A coupled Maxwell integrodifferential model for magnetization processes

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Abstract

A mathematical model for instationary magnetization processes is considered, where the underlying spatial domain includes electrically conducting and nonconducting regions. The model accounts for the magnetic induction law that couples the given electrical voltage with the induced electrical current in the induction coil. By a theorem of Showalter on degenerate parabolic equations, theorems on existence, uniqueness, and regularity of the solution to the associated Maxwell integrodifferential system are proved.

AMS (MOS) subject classification. 35K65, 35B37, 78A25
Key Words. Evolution Maxwell equations, induction law, degenerate parabolic equation

1 Introduction

Due to their paramount importance for various electromagnetic processes in different applications, Maxwell equations attracted increasing interest in the past years. In particular, the control of magneto-hydrodynamic processes led to the discussion of new mathematical aspects. There is already an extended literature on the whole field.

Let us mention first the classical monographs by Bossavit [6] or Monk [13], where the foundations of the underlying numerical analysis are contained. We also refer to the recent book by Rodriguez and Valli [1] on stationary Maxwell equations.

Our paper is close to recent contributions on evolution Maxwell equations of degenerate parabolic type by H. Ammari, et al [2], Arnold and von Harrach [4], Bachinger et al [5], Hömberg and Sokołowski [10], and Kolmbaur [12]. A characteristic feature of these papers is the presence of conducting and nonconducting regions in the spatial domain. While [5] and [12] consider the model in bounded regions and [5] also sketches a quasilinear system, in

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[2] and [4] the problem is discussed in the whole space. The paper [10] deals with induction heating and considers a coupled system of the evolution Maxwell and heat equations.

In [5], the evolution Maxwell equations

$$\left\{\begin{array}{rcll} \sigma\dfrac{\partial y}{\partial t} + \operatorname{curl}\mu^{-1}\operatorname{curl} y &=& f(t) & \text{in} \quad \Omega\times(0,T) \\ y\times n &=& 0 & \text{on} \quad \partial\Omega\times(0,T) \\ y(0) &=& y_0 & \text{in} \quad \Omega_1 \end{array}\right. \tag{1}$$

are considered, where $\Omega \subset \mathbb{R}^3$ is a bounded domain that is the union of two subdomains Ω_1 (conducting region) and Ω_2 (nonconducting region) such that $\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$. The electrical conductivity $\sigma : \Omega \to \mathbb{R}$ vanishes on Ω_2 so that the equation (1) is of degenerate parabolic type. Therefore, an initial condition can only be prescribed in Ω_1. By $\mu : \Omega \to \mathbb{R}_+$, the magnetic permeability is denoted.

In the application to the magnetization processes we have in mind, the real quantity of interest is the magnetic field $B : \bar\Omega \times (0,T) \to \mathbb{R}^3$ that is represented by a vector potential $y : \bar\Omega \times (0,T) \to \mathbb{R}^3$ to be determined by equation (1). The given right-hand side $f : \Omega \times (0,T) \to \mathbb{R}^3$ has to obey some special regularity properties. In particular, $\operatorname{div} f = 0$ is required on Ω . For us, the special choice

$$f(x,t) = \left\{\begin{array}{ll} 0 & \text{in} \quad \Omega_1\times(0,T), \\ e(x)i(t) & \text{in} \quad \Omega_2\times(0,T) \end{array}\right. \tag{2}$$

is of particular interest, where $e : \Omega_2 \to \mathbb{R}^3$ is a given divergence free vector field and $i : [0,T] \to \mathbb{R}$ stands for the electrical current in an induction coil.

By the induction coil, magnetic fields are generated, but in practice the quantity under control is the electrical voltage $u : [0,T] \to \mathbb{R}$. According to Faraday's induction law, the total magnetic flux Ψ is coupled with the current i and the voltage u by the equation $\partial\psi/\partial t + Ri = u$. By the divergence theorem, after some steps the model below is obtained. We refer exemplarily to Kaltenbacher [11], chpt. 7.3, who also uses this well known idea. After splitting the equations in their parts in Ω_1 and Ω_2, the related model amounts to the following equations:

$$\left\{\begin{array}{rcll} \sigma\dfrac{\partial y}{\partial t} + \operatorname{curl}\mu^{-1}\operatorname{curl} y &=& 0 & \text{in} \quad \Omega_1\times(0,T) \\ \operatorname{curl}\mu^{-1}\operatorname{curl} y &=& e(x)i(t) & \text{in} \quad \Omega_2\times(0,T) \\ y\times n &=& 0 & \text{on} \quad \partial\Omega\times(0,T) \\ y(0) &=& y_0 & \text{in} \quad \Omega_1, \\ && & \\ \displaystyle\int_\Omega \frac{\partial y}{\partial t}\cdot e(x)\,dx + R\,i(t) &=& u(t) & \text{in} \quad (0,T) \\ i(0) &=& i_0. & \end{array}\right. \tag{3}$$

Here, $R > 0$ is the resistance of the induction coil and i_0 denotes the initial value for the electrical current. To allow for more generality, we will discuss the model in the form

$$\left\{\begin{array}{rcll} \sigma \dfrac{\partial y}{\partial t} + \operatorname{curl} \mu^{-1} \operatorname{curl} y + \varepsilon y & = & f(t) & \text{in} \quad \Omega \times (0,T) \\ y \times n & = & 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(0) & = & y_0 & \text{in} \quad \Omega_1, \\ & & & \\ \displaystyle\int_\Omega \frac{\partial y}{\partial t} \cdot e(x)\, dx + R\, i(t) & = & u(t) & \text{in} \quad (0,T) \\ i(0) & = & i_0. & \end{array}\right. \tag{4}$$

In this setting, $\varepsilon \geq 0$ is a regularization parameter that can be taken positive in the numerical solution of the system to enhance better stability of numerical methods, $f : \Omega \times (0,T) \to \mathbb{R}^3$ with

$$f(x,t) = \begin{cases} 0 & \text{in} \quad \Omega_1 \\ i(t)e(x) & \text{in} \quad \Omega_2 \end{cases}$$

has to obey certain regularity assumptions to be specified later.

Our paper is organized as follows: In Section 2, we transform the system (4) to a parabolic model that can be handled by a theorem of Showalter [14]. We will eliminate the electrical current i and arrive at a model that covers also the model of [5] as a particular case. In this way, we are able to provide an alternative proof of existence and uniqueness for the equation (1). Moreover, this section contains basic definitions of spaces and bilinear forms. Here and in the next sections, we heavily rely on results of Costabel et al. [8] on the existence and regularity of solutions to elliptic equations of Maxwell type.

In Section 3, we discuss the well-posedness of our general system (4). Moreover, here we discuss an associated adjoint equation as a prerequisite for later applications to the optimal control of magnetization processes.

2 Transformation to a degenerate parabolic equation

2.1 Geometrical configuration and assumptions on the data

In our paper, $\Omega \subset \mathbb{R}^3$ is a bounded open set. This is the hold-all domain that covers an electrical conducting domain Ω_1 and an electrical nonconducting domain Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$.

We first fix an illustrating example of a geometrical configuration. Our theory will hold, however, for any other configuration that obeys our assumptions on Ω_1 and Ω_2 stated after the example.

Let $\Omega \subset \mathbb{R}^3$ be an open cube, while Ω_1 is an open tube of finite length,

$$\Omega_1 = \{x \in \mathbb{R}^3 \,:\, 0 < r_1 < x_1^2 + x_2^2 < r_2,\ z_1 < x_3 < z_2\}.$$

We assume that Ω is sufficiently large such that $\bar{\Omega}_1 \subset \Omega$ and take $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. Moreover, a subdomain $\Omega_c \subset \Omega_2$ is given by

$$\Omega_c = \{x \in \mathbb{R}^3 \,:\, 0 < r_2 < x_1^2 + x_2^2 < r_3,\, c_1 < z < c_2\},$$

where $r_3 > r_2$ and $z_1 \le c_1 < c_2 \le z_2$ are given numbers. In the application, Ω_c stands for an induction coil that – due to our modelling – belongs to the nonconducting domain Ω_2.

Notice that Ω_2 contains exactly one hole given by Ω_1 and that the boundary of Ω_2 is composed of two disjoint connected sets.

Let us call this example geometry for later reference as *tube with coil*. Our theory is is true for the following more general setting: We assume once and for all that Ω, Ω_1, Ω_2, and $\Omega_c \subset \Omega_2$ are (open) bounded Lipschitz domains such that $\bar{\Omega}_1 \subset \Omega$ (i.e. Ω_1 is strictly included in Ω), Ω_2 has exactly one hole formed by $\bar{\Omega}_1$ and that the boundary $\partial\Omega_2$ is composed of two connected components.

Let $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$ denote the interface between Ω_1 and Ω_2. The *electrical conductivity* $\sigma : \Omega \to \mathbb{R}$ is given with some constant $\sigma_0 > 0$ by

$$\sigma(x) := \begin{cases} \sigma_0 & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2. \end{cases}$$

The *magnetic permeability* $\mu : \Omega \to \mathbb{R}$ is assumed to be bounded and measurable and uniformly positive such that

$$\mu(x) \ge \mu_0 > 0 \quad \text{for a.a. } x \in \Omega.$$

Let $\mathcal{O} \subset \mathbb{R}^3$ be an open domain. We use the standard Sobolev spaces $H(\text{curl}, \mathcal{O})$ and $H(\text{div}, \mathcal{O})$ and the space

$$H(\text{div} = 0, \mathcal{O}) := \{y \in L^2(\mathcal{O})^3 : \text{div}\, y = 0 \text{ in } \mathcal{O}\},$$

the space of divergence free vector functions equipped with the inner product of $L^2(\mathcal{O})^3$. It is well known that this is a Hilbert space. Moreover, we need the space

$$H_0(\text{curl}, \Omega) := \{y \in L^2(\Omega)^3 : \text{curl}\, y \in L^2(\Omega)^3 \text{ and } y \times n = 0 \text{ on } \partial\Omega\}.$$

Note that from Theorem I.2.11 of [9], the mapping

$$\mathcal{D}(\Omega)^3 \to L^2(\Gamma)^3 : y \to y \times n,$$

can be extended as a continuous mapping from $H_0(\text{curl}, \Omega)$ to $H^{-\frac{1}{2}}(\Gamma)^3$.

