SUFFICIENT OPTIMALITY IN A PARABOLIC CONTROL PROBLEM

Hans D. Mittelmann

Department of Mathematics, Arizona State University, Box 871804, Tempe, AZ 85287-1804, U.S.A.

mittelmann@asu.edu

Fredi Tröltzsch

Technische Universität Berlin, Fakultät II - Mathematik und Naturwissenschaften, Sekretariat Ma 4-5, Str. des 17. Juni 136, D-10623 Berlin, Germany.

troeltz@math.tu-berlin.de

Abstract We define a class of parabolic problems with control and state constraints and identify a problem within this class which possesses a locally unique critical point satisfying the second order sufficient optimality conditions. The second derivative of the Lagrangian is not globally coercive, since active equality and strongly active inequality constraints are considered. This is both shown analytically as well as verified numerically for a finite difference discretization.

Keywords: Optimal control, nonlinear parabolic equation, second-order sufficient optimality condition, numerical verification

1. Introduction

The theory of second-order sufficient optimality conditions (SSC) for the optimal control of semilinear elliptic and parabolic equations is a field of active research. Conditions of this type play an important role in the associated numerical analysis. Their verification is a basic and important issue. Although a numerical confirmation of SSC cannot yet give a definite answer whether they really hold in the infinite-dimensional problem, it provides some evidence about their validity. We refer to recent papers by Mittelmann [7; 8], who confirmed that second order sufficient conditions can be checked effectively by numerical techniques. Here, we consider the numerical verification of second order sufficient optimality conditions for the following class of nonlinear optimal control problems of parabolic equations with constraints on the control and the state.

(P) Minimize

$$J(y,u) = \frac{1}{2}\int\int_Q \alpha(x,t)(y(x,t)-y_d(x,t))^2\,dxdt + \frac{\nu}{2}\int_0^T u^2(t)\,dt + \int_0^T [a_y(t)y(l,t)+a_u(t)u(t)]\,dt \tag{1.1}$$

subject to

$$\begin{aligned} y_t - y_{xx} &= e_Q && \text{in } Q\\ y(x,0) &= 0 && \text{in } (0,l)\\ y_x(0,t) &= 0 && \text{in } (0,T)\\ y_x(l,t)+y^2(l,t) &= e_\Sigma(t)+u(t) && \text{in } (0,T) \end{aligned} \tag{1.2}$$

and to

$$u_a \le u(t) \le u_b, \quad \text{a.e. in } (0,T), \tag{1.3}$$

$$\int\int_Q y(x,t)\,dxdt \le 0. \tag{1.4}$$

In this setting, T, ν, $l>0$, $u_a<u_b$ are fixed real numbers, $Q=(0,l)\times(0,T)$. Functions α, y_d, and e_Q are given in $L^\infty(Q)$, and a_y, a_u, e_Σ are fixed in $L^\infty(0,T)$. We shall denote the set of admissible controls by $U_{ad}=\{u\in L^\infty(0,T)\,|\,u_a\le u\le u_b, \quad \text{a.e. in } (0,T)\}$. Problem (P) is nonconvex, since the state equation is semilinear. Its nonlinearity y^2 is not of monotone type, hence standard results on existence and uniqueness of solutions to (1.2) do not apply. However, in our test example we shall construct a pair $(\bar y,\bar u)$ solving (1.2). The linearization of (1.2) at $\bar y$ is uniquely solvable for all $u\in U_{ad}$ with continuity of the solution mapping $u\mapsto y(u)$ from $L^p(0,T)$ to $W(0,T)\cap C(\bar Q)$, $p>2$, where

$$W(0,T)=\{y\in L^2(0,T;H^1(0,l))\,|\,y_t\in L^2(0,T;H^1(0,l)^*)\},$$

see Raymond and Zidani [11]. Thus the implicit function theorem guarantees existence and uniqueness of the solution $y=y(u)$ of (1.2) in $W(0,T)\cap C(\bar Q)$ for all u in a sufficiently small L^p-neighborhood of $\bar u$.

