathbb{R}^n)^2 \times L^2(\Omega;\mathbb{R}^n)$:

Theorem 3.4 *Let (1.3) hold and suppose that the triple* $(\bar{u},\bar{w},\bar{f})$ *satisfies the first-order necessary optimality conditions together with the second-order sufficient optimality conditions (SSC). Then there exist* $\rho_i > 0$, $i = 1, 2$ *such that, for all* $\varepsilon \in B_Y(0,\rho_1)$, *the perturbed inclusion (3.1) in the ball* $B_Z((\bar{u},\bar{w},\bar{f}),\rho_2)$ *admits a unique solution* $(u_\varepsilon, w_\varepsilon, f_\varepsilon)$ *and the mapping* $\varepsilon \mapsto (u_\varepsilon, w_\varepsilon, f_\varepsilon) : B_Y(0,\rho_1) \to B_Z((\bar{u},\bar{w},\bar{f}),\rho_2)$ *is Lipschitz continuous.*

This stability result refers to solutions of the perturbed inclusion (3.1). It does not automatically guarantee that these are local solutions of $(\mathfrak{P}_\varepsilon)$. To have this, we need that $(u_\varepsilon, w_\varepsilon, f_\varepsilon)$ satisfies a second order sufficient optimality condition. It is a nontrivial but standard exercise to show that the second order condition (SSC) is stable under small perturbations of $(\bar{u},\bar{w},\bar{f})$ (notice that also the linearized equation is shifted by the perturbation). The continuity estimate of L'' is the main tool to do this. Therefore, we only state the following result and skip its proof.

Corollary 3.5 *Under the assumptions of Theorem 3.4, there is* $0 < \tilde{\rho}_1 \le \rho_1$, *such that for all* $\varepsilon \in B_Y(0,\tilde{\rho}_1)$ *the perturbed optimal control problem* $(\mathfrak{P}_\varepsilon)$ *has a unique local solution* $(u_\varepsilon, f_\varepsilon)$ *in* $B_Z((\bar{u},\bar{w},\bar{f}),\rho_2)$.

Since $W^{2,2}(\Omega;\mathbb{R}^n)$ is continuously embedded into $C(\overline{\Omega})$, this implies L^∞-Lipschitz stability of the optimal state and adjoint state with respect to perturbations in L^2, while stability of optimal controls is only shown in L^2. If, however, the perturbation ε^f varies in L^r, $2 < r \leq \infty$, then also L^r-stability of the controls can be expected. To formulate this result, we put $Y_r := L^2(\Omega;\mathbb{R}^n)^2 \times L^r(\Omega;\mathbb{R}^n)$ and $Z_r := W^{2,2}(\Omega;\mathbb{R}^n)^2 \times L^r(\Omega;\mathbb{R}^n)$.

Corollary 3.6 *Theorem 3.4 remains true, if Z_r and Y_r are substituted for Z and Y, respectively, for all $2 < r \leq \infty$.*

Proof. Adapting formula (2.10) to the perturbed case (3.6c), it can immediately be seen that

$$f_\varepsilon(x) = \mathrm{Proj}_{S(x)}\{\gamma^{-1}(w_\varepsilon(x) - \varepsilon^f(x))\}$$

holds for a.a. $x \in \Omega$. Adopting the notation of the proof of Lemma 3.2, this implies

$$\begin{aligned} |F(x)| &\leq \left|\mathrm{Proj}_{S(x)}\{\gamma^{-1}(w_1(x) - \varepsilon_1^f(x))\} - \mathrm{Proj}_{S(x)}\{\gamma^{-1}(w_2(x) - \varepsilon_2^f(x))\}\right| \\ &\leq C\left|W(x) - \mathcal{E}^f(x)\right| \leq C\left(\|W\|_{C(\overline{\Omega})} + |\mathcal{E}^f(x)|\right) \end{aligned}$$

for a.a. $x \in \Omega$, since the projection mapping is Lipschitz continuous on $\mathbb{R}^n$. In view of Theorem 3.4, we continue by