Further, a nontrivial divergence free function

$$e \in H(\text{div}, \Omega_c) \cap H(\text{curl}, \Omega_c)$$

is given such that

$$e(x) \cdot n = 0 \quad \forall x \in \partial\Omega_c. \tag{5}$$

For all $x \in \Omega \setminus \bar{\Omega}_c$, we extend e by $e(x) = 0$ and denote the extended function by the same symbol e. For the tube with coil Ω_c, we define for all $x \in \Omega_c$

$$e(x_1, x_2, x_3) = \frac{N_c}{|\Omega_c|\sqrt{x_1^2 + x_2^2}} \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \tag{6}$$

and extend e by zero to the cube Ω. The natural number N_c is the number of windings of the induction coil, and $|\Omega_c|$ is the area of the cross section of the coil that is perpendicular to the windings. The extension, still denoted by e, belongs to $H(\operatorname{div}, \Omega)$ but not to $H(\operatorname{curl}, \Omega)$. Notice that $e \cdot n = 0$ holds on $\partial\Omega_c$ and on Γ.

2.2 Simplification of the equations

Next, we simplify the system (4). As $R > 0$, we can eliminate i from the fifth identity of (4) and find

$$i(t) = -R^{-1} \int_\Omega \frac{\partial y}{\partial t}(x,t) \cdot e(x)\, dx + R^{-1} u(t) \quad \text{in } (0,T). \tag{7}$$

In that way the initial condition $i(0) = i_0$ is formally equivalent to

$$R^{-1} \int_\Omega \frac{\partial y}{\partial t}(x,0) \cdot e(x)\, dx = R^{-1} u(0) - i_0.$$

However, in associated optimal control problems, the voltage u might be chosen as a control function of $L^2(0,T)$ so that $u(0)$ is not defined. Since this is not satisfactory, we replace the last condition by

$$R^{-1} \int_\Omega y(x,0) \cdot e(x)\, dx = \alpha_0, \tag{8}$$

where α_0 has to be chosen properly to comply with the given initial condition $i(0) = i_0$. In Theorem 3.14, we formulate smoothness assumptions on u and y_0 that guarantee the continuity of i so that the initial value $i(0)$ is well defined.

Inserting the expression (7) of i in the first identity of (4), we arrive at

$$R^{-1} \int_\Omega \frac{\partial y}{\partial t}(t) \cdot e\, dx\, e + \operatorname{curl} \mu^{-1} \operatorname{curl} y + \varepsilon y = R^{-1} u(t)\, e \quad \text{in } \Omega_2 \times (0,T).$$

These considerations show that (4) is formally equivalent to

$$\left\{ \begin{array}{rcll} \sigma \dfrac{\partial y}{\partial t} + \operatorname{curl} \mu^{-1} \operatorname{curl} y + \varepsilon y &=& 0 & \text{in} \quad \Omega_1 \times (0,T) \\ R^{-1} \left(\displaystyle\int_\Omega \frac{\partial y}{\partial t} \cdot e\, dx \right) e + \operatorname{curl} \mu^{-1} \operatorname{curl} y + \varepsilon y &=& R^{-1} u\, e & \text{in} \quad \Omega_2 \times (0,T) \\ y \times n &=& 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(0) &=& y_0 & \text{in} \quad \Omega_1, \\ R^{-1} \displaystyle\int_\Omega y(x,0) \cdot e(x)\, dx &=& \alpha_0. & \end{array} \right. \tag{9}$$

In this form, we shall investigate the degenerate parabolic system, where we allow for a more general right-hand sides in the first two equations of (9).

2.3 Relation between α_0 and i_0

We assume in this subsection that i is a continuous function so that i has a well defined initial value i_0.

Let us explain that i_0 is uniquely determined by α_0 and vice versa, if y_0 is smooth enough, say $y_0 \in H(\text{curl}, \Omega_1)$. Given initial data y_0 in Ω_1, an extension y_{20} to Ω_2 that is compatible with the boundary value problem included in (3) should solve the equations

$$\left\{\begin{array}{rcll} \text{curl}\,\mu^{-1}\,\text{curl}\,y_{20}+\varepsilon\,y_{20} & = & e(x)\,i_0 & \text{in} \quad \Omega_2, \\ y_{20}\times n & = & 0 & \text{on} \quad \partial\Omega, \\ y_{20}\times n & = & y_0\times n & \text{on} \quad \Gamma. \end{array}\right. \tag{10}$$

The second boundary condition in (10) is due to the continuity of the trace $y\times n$ across Γ if $y \in H(\text{curl}, \Omega)$. If this boundary value problem is uniquely solvable, then $y_{20} := y_2(\cdot, 0) = y(\cdot, 0)|_{\Omega_2}$ can be taken as the initial datum in Ω_2.

Lemma 2.1 *If y_0 belongs to $H(\text{curl}, \Omega_1)$, then for all $\varepsilon > 0$ and $i_0 \in \mathbb{R}$, the boundary value problem* (10) *has a unique solution $y_{20} \in H(\text{curl}, \Omega_2) \cap H(\text{div}, \Omega_2)$. If $\varepsilon = 0$ and $i_0 \in \mathbb{R}$, then the boundary value problem* (10) *has a unique solution $y_{20} \in H(\text{curl}, \Omega_2) \cap H(\text{div}, \Omega_2)/K_N(\Omega_2)$, where $K_N(\Omega_2)$ is defined by (27).*

Proof. We construct y_{20} as the sum of a function z satisfying the homogeneous boundary conditions and of a function $R_1 y_0$ fulfilling the inhomogeneous boundary condition on Γ.

First, we take an extension $Ry_0 \in H(\text{curl}, \Omega_2)$ of y_0 such that

$$Ry_0 \times n = \left\{\begin{array}{ll} y_0\times n & \text{on} \quad \Gamma, \\ 0 & \text{on} \quad \partial\Omega. \end{array}\right.$$

This extension exists, since the trace mapping $y \mapsto y\times n$ from $H(\text{curl}, \Omega_j), j = 1$ or 2 to $H^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ is surjective, [15, section 7] or [7, p. 848].

This function Ry_0 is not necessarily divergence free. Therefore, we subtract $\nabla\theta$ from Ry_0, where $\theta \in H_0^1(\Omega_2)$ is the unique solution of

$$\int_{\Omega_2} \nabla\theta\cdot\nabla\bar{\psi}\,dx = \int_{\Omega_2} Ry_0\cdot\nabla\bar{\psi}\,dx \quad \forall\psi \in H_0^1(\Omega_2).$$

In view of $\text{curl}\,\nabla\theta = 0$ we have $\nabla\theta \in H(\text{curl}, \Omega_2)$. Moreover, the function $R_1 y_0 = Ry_0 - \nabla\theta \in H(\text{curl}, \Omega_2)$ is divergence free as a simple computation shows. It satisfies the same boundary conditions than Ry_0, namely

$$R_1 y_0 \times n = \left\{\begin{array}{ll} y_0\times n & \text{on} \quad \Gamma, \\ 0 & \text{on} \quad \partial\Omega. \end{array}\right.$$

This follows from the implication $\theta \in H_0^1(\Omega_2) \Rightarrow \nabla\theta \in H_0(\text{curl}, \Omega_2)$.

Now we consider the variational equation

$$\begin{aligned}\int_{\Omega_2} &(\mu^{-1}\operatorname{curl} z \cdot \operatorname{curl} \bar{w} + \operatorname{div} z \operatorname{div} \bar{w} + \varepsilon\, z \cdot \bar{w})\, dx \\ &= \int_{\Omega_2} (i_0\, e \cdot \bar{w} - \mu^{-1}\operatorname{curl}(R_1 y_0) \cdot \operatorname{curl} \bar{w} - \varepsilon\, R_1 y_0 \cdot \bar{w})\, dx \quad \forall w \in X_N(\Omega_2)\end{aligned} \tag{11}$$

for $z \in X_N(\Omega_2) := H_0(\operatorname{curl}, \Omega_2) \cap H(\operatorname{div}, \Omega_2)$.

Due to the regularizing divergence term, the sesquilinear form on the left-hand side of (11) is coercive in $X_N(\Omega_2)$ if ε is positive. If, however, $\varepsilon = 0$, then this problem has a unique solution $z \in X_N(\Omega_2)/K_N(\Omega_2)$ since the same sesquilinear form is coercive on $X_N(\Omega_2)/K_N(\Omega_2)$ due to the compact embedding of $X_N(\Omega_2)$ into $L^2(\Omega_2)^3$ [16] and since the right-hand side of (11) is equal to zero for any element of $K_N(\Omega_2)$ (see (28)) due to the divergence free property of e and the fact that $\int_\Gamma e \cdot n\, d\sigma = 0$.

As e and $R_1 y_0$ are divergence free in Ω_2, the same holds true for z. This is confirmed by taking test functions $w = \nabla\chi$ in (11), where $\chi \in H_0^1(\Omega_2)$ is the weak solution of

$$\Delta\chi - \varepsilon\chi = g \in L^2(\Omega_2).$$

Then we find that

$$\int_{\Omega_2} \operatorname{div} z\, \bar{g}\, dx = 0 \quad \forall g \in L^2(\Omega_2),$$

hence $\operatorname{div} z = 0$. Therefore we deduce that

$$\operatorname{curl}(\mu^{-1}\operatorname{curl}(z + R_1 y_0)) + \varepsilon(z + R_1 y_0)) = i_0\, e \text{ in } \mathcal{D}'(\Omega_2)$$

and this yields the desired field $y_{20} := z + R_1 y_0$.