We shall consider a particular example of (P), where SSC are fulfilled, although the second order derivative $\mathcal{L}''$ of the Lagrange function is not positive definite on the whole space. This is possible, since we consider strongly active control constraints. Therefore, the construction of this example is more involved than the analysis of a similar one presented by Arada, Raymond and Tröltzsch in [1], where $\mathcal{L}''$ was coercive on the whole space. As a natural consequence, the numerical verification is more difficult. In fact, the example from [1] was verified numerically in [7] for coarser and in [8] for finer discretizations establishing the definiteness of a projected Hessian matrix while even the full matrix has this property. This gave rise to our search for the example presented below. The analysis of SSC for semilinear elliptic and parabolic control problems with pure control constraints is already quite well elaborated. We refer to the referenes in [5], [10]. The more difficult case of pointwise state-constraints is investigated, by Casas, Tröltzsch, and Unger [5], or Raymond and Tröltzsch [10], and in further papers cited therein. However, the discussion of SSC for state constraints is still rather incomplete. Problems with finitely many inequality and equality constraints of functional type are discussed quite completely in a recent paper by Casas and Tröltzsch [4].

2. First and second-order optimality condition

2.1. First order necessary conditions

Let the control $\bar{u}$ be locally optimal for (P) with associated state $\bar{y}$, i.e.

$$J(y,u) \geq J(\bar{y},\bar{u}) \tag{2.1}$$

holds for all (y,u) satisfying the constraints (1.2-1.4), where u belongs to a sufficiently small L^∞-neighborhood of $\bar{u}$. Suppose further that $(\bar{y},\bar{u})$ is regular. Then there exist Lagrange multipliers $\bar{p} \in W(0,T) \cap C(\bar{Q})$ (the adjoint state) and $\bar{\lambda} \geq 0$ such that the *adjoint equation*

$$\begin{array}{rll} -\bar{p}_t - \bar{p}_{xx} = \alpha(\bar{y} - y_d) + \bar{\lambda} & \text{in} & Q \\ \bar{p}(x,T) = 0 & \text{in} & (0,l) \\ \bar{p}_x(0,t) = 0 & \text{in} & (0,T) \\ \bar{p}_x(l,t) + 2\bar{y}(l,t)\,\bar{p}(l,t) = a_y(t) & \text{in} & (0,T), \end{array} \tag{2.2}$$

the *variational inequality*

$$\int_0^T (\nu\bar{u}(t) + \bar{p}(l,t) + a_u(t))(u(t) - \bar{u}(t))\,dt \geq 0 \quad \forall u \in U_{ad}, \tag{2.3}$$

and the *complementary slackness condition*

$$\bar{\lambda} \int\int_Q \bar{y}(x,t)\,dxdt = 0 \tag{2.4}$$

are fulfilled, see [3] or [11]. We mention that (2.3) is equivalent to the well-known projection property

$$\bar{u}(t) = \Pi_{[u_a,u_b]}\{-\frac{1}{\nu}(\bar{p}(l,t) + a_u(t))\}, \tag{2.5}$$

where $\Pi_{[u_a,u_b]} : \mathbb{R} \to [u_a, u_b]$ denotes projection onto $[u_a, u_b]$. Moreover, we recall that these conditions can be derived by variational principles applied to the Lagrange function $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}(y,u,p,\lambda) = J(y,u) - \int\int_Q (y_t - y_{xx} - e_Q)\,dxdt + \int\int_Q \bar{\lambda} y(x,t)\,dxdt \\ - \int_0^T (y_x(l,t) + y^2(l,t) - u(t) - e_\Sigma(t))p(l,t)\,dt.\end{aligned}$$

Defining $\mathcal{L}$ in this way, we tacitly assume that the homogeneous initial and boundary conditions of y are formally included in the state space. The conditions (2.2–2.3) follow from $\mathcal{L}_y(\bar{y},\bar{u},\bar{p},\bar{\lambda})y = 0$ for all admissible increments y and $\mathcal{L}_u(\bar{y},\bar{u},\bar{p},\bar{\lambda})(u-\bar{u}) \geq 0 \quad \forall u \in U_{ad}$. Let $\tau > 0$ be given. We define

$$\begin{aligned}A^+(\tau) &= \{t \in (0,T) \,|\, \nu\bar{u}(t) + \bar{p}(l,t) + a_u(t) \leq -\tau\} \\ A^-(\tau) &= \{t \in (0,T) \,|\, \nu\bar{u}(t) + \bar{p}(l,t) + a_u(t) \geq \tau\}.\end{aligned}$$

It holds $\bar{u} = u_b$ on A^+ and $\bar{u} = u_a$ on A^-. These sets indicate strongly active control constraints.