$$|F(x)| \leq C\,(\|\mathcal{E}\|_{L^2(\Omega;\mathbb{R}^n)^3} + |\mathcal{E}^f(x)|),$$

which in turn implies

$$\|F\|_{L^r(\Omega;\mathbb{R}^n)} \leq C\,(\|\mathcal{E}^q\|_{L^2(\Omega;\mathbb{R}^n)} + \|\mathcal{E}^u\|_{L^2(\Omega;\mathbb{R}^n)} + \|\mathcal{E}^f\|_{L^r(\Omega;\mathbb{R}^n)}). \qquad \blacksquare$$

Stability results of this type are of particular interest for the convergence analysis of numerical methods. For instance, the convergence of Lagrange-Newton-SQP methods can be proved by the Kantorovich-Newton theorem for generalized equations provided that they are strongly regular. In our case, strong regularity is given by Lemma 3.2. Moreover, Lipschitz stability of optimal solutions is interesting in itself and can be used to answer other questions of optimal control theory.

4 Global analysis of ($\mathfrak{P}$)

In this section, we show that Lipschitz continuity can be obtained without assuming a second-order sufficient optimality condition. Instead, we proceed under the condition, pointed out already in Málek and Roubíček (1999) that

$$N_q \frac{N_2}{\nu}\|\rho\|_{L^2(\Omega)} + N_{q,n}\|u_\mathrm{d}\|_{L^q(\Omega;\mathbb{R}^n)} \leq \frac{1}{2C_1} - \eta \tag{4.1}$$

with $\eta \geq 0$ being a tolerance (allowing us to distinguish the case $\eta = 0$), N_q again denoting the norm of the embedding $W^{1,2}(\Omega) \subset L^q(\Omega)$ while $N_{q,n}$ is the norm of the embedding $L^q(\Omega) \subset L^n(\Omega)$ and C_1 is from (2.6). For given $u_{\rm d}$ and ρ, the condition (4.1) requires ν to be sufficiently large or, in other words, the Reynolds number to be sufficiently small. As the fluid is usually given with its viscosity ν, we rather need a sufficiently small driving force (i.e. being restricted by ρ) and desired velocity profile $u_{\rm d}$, as (4.1) indeed expresses. Cf. Bubák (2002) for quantitative analysis of these aspects.

This condition (4.1) causes that the last term in (2.7) is dominated by the term $\frac{1}{2}\|\tilde{u} - u\|^2_{L^2(\Omega;\mathbb{R}^n)}$ because of the following estimate:

$$\begin{aligned}
(((\tilde{u} - u) \cdot \nabla)w, \tilde{u} - u) &\leq \|\nabla w\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} ||\tilde{u} - u||^2_{L^2(\Omega;\mathbb{R}^n)} \\
&\leq C_1 \|u - u_{\rm d}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)} ||\tilde{u} - u||^2_{L^2(\Omega;\mathbb{R}^n)} \\
&\leq C_1 \Big(\|u\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)} + \|u_{\rm d}\|_{L^{n+\epsilon}(\Omega;\mathbb{R}^n)} \Big) ||\tilde{u} - u||^2_{L^2(\Omega;\mathbb{R}^n)} \\
&\leq C_1 \Big(N_q \frac{N_2}{\nu} \|\rho\|_{L^2(\Omega)} + N_{q,n} \|u_{\rm d}\|_{L^q(\Omega;\mathbb{R}^n)} \Big) ||\tilde{u} - u||^2_{L^2(\Omega;\mathbb{R}^n)};
\end{aligned} \tag{4.2}$$

cf. (1.9) and (2.6).