The uniqueness of the solution is a consequence of the coercivity of the sesquilinear form

$$a(y, z) := \int_{\Omega_2} \mu^{-1}\operatorname{curl} y \cdot \operatorname{curl} \bar{z} + \operatorname{div} y\ \operatorname{div} \bar{z} + \varepsilon\, y\, z\, dx$$

in $X_N(\Omega_2)$. Given two divergence free solutions v, w of (10), their difference $v - w$ belongs to $X_N(\Omega_2)$ and solves the homogeneous variational equation $a(v - w, z) = 0$; then $v = w$ follows (modulo $K_N(\Omega_2)$ in the case $\varepsilon = 0$). ■

Assuming the continuity of the function $t \mapsto \int_\Omega y(x,t) \cdot e(x)\, dx$ at $t = 0$ we deduce

$$\alpha_0 = R^{-1} \int_\Omega y(x,0) \cdot e(x)\, dx = R^{-1} \int_{\Omega_2} y_{20}(x) \cdot e(x)\, dx.$$

In other words, the constant α_0 in (8) can be obtained from the initial value y_0 of y in Ω_1 and from the initial value i_0 of the electric current that determines y_{20} by (10).

Conversely, let us determine i_0 such that y_{20} satisfies the initial condition (8). To this aim, we split y_{20} as

$$y_{20} = i_0\, y_e + y_\Gamma, \tag{12}$$

where y_e solves

$$\left\{\begin{array}{rcll} \operatorname{curl}\mu^{-1}\operatorname{curl} y_e + \varepsilon\, y_e &=& e(x) & \text{in} \quad \Omega_2, \\ y_e \times n &=& 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_2, \\ y_e \times n &=& 0 & \text{on} \quad \Gamma \end{array}\right. \tag{13}$$

and y_Γ is obtained from

$$\left\{\begin{array}{rcll} \operatorname{curl}\mu^{-1}\operatorname{curl} y_\Gamma + \varepsilon\, y_\Gamma &=& 0 & \text{in} \quad \Omega_2, \\ y_\Gamma \times n &=& 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_2, \\ y_\Gamma \times n &=& y_0 \times n & \text{on} \quad \Gamma. \end{array}\right. \tag{14}$$

Both functions exists thanks to Lemma 2.1.

Lemma 2.2 *If $e \neq 0$ in the sense of $L^2(\Omega_2)^3$, then*

$$\int_{\Omega_2} y_e \cdot e\, dx \neq 0.$$

Proof. The function y_e belongs to $H_0(\operatorname{curl}, \Omega_2)$, hence it can be taken as test function in (13). We get

$$\int_{\Omega_2} \mu^{-1} \operatorname{curl} y_e \cdot \operatorname{curl} y_e\, dx + \varepsilon \int_{\Omega_2} |y_e|^2 dx = \int_{\Omega_2} y_e \cdot e\, dx.$$

If $\varepsilon > 0$, then a vanishing right-hand side would instantly imply $y_e = 0$ and, via (13), also $e = 0$ in contrary to the assumption. If $\varepsilon = 0$, then we find $\operatorname{curl} y_e = 0$ and again $e = 0$ by (13). ■

Inserting the ansatz $y_{20} = i_0 y_e + y_\Gamma$ in (8), we directly obtain the desired value for i_0 by

$$i_0 = \frac{R\alpha_0 - \int_{\Omega_2} y_\Gamma \cdot e\, dx}{\int_{\Omega_2} y_e \cdot e\, dx}. \tag{15}$$

The equivalence of the system (4) (resp. (3) for $\varepsilon = 0$) with (9) will be discussed at the end of our paper.

3 Existence and uniqueness of solutions

3.1 Preparations for the application of a theorem by Showalter

Our results on existence and uniqueness rely on the following theorem:

Theorem 3.1 ([14], Theorem V4.A) *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot,\cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V_m')$ be the corresponding operator given by $\mathcal{M}x(y) = m(x,y)$, $x, y \in V_m$. Let D be a subspace of V_m and $L : D \to V_m'$ be linear and monotone.*

(a) If $\ker \mathcal{M} \cap D \subset \ker L$ *and if* $\mathcal{M} + L : D \to V'_m$ *is a surjection, then for every* $f \in C^1([0,\infty), V'_m)$ *and* $u_0 \in D$ *there exists a solution of*

$$(\mathcal{M}u)_t + Lu(t) = f(t), \quad t > 0$$

with $(\mathcal{M}u)(0) = \mathcal{M}u_0$.

(b) If $\ker \mathcal{M} \cap \ker L = \{0\}$*, then there is at most one solution.*

To apply this theorem, we show that problem (4) fits in the associated framework. For this purpose, we first define the linear and continuous operators $\mathcal{M}$ and L used in that theorem.

We define the linear bounded operator $\mathcal{M} : L^2(\Omega)^3 \to L^2(\Omega)^3$ as follows:

$$\mathcal{M}y := \begin{cases} \sigma y & \text{in } \Omega_1, \\ R^{-1}\left(\int_{\Omega_2} y(x)\cdot e(x)\,dx\right) e & \text{in } \Omega_2. \end{cases}$$

The operator L, whose domain D will be specified below, is introduced by

$$Ly = \operatorname{curl} \mu^{-1} \operatorname{curl} y + \varepsilon y.$$

By these operators, problem (9) can be shortly and still formally written as

$$\begin{cases} (\mathcal{M}y)_t + Ly &= f \quad \text{in} \quad \Omega \times (0,T), \\ \mathcal{M}y(0) &= g_0 \quad \text{in} \quad \Omega, \end{cases} \tag{16}$$

where f is defined by

$$f(x,t) = \begin{cases} 0 & \text{in } \Omega_1 \times (0,T), \\ R^{-1}u(t)e(x) & \text{in } \Omega_2 \times (0,T). \end{cases}$$

Later, we shall admit more general functions f on the right-hand side. The initial datum g_0 is given by

$$g_0 = \begin{cases} \sigma\, y_0 & \text{in } \Omega_1, \\ \alpha_0 e & \text{in } \Omega_2. \end{cases}$$

We shall apply the framework of section V.4 of [14] to the system (16) in the space

$$V = \{y \in L^2(\Omega)^3 : \operatorname{div} y_1 = 0 \text{ in } \Omega_1, \operatorname{div} y_2 = 0 \text{ in } \Omega_2 \text{ and } \langle y_2 \cdot n, 1\rangle_\Gamma = 0\},$$

equipped with the semi-inner product

$$m(y,z) = \int_{\Omega_1} \sigma(x)y(x)\cdot \bar{z}(x)\,dx + R^{-1}\left(\int_{\Omega_2} y(x)\cdot e(x)\,dx\right)\left(\int_{\Omega_2} \bar{z}(x)\cdot e(x)\,dx\right).$$

Here, we used the notation $y_i := y_{|\Omega_i}$, $i = 1, 2$, that will be applied throughout our paper. Moreover, $\langle\cdot;\cdot\rangle_\Gamma$ means the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Let us denote by V_m the corresponding seminorm space.

Lemma 3.2 *The dual space* V'_m *is the Hilbert space*

$$V'_m = \{y \in V_m \,:\, \exists \alpha \in \mathbb{C} : y_2 = y_{|\Omega_2} = \alpha e\},$$

that is equipped with the inner product m.

Proof. Denote by S the right-hand side above. Since the embedding $S \hookrightarrow V'_m$ is trivial, it remains to show the converse one. Let $\ell \in V'_m$, then there exists $C > 0$ such that

$$|\ell(z)| \leq C \left(\|z\|_{L^2(\Omega_1)^3} + \left| \int_{\Omega_2} z(x) \cdot e(x)\, dx \right| \right) \quad \forall z \in V. \tag{17}$$

Further, we introduce the space

$$W := \{y \in H(\text{div} = 0, \Omega_2) : \langle y_2 \cdot n, 1 \rangle_\Gamma = 0\}$$

that is a closed subspace of $H(\text{div} = 0, \Omega_2)$.

Take $z_1 \in H(\text{div} = 0, \Omega_1)$ and $z_2 \in W$ and denote by $\tilde{z}_1$ (resp. $\tilde{z}_2$) the extension by zero of z_1 (resp. z_2) to the domain outside of Ω_1 (resp. Ω_2). We rapidly confirm that $\tilde{z}_1$ and $\tilde{z}_2$ belong to V. By (17) we further have

$$\begin{aligned} |\ell(\tilde{z}_1)| &\leq C\|\tilde{z}_1\|_{L^2(\Omega_1)^3} = C\|z_1\|_{L^2(\Omega_1)^3}, \\ |\ell(\tilde{z}_2)| &\leq C\left| \int_{\Omega_2} \tilde{z}_2(x) \cdot e(x)\, dx \right| = C\left| \int_{\Omega_2} z_2(x) \cdot e(x)\, dx \right|. \end{aligned} \tag{18}$$

The first estimate means that the mapping

$$z_1 \mapsto \ell(\tilde{z}_1)$$

is linear and continuous from $H(\text{div} = 0, \Omega_1)$ to $\mathbb{C}$; hence, there exists $h \in H(\text{div} = 0, \Omega_1)$ such that

$$\ell(\tilde{z}_1) = \int_{\Omega_1} \bar{h} \cdot z_1\, dx \quad \forall z_1 \in H(\text{div} = 0, \Omega_1). \tag{19}$$

Let us show that the second estimate implies the existence of $\alpha \in \mathbb{C}$ such that

$$\ell(\tilde{z}_2) = \alpha \int_{\Omega_2} z_2 \cdot e\, dx \quad \forall z_2 \in H(\text{div} = 0, \Omega_2). \tag{20}$$

Indeed, as $e \in W$, we can split any $z_2 \in W$ in the form

$$z_2 = \Pi_e z_2 + (Id - \Pi_e) z_2,$$

where Π_e is the projection on span$\{e\}$ with respect to the inner product of $L^2(\Omega_i)^3$, namely

$$\Pi_e z_2 = \frac{\int_{\Omega_2} z_2 \cdot e\, dx}{\int_{\Omega_2} |e|^2\, dx}\, e.$$

By the estimate (18) we get

$$|\ell(\widetilde{z_2 - \Pi_e z_2})| \leq C \left| \int_{\Omega_2} (z_2 - \Pi_e z_2)(x) \cdot e(x)\, dx \right| = 0,$$

we then deduce that

$$\ell(\tilde{z}_2) = \ell(\widetilde{\Pi_e z_2}) = \frac{\int_{\Omega_2} z_2 \cdot e\, dx}{\int_{\Omega_2} |e|^2\, dx} \ell(e).$$

This proves (20).