2.2. Second order sufficient optimality conditions

Let $(\bar{y},\bar{u},\bar{p},\bar{\lambda})$ be given such that the system of first order necessary conditions is satisfied, i.e. the relations (1.2-1.4), (2.2-2.4) and $\bar{\lambda} \geq 0$ are fulfilled. Now we state second order conditions, which imply local optimality of $\bar{u}$. For this purpose, we need the second order derivative of $\mathcal{L}$ with respect to (y,u),

$$\mathcal{L}''(\bar{y},\bar{u},\bar{p},\bar{\lambda})[y,u]^2 = \int\int_Q \alpha\, y^2\,dxdt + \nu \int_0^T u^2\,dt + 2\int_0^T \bar{p}(l,t)y^2(l,t)\,dt. \tag{2.6}$$

Let us assume as in the example below that the state-constraint (1.4) is active at $\bar{y}$ and $\bar{\lambda} = 1$. Then we require the following *second-order sufficient optimality condition*:

(SSC) There exist positive δ and τ such that

$$\mathcal{L}''(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda})[y, u]^2 \geq \delta \int_0^T u^2\, dt \tag{2.7}$$

holds for all $y \in W(0,T)$, $u \in L^2(0,T)$ such that

$$\begin{aligned} y_t - y_{xx} &= 0 \\ y(x,0) &= 0 \\ y_x(0,t) &= 0 \\ y_x(l,t) + 2\bar{y}(l,t)y(l,t) &= u(t) \end{aligned} \tag{2.8}$$

and

$$u(t) = 0 \qquad \text{on } A^+(\tau) \cup A^-(\tau) \tag{2.9}$$

$$u(t) \geq 0 \;\text{ if } \bar{u}(t) = u_a \text{ but } t \notin A^-(\tau) \tag{2.10}$$

$$u(t) \leq 0 \;\text{ if } \bar{u}(t) = u_b \text{ but } t \notin A^+(\tau) \tag{2.11}$$

$$\int\int_Q y(x,t)\, dxdt = 0. \tag{2.12}$$

It is known that (SSC) implies local optimality of $\bar{u}$ in a neighborhood of $L^\infty(0,T)$, see [4]. In our example, we shall verify a slightly stronger condition. We require (2.7) for all (y,u), which satisfy only (2.8-2.9).

3. The test example

We fix here the following quantities in (P):

$$T = 1,\; l = \pi,\; u_a = 0,\; u_b = 1, \nu = 0.004$$

$$\alpha(x,t) = \begin{cases} \alpha_o \in \mathbb{R}, & t \in [0,1/4] \\ 1, & t \in (1/4, 1], \end{cases}$$

$$y_d(x,t) = \begin{cases} \frac{1}{\alpha(x,t)}(1 - (2-t)cosx), & t \in [0,1/2] \\ \frac{1}{\alpha(x,t)}(1 - (2 - t - \alpha(x,t)(t-1/2)^2))cosx, & t \in (1/2, 1], \end{cases}$$

$$a_y(t) = \begin{cases} 0, & t \in [0,1/2] \\ 2(t-1/2)^2(1-t), & t \in (1/2,1], \end{cases}$$

$$a_u(t) = \nu + 1 - (1 + 2\nu)t,$$

$$e_Q(t) = \begin{cases} 0, & t \in [0, 1/2] \\ (t^2 + t - 3/4)cosx, & t \in (1/2, 1], \end{cases}$$

$$e_\Sigma(t) = \begin{cases} 0, & t \in [0, 1/2] \\ (t - 1/2)^4 - (2t - 1), & t \in (1/2, 1]. \end{cases}$$

Theorem 1 *The quantities*

$$\begin{aligned} \bar{u} &= max\{0, 2t - 1\} \\ \bar{y} &= \begin{cases} 0, & t \in [0, 1/2] \\ (t - 1/2)^2 cosx, & t \in (1/2, 1] \end{cases} \\ \bar{p} &= (1 - t)cosx \\ \bar{\lambda} &= 1 \end{aligned}$$

satisfy the system of first order necessary conditions.