Then, as $\gamma \geq 0$, Φ is convex on $\mathcal{F}_{\rm ad}$ and the 1st-order optimality condition (2.4) is even sufficient for global optimality, as already observed in Málek and Roubíček (1999). One can deduce even more:

Proposition 4.1 *Let (1.1)-(1.3) hold. If $\gamma = 0$ but (4.1) holds with $\eta > 0$, then the optimal state $\bar{u}$ is unique while the optimal control $\bar{f}$ is unique only up to rotation-free functions, i.e. modulo the linear space $\{\nabla p;\ p \in W^{1,2}(\Omega)\}$. If $\gamma > 0$ and (4.1) holds (possibly with $\eta = 0$), then the optimal control $\bar{f}$ as well as the optimal state $\bar{u}$ are unique and satisfy the second-order condition (2.18).*

Proof. The increment formula (2.7), together with the calculation (4.2), yields the estimate $(\Phi'(f_1) - \Phi'(f_2), f_1 - f_2) \geq C_1\eta||u_1 - u_2||^2_{L^2(\Omega;\mathbb{R}^n)}$. Hence, the optimal $\bar{u}$ must be unique. Then, as $\bar{f}$ satisfies (1.8), it is determined uniquely but only up to ∇p for $p \in W^{1,2}(\Omega)$ arbitrary (such that $\bar{f} + \nabla p \in \mathcal{F}_{\rm ad}$, of course). Indeed, by Green's formula, $(\bar{f} + \nabla p, v) = (\bar{f}, v) - (p, {\rm div}\, v) = (\bar{f}, v)$ if tested by $v \in W^{1,2}_{0,{\rm DIV}}(\Omega;\mathbb{R}^n)$ so that the control $\bar{f} + \nabla p$ has the same effect as $\bar{f}$ if $\gamma = 0$.

If $\gamma > 0$, then (4.1) implies uniform convexity of Φ on $L^2(\Omega;\mathbb{R}^n)$ hence the uniqueness of $\bar{f}$ and hence also of $\bar{u}$ is obvious.

Furthermore, note that (4.1) also ensures that (2.17) holds even for any (u, f). In particular, it holds for those (u, f) which satisfy (2.18). ∎

Note that, in view of the automatic validity of the second-order condition (2.17), Proposition 2.6 says that any admissible (u, f) satisfying the 1st-order optimality conditions is automatically locally optimal. This is, however, not a surprising effect as we already proved that (4.1) guarantees even much more, namely that this (u, f) is even the unique globally optimal pair.

Let us now investigate the Lipschitz stability of this globally optimal pair, denoted by (u, f), under the perturbations of the cost functional involved in $(\mathfrak{P}_\varepsilon)$, i.e. we confine ourselves to $\varepsilon = (0, \varepsilon^u, \varepsilon^f,)$, considering $\varepsilon^q \equiv 0$. By an appropriate modification of (3.1), the 1st-order optimality condition for $(\mathfrak{P}_\varepsilon)$ is now

$$\mathcal{F}(u, w, f) + (0, 0, N_{\mathcal{F}_{\rm ad}}(f)) \ni (0, \varepsilon^u, \varepsilon^f). \tag{4.3}$$

Let us mention for convenience that (4.3), in its classical formulation, represents the following system:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \tag{4.4a}$$
$$\operatorname{div} u = 0\,.$$
$$-\nu\Delta w + (\nabla u)^\top w - (u\cdot\nabla)w + \nabla\pi = u - u_{\rm d} + \varepsilon^u, \tag{4.4b}$$
$$\operatorname{div} w = 0\,,$$
$$\forall \tilde f \in \mathcal{F}_{\rm ad}: \qquad (w - \gamma f, \tilde f - f) \le (\varepsilon^f, \tilde f - f) \tag{4.4c}$$

Assuming again (1.1) with $q > n$ and (4.1), the triple (u, f, w) solving (4.3) is determined uniquely.