For each $z \in V$, it holds $z_1 \in H(\mathrm{div} = 0, \Omega_1)$ and $z_2 \in W$. In view of

$$z = \tilde{z}_1 + \tilde{z}_2,$$

and (19), (20), we conclude that

$$\ell(z) = \int_{\Omega_1} h \cdot z_1\, dx + \alpha \int_{\Omega_2} z_2 \cdot e\, dx,$$

implying the claim of the theorem. ∎

To define the domain of the operator L, we recall a next result from [8]. To this aim, we introduce the space

$$Y(\Omega) := \{y \in H_0(\mathrm{curl}, \Omega) : \mathrm{div}\, y_i \in L^2(\Omega_i), i = 1, 2 \text{ and } \langle y_2 \cdot n; 1\rangle_\Gamma = 0\},$$

where we recall that

$$H_0(\mathrm{curl}, \Omega) := \{y \in L^2(\Omega)^3 : \mathrm{curl}\, y \in L^2(\Omega)^3 \text{ and } y \times n = 0 \text{ on } \partial\Omega\}.$$

The space $Y(\Omega)$ is a Hilbert space with the norm

$$\|y\|^2_{Y(\Omega)} = \|y\|^2_{L^2(\Omega)^3} + \|\mathrm{curl}\, y\|^2_{L^2(\Omega)^3} + \|\mathrm{div}\, y_1\|^2_{L^2(\Omega_1)} + \|\mathrm{div}\, y_2\|^2_{L^2(\Omega_2)}.$$

In $Y(\Omega)$, we define two sesquilinear forms: for $y, z \in Y(\Omega)$, let

$$a_R(y, z) = \int_{\Omega_1} \sigma\, y_1 \cdot \bar{z}_1\, dx + \int_\Omega (\mu^{-1} \mathrm{curl}\, y \cdot \mathrm{curl}\, \bar{z} + \varepsilon\, y \cdot \bar{z})\, dx$$
$$+ e^{i\frac{\pi}{4}} \int_{\Omega_1} \mathrm{div}\, y_1\, \mathrm{div}\, \bar{z}_1\, dx + e^{i\frac{\pi}{4}} \int_{\Omega_2} \mathrm{div}\, y_2\, \mathrm{div}\, \bar{z}_2\, dx,$$

and

$$a_0(y, z) = \int_\Omega (\mu^{-1} \mathrm{curl}\, y \cdot \mathrm{curl}\, \bar{z} + \varepsilon\, y \cdot \bar{z})\, dx$$
$$+ e^{i\frac{\pi}{4}} \int_{\Omega_1} \mathrm{div}\, y_1\, \mathrm{div}\, \bar{z}_1\, dx + e^{i\frac{\pi}{4}} \int_{\Omega_2} \mathrm{div}\, y_2\, \mathrm{div}\, \bar{z}_2\, dx.$$

Recall that $\varepsilon \geq 0$ was assumed. According to Lemma 2.2 of [8] we know that there exists a positive constant C such that

$$\Re a_R(y, y) \geq C \|y\|^2_{Y(\Omega)} \quad \forall y \in Y(\Omega). \tag{21}$$

where $\Re$ denotes the real part of a complex number. Thanks to this coercivity property, for any $F \in L^2(\Omega)^3$, there exists a unique $y \in Y(\Omega)$ solution of

$$a_R(y, z) = \int_\Omega F \cdot \bar{z}\, dx \quad \forall z \in Y(\Omega). \tag{22}$$

In particular if $F \in V$, we have the next result (compare with Theorem 2.3 of [8]).

Theorem 3.3 *If $F \in L^2(\Omega)^3$ satisfies $\operatorname{div} F_1 = 0$, $\operatorname{div} F_2 = 0$ and $\langle F_2 \cdot n, 1\rangle_\Gamma = 0$, then the unique solution $y \in Y(\Omega)$ of (22) satisfies the system*

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl} y_1) + (\sigma + \varepsilon) y_1 = F_1 \ \text{in } \Omega_1, \\ \operatorname{div} y_1 = 0 \ \text{in } \Omega_1, \\ \operatorname{curl}(\mu^{-1} \operatorname{curl} y_2) + \varepsilon y_2 = F_2 \ \text{in } \Omega_2, \\ \operatorname{div} y_2 = 0 \ \text{in } \Omega_2, \\ [\varepsilon y \cdot n] + \sigma y_1 \cdot n = 0 \ \text{on } \Gamma. \end{cases} \tag{23}$$

In particular this implies that y belongs to V.

In the theorem, the expression $[\varepsilon y \cdot n]$ denotes the jump of $\varepsilon y \cdot n$ across Γ. The proof of this theorem is the same as the proof of Theorem 2.3 of [8] and is therefore omitted.

Now we are able to explain the operator L more precisely. Its domain is

$$D := \{y \in Y(\Omega) \cap V : \exists f \in V'_m \text{ such that } a_0(y, z) = m(f, z) \quad \forall z \in Y(\Omega)\}.$$

Notice that we are justified to identify V'_m with a subspace of V_m in view of Lemma 3.2.

For any $y \in D$, define

$$Ly = f$$

with the unique f appearing in the definition of D.

Lemma 3.4 *The operator L is linear and monotone from D into V'_m. Moreover $\mathcal{M} + L$ is surjective from D onto V'_m.*

Proof. By the definition of L, it is obvious that for any $y \in D$

$$m(Ly, y) = m(f, y) = a_0(y, y),$$

hence it follows

$$\Re\, m(Ly, y) = \Re a_0(y, y) \geq 0.$$

In other words, L is linear and monotone from D into V'_m.

Let us now prove the surjectivity of $\mathcal{M} + L$ from D onto V'_m. Introduce a sesquilinear form b on $Y(\Omega)$ by

$$b(y, z) = a_R(y, z) + R^{-2} \|e\|^2_{L^2(\Omega)^3} \left(\int_{\Omega_2} y(x) \cdot e(x)\, dx\right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx\right).$$

For any $y \in Y(\Omega)$, it holds

$$\Re b(y, y) \geq \Re a_R(y, y)$$

and hence the form b is strongly coercive on $Y(\Omega)$ by the coercivity property of a_R. Therefore, for any $f \in V'_m$ there exists a unique solution $y_1 \in Y(\Omega)$ of

$$b(y_1, z) = m(f, z) \quad \forall z \in Y(\Omega). \tag{24}$$

Notice that $z \mapsto m(f, z)$ defines a linear and continuous functional on $Y(\Omega)$. This identity is equivalent to

$$a_R(y_1, z) = m(g, z) \quad \forall z \in Y(\Omega)$$

with

$$g = f - R^{-2}\|e\|^2_{L^2(\Omega)^3} \left(\int_\Omega y_1(x) \cdot e(x)\, dx \right) e,$$

Since g belongs to V'_m, we deduce by Theorem 3.3 that $y_1 \in V$.

Similarly, (24) is equivalent to

$$a_0(y_1, z) = m(h, z) \quad \forall z \in Y(\Omega),$$

with

$$h = f - R^{-2}\|e\|^2_{L^2(\Omega)^3} \left(\int_\Omega y_1(x) \cdot e(x)\, dx \right) e - \sigma y_1.$$

Since again $h \in V'_m$, by the definition of D, we deduce that y_1 belongs to D.

Finally for any $z \in Y(\Omega)$, we see that

$$m((\mathcal{M} + L)y_1, z) = b(y_1, z),$$

and by the previous considerations, we deduce that the solution $y_1 \in Y(\Omega)$ of (24) belongs to D and is solution of

$$(\mathcal{M} + L)y_1 = f.$$

This proves the surjectivity of $\mathcal{M} + L$. ∎

Lemma 3.5 *We have* $\ker \mathcal{M} \cap D = \{0\}$.

Proof. Let y be in $\ker \mathcal{M} \cap D$. Then it follows from $\mathcal{M}y = 0$ that $y = 0$ in Ω_1 and y is orthogonal to e in Ω_2,

$$\int_{\Omega_2} y \cdot e\, dx = 0. \tag{25}$$

On the other hand, the fact that y belongs to D means that there exists $f \in V'_m$ such that

$$a_0(y, z) = m(f, z) \quad \forall z \in Y(\Omega).$$

Since y is zero in Ω_1, we have equivalently

$$\int_{\Omega_2} (\varepsilon\, y \cdot \bar{z} + \mu^{-1} \operatorname{curl} y \cdot \operatorname{curl} \bar{z})\, dx + e^{i\frac{\pi}{4}} \int_{\Omega_2} \operatorname{div} y \operatorname{div} \bar{z}\, dx = \int_{\Omega_1} f_1 \cdot \bar{z}\, dx \tag{26}$$
$$+R^{-1} \left(\int_{\Omega_2} f_2(x) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \quad \forall z \in Y(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega_1)^3$ be an arbitrary test function and define $\tilde{z}$ by $\tilde{z} = \varphi$ in Ω_1 and zero outside. Then $\tilde{z}$ belongs to $Y(\Omega)$ and therefore the previous identity implies that

$$\int_{\Omega_1} f_1 \cdot \bar{\varphi}\, dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega_1)^3.$$

Therefore, we have $f_1 = 0$. Coming back to (26), we insert $z = y$ and find that

$$\int_{\Omega_2} (\varepsilon\, |y_2|^2 + \mu^{-1} |\operatorname{curl} y_2|^2)\, dx + e^{i\frac{\pi}{4}} \int_{\Omega_2} |\operatorname{div} y_2|^2\, dx$$
$$= R^{-1} \left(\int_{\Omega_2} f_2(x) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{y}_2(x) \cdot e(x)\, dx \right) = 0,$$

by (25). Notice that we introduced the notation $y_2 = y_{|\Omega_2}$. This implies that

$$\operatorname{curl} y_2 = 0 \text{ and } \operatorname{div} y_2 = 0.$$

Since y is in $H_0(\operatorname{curl}, \Omega)$ and $y = 0$ in Ω_1, we deduce that y_2 belongs to

$$K_N(\Omega_2) = \{ y \in H_0(\operatorname{curl}, \Omega_2) \cap H(\operatorname{div}, \Omega_2) : \operatorname{curl} y_2 = 0 \text{ and } \operatorname{div} y_2 = 0 \}. \tag{27}$$

According to Proposition 3.18 of [3], the dimension of $K_N(\Omega_2)$ is the number of holes in Ω_2 which is 1 in our case.

Thanks to the same proposition, $K_N(\Omega_2)$ is spanned by all ∇q with $q \in H^1(\Omega_2)$ the unique solution of

$$\begin{cases} \Delta q = 0 & \text{in } \Omega_2, \\ q = 0 & \text{on } \partial\Omega, \\ q = \textit{constant} & \text{on } \Gamma, \\ \langle \partial_n q, 1 \rangle_\Gamma = 1, & \\ \langle \partial_n q, 1 \rangle_{\partial\Omega} = -1. & \end{cases} \tag{28}$$

Hence there exists $\alpha \in \mathbb{C}$ and a $q \in H^1(\Omega_2)$ with (28) such that

$$y_2 = \alpha \nabla q.$$

Since y belongs to $Y(\Omega)$ it satisfies

$$\langle y_2 \cdot n, 1 \rangle_\Gamma = 0,$$

or equivalently

$$\alpha \langle \partial_n q, 1 \rangle_\Gamma = 0.$$

In view of $\langle \partial_n q, 1 \rangle_\Gamma = 1$, this implies $\alpha = 0$. ■

3.2 Existence and regularity of solutions to (16)

Now all the hypotheses of Theorem V.4.A of [14] are fulfilled and it holds $\ker \mathcal{M} \cap \ker L \subset \ker \mathcal{M} \cap D = \{0\}$. According to this theorem, we obtain the following existence result.

Theorem 3.6 *For all $f \in C^1([0,\infty), V_m')$ and all $g_0 \in V_m'$ there exists a unique solution $y : [0,\infty) \to L^2(\Omega)^3$ of problem (16) with the regularity*

$$\mathcal{M}y \in C([0,\infty), V_m') \cap C^1((0,\infty), V_m')$$

and such that

$$y(t) \in D, \forall t > 0.$$

Here, the first identity of (16) has to be understood as follows:

$$(\mathcal{M}y)_t(t) + Ly(t) = f(t) \text{ in } V_m' \quad \forall t > 0. \tag{29}$$

Notice that the derivative $(\mathcal{M}y)_t(t)$ of the abstract function $t \mapsto (\mathcal{M}y)(t)$ is defined in the strong sense.

The assumption on f requires in particular that $f(t)$ is divergence free in Ω_1 for all t. The same holds true in Ω_2, because $f(t) = \varphi(t)\, e$ with some real valued function φ and e is divergence free.

Thanks to this theorem, we have $\mathcal{M}y \in C([0,\infty), V_m')$. Lemma 3.2 on the characterization of V_m' yields that $(\mathcal{M}y)(t) = z(t)$, where $z(t) \in V_m$. In Ω_1, it follows $\sigma\, y_1(t) = z_1(t)$, hence continuity of z_1 yields $y_1 \in C([0,T], L^2(\Omega_1)^3)$. Moreover, we have

$$R^{-1} \int_{\Omega_2} y_2(t) \cdot e \, dx \, e = z_2(t)$$

so that continuity of z_2 implies that $t \mapsto \int_{\Omega_2} y_2(t) \cdot e \, dx$ is continuous on $[0,\infty)$. The continuous dependence of their norms on the data is part of Corollary 3.7 below.

The differential equation (29) is satisfied for each $t > 0$, but the theorem above does not provide sufficient information on the regularity of y. This is the task of the next result.

Corollary 3.7 *Assume that $f \in C^1([0,\infty), V_m')$ and that $g_0 \in V_m'$. Then, for any $T > 0$, the unique solution $y : [0,\infty) \to L^2(\Omega)^3$ of problem (16) satisfies*

$$y \in L^2(0,T; Y(\Omega)). \tag{30}$$

Moreover, it holds

$$y_1 \in C([0,T]; L^2(\Omega_1)^3), \tag{31}$$

$$\sigma y_t \in L^1(0,T; Y(\Omega)') \text{ and } \int_{\Omega_2} y_2(x,\cdot) \cdot e(x) \, dx \in W^{1,2}(0,T). \tag{32}$$

There is a constant $c > 0$ not depending on f and g_0 such that

$$\begin{aligned} &\|\mathcal{M}y\|_{C([0,T],V_m')} + \|y\|_{L^2(0,T;Y(\Omega))} + \|\sigma y_t\|_{L^1(0,T;Y(\Omega)')} \\ &\quad + \|\textstyle\int_{\Omega_2} y(x,\cdot) \cdot e(x) \, dx\|_{W^{1,2}(0,T)} \le c\,(\|f\|_{L^2(0,T;V_m')} + \|g_0\|_{V_m'}). \end{aligned} \tag{33}$$

Proof. (i) *Estimation of* $\|\mathcal{M}y(t)\|_{V_m'}$*:* As for all $t > 0$, $y(\cdot,t) \in D \subset V$, the existence result implies that we have

$$\int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{y}(x,t)\,dx + R^{-1}\left(\int_{\Omega_2} y_t(x,t)\cdot e(x)\,dx\right)\left(\int_{\Omega_2} \bar{y}(x,t)\cdot e(x)\,dx\right) \qquad (34)$$
$$+a_0(y(\cdot,t),y(\cdot,t)) = \int_{\Omega} f(x,t)\cdot \bar{y}(x,t)\,dx.$$

Notice that we have $f(\cdot,t)|_{\Omega_2} = \varphi(t)\,e$ with some $\varphi \in C^1[0,T]$, hence

$$\int_{\Omega} f(x,t)\cdot\bar{y}(x,t)\,dx = \int_{\Omega_1} f(x,t)\cdot\bar{y}(x,t)\,dx + \varphi(t)\int_{\Omega_2} e(x)\cdot\bar{y}(x,t)\,dx.$$

Both integrals in the right-hand side are continuous functions on $[0,T]$ (cf. Remark 3.2), thus the right-hand side of (34) is well defined and bounded.

Let us introduce the real function

$$h(t) = \int_{\Omega_1} \sigma|y(x,t)|^2\,dx + R^{-1}\left|\int_{\Omega_2} y(x,t)\cdot e(x)\,dx\right|^2 = \|\mathcal{M}y(t)\|^2_{V_m'}.$$

Its derivative

$$\frac{d}{dt}h(t) = \int_{\Omega_1}\sigma y_t(x,t)\cdot\bar{y}(x,t)\,dx + R^{-1}\left(\int_{\Omega_2} y_t(x,t)\cdot e(x)\,dx\right)\left(\int_{\Omega_2}\bar{y}(x,t)\cdot e(x)\,dx\right)$$

appears in the left-hand side of (34). Taking the real part of this previous identity and using the fact that

$$\Re a_0(y(\cdot,t),y(\cdot,t)) = a_0(y(\cdot,t),y(\cdot,t)) \geq 0,$$

we get in view of the identity $\sqrt{h(t)} = \|y(\cdot,t)\|_{V_m}$ that

$$\frac{d}{dt}h(t) \leq 2\Re\int_{\Omega} f(t)\cdot\bar{y}(x,t)\,dx \leq 2\|f(\cdot,t)\|_{V_m'}\sqrt{h(t)} \leq \|f(\cdot,t)\|^2_{V_m'} + h(t),$$

for all $t > 0$.

Thanks to the regularity of y stated in Theorem 3.6, h belongs to $W^{1,1}(\eta,T)$ for all $\eta > 0$. Therefore Gronwall's inequality yields

$$h(t) \leq h(\eta)e^{t-\eta} + \int_{\eta}^{t} e^{t-s}\,\|f(\cdot,s)\|^2_{V_m'}\,ds \quad \forall t > \eta.$$

The regularity of y also implies that h is continuous at zero. Hence, passing to the limit $\eta \to 0$ we find

$$h(t) \leq h(0)e^{t} + \int_0^t e^{t-s}\,\|f(\cdot,s)\|^2_{V_m'}\,ds \quad \forall t > 0.$$

By the definition of h, there holds $h(0) = \|g_0\|^2_{V_m'}$. Therefore, we have found that

$$\max_{0\leq t\leq T}\|\mathcal{M}y\|_{V_m'} = \sqrt{h(t)} \leq C(T)(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')}) \qquad (35)$$

holds for a positive constant $C(T)$ that depends on T but not on the data and on y. This proves (31).