Lemma 4.2 *Let (1.1) hold with $q > n$, let (1.2) and (1.3) be satisfied, and assume that also (4.1) holds with $\eta > 0$. Then the mapping $(\varepsilon^u, \varepsilon^f) \mapsto (f, u) : L^2(\Omega;\mathbb{R}^n)^2 \to L^2(\Omega;\mathbb{R}^n)^2$ is Lipschitz continuous.*

Proof. We take again two vectors of perturbation parameters $\varepsilon_i \equiv (0, \varepsilon_i^u, \varepsilon_i^f)$, $i = 1, 2$, and denote by (u_i, f_i, w_i), $i = 1, 2$, the corresponding optimal solution f_i, the optimal velocity u_i, and the adjoint velocity w_i to $(\mathfrak{P}_{\varepsilon_i})$. We subtract (4.4c), written for $\varepsilon^f = \varepsilon_1^f$ with $\tilde f = f_2$, from (4.4c) for $\varepsilon^f = \varepsilon_2^f$ with $\tilde f = f_1$. This gives again (3.8). Subtracting the perturbed state equations (4.4a) in the weak formulation (cf. (1.8)) for $\varepsilon = \varepsilon_1$ from that for $\varepsilon = \varepsilon_1$, and testing it by $W := w_1 - w_2$, we get the following expression for the first right-hand-side term in (3.8):

$$(W, F) = ((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2, W) + \nu(\nabla U : \nabla W). \tag{4.5}$$

Here, we adopt again the shorthand notation $U := u_1 - u_2$, $F := f_1 - f_2$, etc, used in the proof of Lemma 3.2. Subtracting the perturbed adjoint equations (4.4b) in the weak formulation (cf. (2.3)) for $\varepsilon = \varepsilon_1$ from that for $\varepsilon = \varepsilon_2$, and testing it by U, we get

$$\begin{aligned}\nu(\nabla W : \nabla U) - ((u_1\cdot\nabla)w_1, U) + (w_1, (U\cdot\nabla)u_1)&\\ + ((u_2\cdot\nabla)w_2, U) - (w_2, (U\cdot\nabla)u_2) &= -(U + \mathcal{E}^u, U).\end{aligned} \tag{4.6}$$

Comparing (4.6) with (4.5), after using several times the Green formula in the form $((u\cdot\nabla)v,w) = -((u\cdot\nabla)w,v)$ and performing an algebraic manipulation, we get

$$\begin{aligned}(W,F) &= -||U||^2_{L^2(\Omega;\mathbb{R}^n)} - (\mathcal{E}^u,U)\\ &\quad +((u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2,W) + ((u_1\cdot\nabla)w_1,U)\\ &\quad -(w_1,(U\cdot\nabla)u_1) - ((u_2\cdot\nabla)w_2,U) + (w_2,(U\cdot\nabla)u_2).\\ &= -||U||^2_{L^2(\Omega;\mathbb{R}^n)} - (\mathcal{E}^u,U) + ((U\cdot\nabla)W,U)\\ &\le -||U||^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{1}{4\delta}||\mathcal{E}^u||^2_{L^2(\Omega;\mathbb{R}^n)}\\ &\quad +\delta||U||^2_{L^2(\Omega;\mathbb{R}^n)} + ||\nabla W||_{L^\infty(\Omega;\mathbb{R}^n)}||U||^2_{L^2(\Omega;\mathbb{R}^n)}. \end{aligned}\tag{4.7}$$

The assumption (4.1) with $\eta > 0$ implies, like in (4.2), that $||\nabla W||_{L^\infty(\Omega;\mathbb{R}^n)} \le ||\nabla w_1||_{L^\infty(\Omega;\mathbb{R}^n)} + ||\nabla w_2||_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 - C_1\eta < 1$, so that (4.7) yields $(W,F) \le C||\mathcal{E}^u||^2$ for some $C = C(\eta)$. Moreover, the second term in the right-hand side of (3.8) can be estimated as

$$(\mathcal{E}^f,F) \le \frac{1}{4\delta}||\mathcal{E}^f||^2_{L^2(\Omega;\mathbb{R}^n)} + \delta||F||^2_{L^2(\Omega;\mathbb{R}^n)},\tag{4.8}$$

so that the last term can be absorbed in the left-hand side of (3.8) if $\delta > 0$ is small enough. Moreover, the term $-||U||^2_{L^2(\Omega;\mathbb{R}^n)}$ in (4.7), if put to the left-hand side, gives the claimed estimate for U. ∎