(ii) *Estimation of* $\|y\|_{L^2(0,T;Y(\Omega))}$*:* Now we return to (34) and take again the real part of this identity. Integrating in (η, T) for $\eta > 0$ we find that

$$\int_\eta^T \Big\{ \int_{\Omega_1} \sigma \frac{d}{dt}|y(x,t)|^2\, dx + R^{-1}\frac{d}{dt}\left|\int_{\Omega_2} y(x,t)\cdot e(x)\, dx\right|^2 + 2\Re a_0(y(\cdot,t), y(\cdot,t))\Big\}\, dt \tag{36}$$
$$= 2\Re \int_\eta^T \int_\Omega f(t)\cdot \bar{y}(x,t)\, dxdt.$$

This shows that $|y_1|^2$ (resp. $\left|\int_{\Omega_2} y(x,\cdot)\cdot e(x)\, dx\right|^2$) belongs to $W^{1,1}(\eta, T; L^2(\Omega_1))$ (resp. $W^{1,1}(\eta, T)$), cf. the remark after (34). Consequently,

$$\int_{\Omega_2} y(x,\cdot)\cdot e(x)\, dx \in W^{1,2}(0,T)$$

follows by passing to the limit $\eta \to 0$. Moreover, we can integrate by parts in (36) and get equivalently

$$\int_{\Omega_1} \sigma |y(x,T)|^2\, dx + R^{-1}\left|\int_{\Omega_2} y(x,T)\cdot e(x)\, dx\right|^2 + 2\int_\eta^T \Re a_0(y(\cdot,t), y(\cdot,t))\, dt$$
$$= 2\Re\int_\eta^T\int_\Omega f(t)\cdot\bar{y}(x,t)\, dxdt + \int_{\Omega_1}\sigma|y(x,\eta)|^2\, dx + R^{-1}\left|\int_{\Omega_2} y(x,\eta)\cdot e(x)\, dx\right|^2.$$

The right-hand side of this identity admits a limit as η tends to zero, thanks to the regularity of y. Hence the same is true for the left-hand side. Passing to the limit, we obtain

$$\int_{\Omega_1} \sigma |y(x,T)|^2\, dx + R^{-1}\left|\int_{\Omega_2} y(x,T)\cdot e(x)\, dx\right|^2 + 2\int_0^T \Re a_0(y(\cdot,t), y(\cdot,t))\, dt$$
$$= 2\Re\int_0^T\int_\Omega f(t)\cdot\bar{y}(x,t)\, dxdt + \int_{\Omega_1}\sigma|y(x,0)|^2\, dx + R^{-1}\left|\int_{\Omega_2} y(x,0)\cdot e(x)\, dx\right|^2.$$

The above identity implies that there exists $c > 0$ such that

$$\int_0^T\int_\Omega \mu^{-1}|\operatorname{curl} y(x,t)|^2\, dxdt \le c\,(\|f\|_{L^2(0,T;V_m')}\|y\|_{L^2(0,T;V_m')} + \|g_0\|^2_{L^2(0,T,V_m')}).$$

As for all $t > 0$, $y(\cdot,t)$ belongs to $Y(\Omega)\cap V$, by Lemma 2.2 of [8] we have

$$\int_{\Omega_2} |y(x,t)|^2\, dx \le C\Big(\int_\Omega \mu^{-1}|\operatorname{curl} y(x,t)|^2\, dx + \int_{\Omega_1}|y(x,t)|^2\, dx\Big),$$

for some $C > 0$ that is independent of t. The estimates (35) and (37) show that the right-hand side of the previous inequality is square integrable in $(0,T)$. We conclude that (30) holds together with the estimate

$$\|y\|_{L^2(0,T;Y(\Omega))} \le C_1(T)\,(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')}), \tag{37}$$

for a positive constant $C_1(T)$ that depends on T but not on the data and on y.

(iii) *Enlarging the set of test functions:* As $Y(\Omega) \cap V$ is included in V, the existence result implies that for all $t > 0$, we have

$$\begin{aligned}\int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{z}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \\ + a_0(y(\cdot,t), z) = \int_{\Omega} f(t) \cdot \bar{z}(x)\, dx \quad \forall z \in Y(\Omega) \cap V.\end{aligned} \tag{38}$$

Our next goal is to show that this identity remains true for all z in $Y(\Omega)$.

Indeed, for any given $z \in Y(\Omega)$ and $i = 1$ or 2, we can consider $\varphi_i \in H_0^1(\Omega_i)$, the solution of

$$\int_{\Omega_i} \nabla\varphi_i \cdot \nabla\chi\, dx = \int_{\Omega_i} z \cdot \nabla\chi\, dx \quad \forall \chi \in H_0^1(\Omega_i).$$

Such a solution satisfies

$$\operatorname{div}(z - \nabla\varphi_i) = 0 \text{ in } \mathcal{D}'(\Omega_i),$$

hence $z - \nabla\varphi_i$ is divergence free in Ω_i. However, we are not sure that

$$\langle (z_2 - \nabla\varphi_2) \cdot n, 1 \rangle_\Gamma = -\langle \nabla\varphi_2 \cdot n, 1 \rangle_\Gamma$$

is zero, which is needed to have $z - \nabla\varphi_i \in Y(\Omega)$. If this quantity is not zero, we define

$$\phi_2 = \varphi_2 - q\, \langle \nabla\varphi_2 \cdot n, 1 \rangle_\Gamma,$$

where q is the unique element in $H^1(\Omega_2)$ that satisfies (28); cf. the characterization of $K_N(\Omega_2)$. An easy computation confirms that

$$\langle (z_2 - \nabla\phi_2) \cdot n, 1 \rangle_\Gamma = 0.$$

Now we define

$$z_1 = z - \nabla\tilde{\varphi}_1 - \nabla\tilde{\phi}_2,$$

where

$$\tilde{\varphi}_1 = \begin{cases} \varphi_1 & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2, \end{cases}$$

$$\tilde{\phi}_2 = \begin{cases} 1 & \text{in } \Omega_1, \\ \phi_2 & \text{in } \Omega_2 \end{cases}$$

and verify that z_1 belongs to $Y(\Omega) \cap V$. We will show in (v) below that there holds

$$\begin{aligned}\int_{\Omega_1} \sigma y_t(x,t) \cdot \nabla\overline{\tilde{\varphi}_1}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \nabla\overline{\tilde{\varphi}_1}(x) \cdot e(x)\, dx \right) \\ + a_0(y(\cdot,t), \nabla\tilde{\varphi}_1) = \int_{\Omega} f(t) \cdot \nabla\overline{\tilde{\varphi}_1}(x)\, dx\end{aligned} \tag{39}$$

and

$$\int_{\Omega_1} \sigma y_t(x,t) \cdot \nabla\overline{\tilde{\phi}_2}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \nabla\overline{\tilde{\phi}_2}(x) \cdot e(x)\, dx \right) \\ + a_0(y(\cdot,t), \nabla\tilde{\phi}_2) = \int_\Omega f(t) \cdot \nabla\overline{\tilde{\phi}_2}(x)\, dx. \tag{40}$$

Inserting $z := z_1$ in (38) and subtracting it from the sum of the previous two identities we obtain that

$$\int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{z}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \\ + a_0(y(\cdot,t), z) = \int_\Omega f(t) \cdot \bar{z}(x)\, dx \quad \forall z \in Y(\Omega). \tag{41}$$

In this way, we have shown that (38) holds true for all test functions $z \in Y(\Omega)$.

Equivalently, we can re-arrange this as

$$\int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{z}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \\ = -a_0(y(\cdot,t), z) + \int_\Omega f(t) \cdot \bar{z}(x)\, dx \quad \forall z \in Y(\Omega).$$

(iv) *Verification of* (32)*:* By the Cauchy-Schwarz inequality we obtain after integration on $[0,T]$

$$\int_0^T \left| \int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{z}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \right| dt \\ \leq C\sqrt{T} \left(\|f\|_{L^2(0,T,V_m')} + \|y\|_{L^2(0,T;Y(\Omega))} \|z\|_{Y(\Omega)} \right) \quad \forall z \in Y(\Omega),$$

for some $C > 0$. Due to (37), we deduce the existence of a constant $C_2(T) > 0$ such that

$$\int_0^T \left| \int_{\Omega_1} \sigma y_t(x,t) \cdot \bar{z}(x)\, dx + R^{-1} \left(\int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right) \left(\int_{\Omega_2} \bar{z}(x) \cdot e(x)\, dx \right) \right| dt \\ \leq C_2(T) \left(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')} \right) \|z\|_{Y(\Omega)} \big) \quad \forall z \in Y(\Omega). \tag{42}$$

In a first step, since e is different from zero in Ω_2, we can fix a function $\varphi \in \mathcal{D}(\Omega_2)^3$ such that

$$\int_{\Omega_2} \bar{\varphi}(x) \cdot e(x)\, dx \neq 0.$$

If not, it would hold

$$\int_{\Omega_2} \bar{\varphi}(x) \cdot e(x)\, dx = 0, \forall \varphi \in \mathcal{D}(\Omega_2)^3,$$

and by the density of $\mathcal{D}(\Omega_2)^3$ into $L^2(\Omega_2)^3$ we would deduce that $e = 0$.

The function $\tilde{\varphi}$ belongs to $Y(\Omega)$ (here we take the extension by zero). Therefore, we deduce by (42) with $z = \tilde{\varphi}$ that

$$R^{-1} \int_0^T \left| \int_{\Omega_2} y_t(x,t) \cdot e(x)\, dx \right| dt \le C_3(T) \left(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')} \right). \tag{43}$$

This proves the second assertion from (32).

To show the first assertion, by taking any z in $Y(\Omega)$ and using (42) and (43) we deduce that

$$\int_0^T \left| \int_\Omega \sigma y_t(x,t) \cdot \bar{z}(x)\, dx \right| dt \le C_4(T) \left(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')} \right) \|z\|_{Y(\Omega)}. \tag{44}$$

This leads to the first assertion by the definition of the norm of $Y(\Omega)'$.

(v) *Verification of (39) and (40):* To complete the proof, we still have to show (39) and (40). For the first identity, we mention that $\mathcal{M}y(t), \mathcal{M}y_t(t)$ and $f(t)$ belong to V_m' for all $t \in (0,T]$. Therefore, it suffices to show that

$$\int_{\Omega_1} g(x) \cdot \nabla \varphi_1(x)\, dx = 0 \quad \forall g \in V_m'.$$

But for such g, by Green's formula we have

$$\int_{\Omega_1} g(x) \cdot \nabla \varphi_1(x)\, dx = -\int_{\Omega_1} \operatorname{div} g(x) \varphi_1(x)\, dx + \langle g \cdot n, \varphi_1 \rangle_\Gamma.$$

The right-hand side is zero because g is divergence free and $\varphi_1 = 0$ holds on $\Gamma = \partial\Omega_1$.

Similarly, (40) holds, if

$$\int_{\Omega_2} g(x) \cdot \nabla \phi_2(x)\, dx = 0 \quad \forall g \in V_m'.$$

We invoke again the Green's formula and obtain

$$\int_{\Omega_2} g(x) \cdot \nabla \phi_2(x)\, dx = -\int_{\Omega_2} \operatorname{div} g(x) \phi_2(x)\, dx + \langle g_2 \cdot n, \phi_2 \rangle_\Gamma + \langle g_2 \cdot n, \phi_2 \rangle_{\partial\Omega}.$$

The right-hand side is again zero because g is divergence free, $\phi_2 = 0$ holds on $\partial\Omega$, and by $\phi_2 = 1$ on Γ, we finally get

$$\langle g_2 \cdot n, \phi_2 \rangle_\Gamma = \langle g_2 \cdot n, 1 \rangle_\Gamma = 0,$$

in view of $g \in V_m'$.

The estimate (33) follows from (35), (37), (43), and (44). To confirm the $W^{1,2}(0,T)$-estimate, we first mention that we have

$$\left| \int_0^T \int_\Omega f(x,t) \cdot \bar{y}(x,t)\, dx dt \right| \le \|f\|_{L^2(0,T;V_m')} \|y\|_{L^2(0,T;Y(\Omega))}$$

$$\le C_1(T) \|f\|_{L^2(0,T;V_m')} \left(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')} \right) \le C_1(T) \left(\|g_0\|_{V_m'} + \|f\|_{L^2(0,T;V_m')} \right)^2$$

by (37). The maximum norm of h is bounded by (35). In view of this, the $W^{1,2}(0,T)$-estimate follows from (35) and (36). ∎

3.3 Existence for data with lower regularity

Our next step is to weaken the regularity assumption on the datum f, for that purpose, we adopt the next definition. For $T > 0$, $g_0 \in V_m'$ and $f \in L^2(0,T;V_m')$, we say that $y : [0,T) \to L^2(\Omega)^3$ is a weak solution of problem (16) if y has the regularity from (30), (31) and (32) and if it satisfies

$$\langle \sigma y_t(\cdot,t); z\rangle_{Y(\Omega)',Y(\Omega)} + R^{-1}\left(\int_{\Omega_2} y_t(x,t)\cdot e(x)\,dx\right)\left(\int_{\Omega_2} \bar z(x)\cdot e(x)\,dx\right) \qquad (45)$$
$$+a_0(y(\cdot,t),z) = \int_\Omega f(x,t)\cdot \bar z(x)\,dx \quad \forall z \in Y(\Omega),$$

as well as

$$y_1(\cdot,0) = g_0 \text{ in } \Omega_1 \text{ and } \int_{\Omega_2} y_2(x,0)\cdot e(x)\,dx = \int_{\Omega_2} g_0(x)\cdot e(x)\,dx.$$

Note that the initial conditions are well defined due to (31) and the embedding $W^{1,2}(0,T) \hookrightarrow C([0,T])$.

The assumption on g_0 means that $\operatorname{div} y_0 = 0$ in Ω_1, $\operatorname{div} e = 0$ in Ω_2, and the integral condition (5) is satisfied on Γ. The assumption on f is equivalent to $f_1 \in L^2(0,T;L^2(\Omega_1)^3)$ with $\operatorname{div} f_1(t) = 0$ a.e. in $(0,T)$, and $f_2(t) = \alpha(t)\,e$ in Ω_2, where $\alpha \in L^2(0,T)$.

Note further that the estimate (33) of our previous Corollary shows the uniqueness of a weak solution. This estimate also justifies the definition of a weak solution and also guarantees the existence of such a weak solution with appropriate data, namely we have the

Theorem 3.8 *Let $T > 0$ be fixed and assume that $f \in L^2(0,T;V_m')$ and $g_0 \in V_m'$. Then problem (16) has a unique weak solution y.*

Proof. Fix a sequence $f_n \in \mathcal{D}((0,T),V_m'), n \in \mathbb{N}$ such that

$$f_n \to f \text{ in } L^2(0,T;V_m') \text{ as } n \to \infty.$$

Then by Corollary 3.7, for all $n \in \mathbb{N}$, problem (16) with right-hand side f_n and an initial datum g_0 has a unique solution y_n that satisfies

$$\|y_n - y_m\|_{L^2(0,T;Y(\Omega))} + \|\sigma((y_n)_t - (y_m)_t)\|_{L^1(0,T;Y(\Omega)')} + \max_{0\le t\le T}\|\mathcal{M}(y_n - y_m)\|_{V_m'}$$
$$+\left\|\int_{\Omega_2}(y_{n,2}(x,\cdot) - y_{m,2}(x,\cdot))\cdot e(x)\,dx\right\|_{W^{1,1}(0,T)} \le C(T)\,\|f_n - f_m\|_{L^2(0,T;V_m')} \qquad (46)$$

for some $C(T) > 0$ and all $n,m \in \mathbb{N}$. Here, we use the estimate (33). But due to the continuous embedding $Y(\Omega) \hookrightarrow Y(\Omega)'$ this also implies that

$$\|\sigma(y_n - y_m)\|_{H^1(0,T;Y(\Omega)')} \le C\,\|f_n - f_m\|_{L^2(0,T;V_m')}. \qquad (47)$$

Since (f_n) is a Cauchy sequence, (46) implies the same for (y_n) in different spaces. Therefore, from the first estimate we deduce that there exist $y \in L^2(0,T;Y(\Omega))$, $z \in C([0,T],L^2(\Omega_1)^3)$, $w \in L^1(0,T;Y(\Omega)')$, and $\alpha \in W^{1,1}(0,T)$ such that

$$y_n \to y \text{ in } L^2(0,T;Y(\Omega)), \tag{48}$$

$$y_{n,1} \to z \text{ in } C([0,T],L^2(\Omega_1)^3), \tag{49}$$

$$(\sigma y_n)_t \to w \text{ in } L^1(0,T;Y(\Omega)'), \tag{50}$$

$$\int_{\Omega_2} y_{n,2}(x,\cdot)\cdot e(x)\,dx \to \alpha(\cdot) \text{ in } W^{1,2}(0,T), \tag{51}$$

as $n \to \infty$, with $z = y_1$.

On the other hand the estimate (47) implies the existence of $z \in H^1(0,T;Y(\Omega)')$ with

$$\sigma y_n \to z \text{ in } H^1((0,T);Y(\Omega)') \tag{52}$$

as $n \to \infty$.

As $L^2(0,T;L^2(\Omega)^3) \hookrightarrow L^2(0,T;Y(\Omega)')$ and $H^1((0,T);Y(\Omega)') \hookrightarrow L^2(0,T;Y(\Omega)')$, we deduce that

$$z = \sigma y.$$

Furthermore as (52) implies that

$$(\sigma y_n)_t \to z_t \text{ in } L^2(0,T;Y(\Omega)'),$$

and comparing with (50) we obtain that

$$w = \sigma y_t.$$

Moreover, as (48) implies that

$$\int_{\Omega_2} y_{n,2}(x,\cdot)\cdot e(x)\,dx \to \int_{\Omega_2} y_2(x,\cdot)\cdot e(x)\,dx \text{ in } L^2(0,T),$$

we deduce that

$$\alpha(\cdot) = \int_{\Omega_2} y_2(x,\cdot)\cdot e(x)\,dx.$$

In summary we have proved that the limit y satifies (30), (31) and (32).

Finally, by the previous Corollary we know that y_n satisfies (41) with f_n instead of f, namely

$$\int_{\Omega_1} (\sigma y_n)_t(x,t)\cdot \bar{z}(x)\,dx + R^{-1}\left(\int_{\Omega_2} (y_n)_t(x,t)\cdot e(x)\,dx\right)\left(\int_{\Omega_2} \bar{z}(x)\cdot e(x)\,dx\right)$$
$$+a_0(y_n(\cdot,t),z) = \int_\Omega f_n\cdot \bar{z}(x)\,dx, \forall z \in Y(\Omega).$$

Passing to the limit we find that y satisfies (45).

In the same manner, starting from the initial conditions satisfied by y_n and passing to the limit, we deduce that y satisfies the same initial conditions as y_n. ■

3.4 Particular cases

Let us apply Theorem 3.8 to some particular settings that fit in the general system (16). Here, we allow again any $\varepsilon \geq 0$.

First, we consider the case $e = 0$. Here, we have

$$\mathcal{M}y = 0 \text{ in } \Omega_2 \text{ and } g_0 = 0 \text{ in } \Omega_2.$$

Therefore, the system (16) reduces to the degenerate parabolic equation (1). Then Theorem 3.8 includes the following Corollary that recovers a result by Bachinger et al. [5].