The (L^2, L^2)-Lipschitz continuity of U obtained in Lemma 4.2 can further be improved:

Lemma 4.3 *Under the same assumptions as in Lemma 4.2, the mapping* $(\varepsilon^u, \varepsilon^f) \mapsto u : L^2(\Omega;\mathbb{R}^n)^2 \to W^{1,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ *is Lipschitz continuous.*

Proof. In the same notation as in the previous proof, we get from (4.5) used with U instead of W the following estimate

$$\begin{aligned}\nu||\nabla U||^2_{L^2(\Omega;\mathbb{R}^{n\times n})} &= ((u_2\cdot\nabla)u_2 - (u_1\cdot\nabla)u_1,U) + (F,U)\\ &\le \frac{N_2N_4^2}{\nu}||\rho||_{L^2(\Omega)}\|\nabla U\|^2_{L^2(\Omega;\mathbb{R}^{n\times n})}\\ &\quad + \frac{N_2}{4\delta}||F||^2_{L^2(\Omega;\mathbb{R}^n)} + \delta||\nabla U||^2_{L^2(\Omega;\mathbb{R}^{n\times n})},\end{aligned}\tag{4.9}$$

cf. also (1.13). From the assumption (1.3), for $\delta > 0$ sufficiently small, one gets the Lipschitz continuity as claimed. ∎

Proposition 4.4 *Under the same assumptions as in Lemma 4.2, the mapping* $(\varepsilon^u, \varepsilon^f) \mapsto u : L^2(\Omega;\mathbb{R}^n)^2 \to W^{2,2}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$ *is Lipschitz continuous.*

Proof. Subtracting (4.4a) written for $i = 1$ and 2, we get

$$\begin{aligned}-\nu\Delta U + \nabla P &= F + (u_2\cdot\nabla)u_2 - (u_1\cdot\nabla)u_1 =: G,\\ \operatorname{div} U &= 0\,.\end{aligned}\tag{4.10}$$

Using the regularity (1.10), one can estimate the right-hand side of (4.10) as

$$\begin{aligned}
||G||_{L^2(\Omega;\mathbb{R}^n)} &= ||F - (U\cdot\nabla)u_2 - (u_1\cdot\nabla)U||_{L^2(\Omega;\mathbb{R}^n)} \\
&\le ||F||_{L^2(\Omega;\mathbb{R}^n)} + ||U||_{L^6(\Omega;\mathbb{R}^n)}||\nabla u_2||_{L^3(\Omega;\mathbb{R}^{n\times n})} \\
&\quad + ||u_1||_{L^\infty(\Omega;\mathbb{R}^n)}||\nabla U||_{L^2(\Omega;\mathbb{R}^{n\times n})} \\
&\le ||F||_{L^2(\Omega;\mathbb{R}^n)} + N_3N_6||U||_{W^{1,2}(\Omega;\mathbb{R}^n)}||u_2||_{W^{2,2}(\Omega;\mathbb{R}^n)} \\
&\quad + c||u_1||_{W^{2,2}(\Omega;\mathbb{R}^n)}||U||_{W^{1,2}(\Omega;\mathbb{R}^n)} \qquad (4.11)
\end{aligned}$$

with c from (1.10). As we can assume u_1 and u_2 ranging a bounded set in $W^{2,2}(\Omega;\mathbb{R}^n)$, by Lemmas 4.2 and 4.3, we have the Lipschitz continuity of the mapping $(\mathcal{E}^u,\mathcal{E}^f)\mapsto G : L^2(\Omega;\mathbb{R}^n)^2 \to L^2(\Omega;\mathbb{R}^n)$. Then, by the $W^{2,2}$-regularity of the Stokes system Galdi (1994), Chpt. IV, Thm.6.1 occurring on the left-hand side of (4.10), we get the claimed assertion. ∎

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