Corollary 3.9 *Suppose that $y_0 \in L^2(\Omega_1)^3$ is divergence free and f belongs to $L^2(0,T;V_m')$. Then the equation (1) has a unique weak solution $y \in L^2(0,T;Y(\Omega))$ with $\sigma y_t \in L^1(0,T;Y(\Omega)')$.*

Notice that the assumption $f(t) \in V_m'$ $\forall t \in [0,T]$ means that $\operatorname{div} f(t)_{|\Omega_1} = 0$ and $f(t)_{|\Omega_2} = 0$ for all $t \in [0,T]$. We have $y \in C([0,T], L^2(\Omega_1)^3)$ as in Remark 3.2.

In particular, it follows for $y_0 = 0$ that the map $f \mapsto y$ is continuous from $L^2(0,T;V_m')$ to $L^2(0,T;Y(\Omega))$ and the mapping $f \mapsto y_1$ is continuous from $L^2(0,T;V_m')$ with values in $C([0,T], L^2(\Omega_1)^3)$. The latter follows from estimate (35).

From now on, e can be nonzero again and our real interest is the case $e \neq 0$. The next result refers to equation (9), where the right-hand side vanishes in Ω_1 and is equal to $R^{-1}u(t)\,e(x)$ in Ω_2.

Corollary 3.10 *For all given $u \in L^2(0,T)$, divergence free $y_0 \in L^2(\Omega_1)^3$, and $\alpha_0 \in \mathbb{R}$, the system (9) has a unique weak solution $y \in L^2(0,T;Y(\Omega))$ that obeys the regularity stated in Corollary 3.9 above.*

The next result provides a sufficient condition for the assumption that $y|_{\Omega_2}$ belongs to $C([0,T], H(\text{curl}, \Omega_1))$ that will be used as assumption in Lemma 3.12 on the continuity of the electrical current i in the system (4).

Theorem 3.11 *Assume in addition to the assumptions stated in Corollary 3.10 that it holds $\operatorname{curl} \mu^{-1} \operatorname{curl} y_0 \in L^2(\Omega_1)^3$, $e \neq 0$, and $u \in H^1(0,T)$. Then the solution y of (9) belongs to $H^1(0,T;H(\text{curl},\Omega))$.*

Proof. As in the previous sections, we denote the restriction of y_0 to Ω_j by y_{j0}. We consider the solution $w \in L^2(0,T;Y(\Omega))$ of the problem

$$\begin{aligned} \sigma w_t(t) + Lw(t) &= 0 && \text{in } \Omega_1 \\ \int_{\Omega_2} w_t(t) \cdot e\,dx\,e + R\,Lw(t) &= u'(t)e && \text{in } \Omega_2 \end{aligned} \tag{53}$$

subject to the initial conditions

$$\begin{aligned} \sigma w(0) &= -Ly_{10} && \text{in } \Omega_1 \\ \int_{\Omega_2} w(0) \cdot e\,dx &= u(0) - R\,i_0 && \text{in } \Omega_2, \end{aligned} \tag{54}$$

where i_0 is fixed according to (15) so that y_0 satisfies the initial condition (8). Notice that $u(0)$ is defined, since $u \in H^1(0,T)$. For the same reason, we have $u' \in L^2(0,T)$. Moreover, Ly_{10} belongs to $L^2(\Omega_1)^3$, hence the regularity assumption on the initial condition for w in Ω_1 is fulfilled.

Thanks to Corollary 3.10, there exists a unique weak solution $w \in L^2(0,T;Y(\Omega))$ to the problem (53), (54). We also know that $w \in C([0,T], L^2(\Omega_1)^3)$ so that the value $w(0)$ is well defined in Ω_1.

Now we define y by

$$y(t) := \int_0^t w(s)\,ds + y_0, \tag{55}$$

where $y_0 = y_{10}$ is defined in Ω_1 and $y_0 = y_{20} = i_0 y_e + y_\Gamma$ in Ω_2 according to (12). The integral is defined in the Bochner sense. The solution y_0 constructed by Lemma 2.1 is contained in $H(\text{curl}, \Omega)$ because it holds $y_{10} \in H(\text{curl}, \Omega_1)$, $y_{10} \in H(\text{curl}, \Omega_2)$ and $y_{10} \times n = y_{20} \times n$ on Γ.

Therefore, we have that $y \in H^1(0,T;H(\text{curl}, \Omega))$. Let us verify that this is a solution to the system (9); then the regularity result follows by the uniqueness of this solution. In Ω_1, we obtain

$$\begin{aligned}
\sigma y_t(t) + Ly(t) &= \sigma \frac{d}{dt}\Big(\int_0^t w(s)\,ds + y_{10}\Big) + L\int_0^t w(s)\,ds + Ly_{10} \\
&= \sigma w(t) + \int_0^t L\,w(s)\,ds + Ly_{10} \\
&= \sigma w(t) - \int_0^t \sigma w'(s)\,ds + Ly_{10} \\
&= \sigma w(t) - \sigma w(t) + \sigma w(0) + Ly_{10} = 0,
\end{aligned}$$

where the last term vanishes thanks to the upper initial condition of (54). Therefore, the first equation of (9) is fulfilled.

Consider now the equation in Ω_2. We have a.e. in $(0,T)$ that

$$\int_{\Omega_2} w_t(t) \cdot e\,dx\,e + R\,Lw(t) = u'(t)e \text{ in } \Omega_2.$$

Integration over $(0,T)$ yields

$$\int_{\Omega_2} w(t) \cdot e\,dx\,e + R\int_0^t Lw(s)ds = u(t)e - u(0)e + \int_{\Omega_2} w(0) \cdot e\,dx\,e. \tag{56}$$

By the lower initial condition of (54), the right-hand side of (56) is equal to $u(t)e - Ri_0 e$, hence

$$\int_{\Omega_2} w(t) \cdot e\,dx\,e + \int_0^t Lw(s)ds + R\,i_0 e = u(t)e.$$

By (55), the condition $Ly_{20} = L(i_0 y_e + y_\Gamma) = i_0 e$, and the last equation, we get

$$\int_{\Omega_2} y_t(t) \cdot e\, dx\, e + RLy(t) = \int_{\Omega_2} w(t) \cdot e\, dx\, e + R\int_0^t Lw(s)ds + Ri_0 e = u(t)e.$$

This confirms the second differential equation of (9). Moreover, the initial condition is satisfied, because

$$R^{-1} \int_{\Omega_2} y(0) \cdot e\, dx\, e = R^{-1} \int_{\Omega_2} y_{20} \cdot e\, dx\, e = \alpha_0.$$

Notice that y_{20} was defined in a way such that $R^{-1} \int_{\Omega_2} y(0) \cdot e\, dx = \alpha_0$ is granted. ■

Let us finally discuss the equivalence of the system (4) with the system (9).

Assume that y is a weak solution of (9) with $u \in H^1(0,T)$, $\alpha_0 \in \mathbb{R}$ and y_0 satisfying the assumptions of Theorem 3.11. Defining i by (7), the identity (45) becomes

$$\langle \sigma y_t(\cdot,t); z\rangle_{Y(\Omega)',Y(\Omega)} + a_0(y(\cdot,t),z) = i(t) \int_\Omega e(x) \cdot \bar{z}(x)\, dx \quad \forall z \in Y(\Omega). \tag{57}$$

Due to Theorem 3.11 and Lemma 3.12 below, i is continuous and by the results of subsection 2.3, $i_0 = i(0)$ is given by formula (15).

This consideration suggests the following definition: Given $u \in H^1(0,T)$, $i_0 \in \mathbb{R}$ and y_0 satisfying the assumptions of Theorem 3.11, we say that (y,i) is a weak solution of (4), if $y \in H^1(0,T; H(\mathrm{curl},\Omega))$ has further the regularity from (30), (31), (32) while $i \in C[0,T]$, and the conditions (57) and (7) hold with $y(\cdot,0) = y_0$ and $i(0) = i_0$.

Again from the results of subsection 2.3, if (y,i) is weak solution of (4) in the above sense, then by (7) and using formula (15), we directly deduce that y is a weak solution of (9).

Lemma 3.12 *Assume that y is a weak solution to (9) such that $y|_{\Omega_2}$ belongs to the space $C([0,T], H(\mathrm{curl},\Omega_2))$. Then the function i is continuous on $[0,T]$.*

Proof. Owing to (57), the solution y satisfies the equation

$$\int_{\Omega_2} \{\mu^{-1} \operatorname{curl} y(t) \cdot \operatorname{curl} \varphi + \varepsilon\, y(t) \cdot \varphi\}\, dx = \int_{\Omega_2} e \cdot \varphi\, dx\; i(t)$$

for all test functions $\varphi \in H_0(\mathrm{curl},\Omega_2)$. As $H_0(\mathrm{curl},\Omega_2)$ is dense in $L^2(\Omega)^3$, there exists a sequence (e_n) of functions of $H_0(\mathrm{curl},\Omega_2)$ converging to e in the sense of $L^2(\Omega)^3$. If n is sufficiently large, then

$$\int_{\Omega_2} e \cdot e_n\, dx > 0$$

holds, since $\|e\|_{L^2(\Omega)^3} > 0$. Now we take e_n as a test function for a sufficiently large n, hence

$$i(t) = \frac{1}{\int_{\Omega_2} e \cdot e_n\, dx} \int_{\Omega_2} \{\mu^{-1} \operatorname{curl} y(t) \cdot \operatorname{curl} e_n + \varepsilon\, y(t) \cdot e_n\}\, dx.$$

Thanks to our assumption on y, the right-hand side is continuous. Hence the same holds true for i. ■

Acknowledgement. The authors are grateful to Dr. S. Stingelin (Endress+Hauser Flowtec AG Reinach, Switzerland) for his cooperation and for drawing their attention to the integro-differential model (3).

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