Proof: Insert the definitions above in the state equation and adjoint equation. Then it is easy to check by elementary calculations that $\bar{u}$, $\bar{y}$ fulfil (1.2) and that $\bar{p}$ satisfies the adjoint equation (2.2). Moreover, the state-inequality constraint (1.4) is fulfilled as an equality, since the integral of $cosx$ over $[0, \pi]$ vanishes. Clearly, $\bar{u}$ is an admissible control. The variational inequality (2.3) is easy to verify by (2.5): We find

$$-1/\nu(\bar{p}(\pi, t) + a_u(t)) = 2t - 1 = \begin{cases} < 0, & t \in [0, 1/2) \\ > 0, & t \in (1/2, 1]. \end{cases}$$

Therefore,

$$\Pi_{[0,1]}\{-1/\nu(\bar{p}(\pi, t) + a_u(t))\} = max\{0, 2t - 1\} = \bar{u}(t).$$

□

Next we consider the second order sufficient condition (SSC) for the example analytically. What conditions must be checked to verify them? Thanks to our construction, $\bar{u}$ is strongly active on $[0, 1/2)$ and $\bar{u} = 0$ holds there. If $b < 1/2$ is given, then

$$\nu\bar{u}(t) + \bar{p}(\pi, t) + a_u(t) = \bar{p}(\pi, t) + a_u(t) = -\nu(2t - 1) > \nu(2b - 1)$$

holds for $t \in [0, b]$. Therefore, $t \in A^-(\tau)$ for $\tau = |\nu(2b - 1)|$. To verify the second order sufficient conditions, it suffices to confirm the coercivity condition (2.7) for all pairs (y, u) coupled through the linearized equation (2.8) and satisfying $u = 0$ on $[0, b]$ ($0 < b < 1/2$ being arbitrary but fixed). Assume that

$$\alpha(x, t) = \begin{cases} \alpha_0, & 0 \le t \le b \\ 1, & b < t \le 1. \end{cases} \tag{3.1}$$

Theorem 2 *Let α have the form (3.1), where $b \in [0, 1/2)$. Then the second order sufficient conditions (SSC) are satisfied by $(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda})$ for arbitrary $\alpha_0 \in \mathbb{R}$.*

Proof: Let u vanish on $[0, b]$ and let y solve (2.8). Then $y(x,t) = 0$ on $[0, b]$. For $\mathcal{L}''$ we get

$$\begin{aligned} \mathcal{L}''(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda})[y,u]^2 &= \int_0^\pi \int_0^b \alpha_0 \cdot 0 \, dx dt + \int_0^\pi \int_b^1 y^2 dx dt + \nu \int_0^1 u^2 dt \\ &\quad -2 \int_0^1 \bar{p}(\pi, t) y^2(\pi, t) \, dt \\ &\geq \nu \int_0^1 u^2 dt - 2 \int_0^1 (-(1-t)) y^2(\pi, t) \, dt \\ &\geq \nu \int_0^1 u^2 dt. \end{aligned} \tag{3.2}$$

Hence the coercivity condition (2.7) is satisfied. □

Notice that α_0 was not assumed to be positive. If $\alpha_0 \geq 0$, then $\mathcal{L}''$ is obviously coercive on the whole space $W(0,1) \times L^2(0,1)$, and (SSC) is satisfied in a very strong sense. However, we might find negative values for α_0 such that $\mathcal{L}''$ is partially indefinite.

Theorem 3 *If $\alpha_0 < 0$ is sufficiently small, then a pair (y, u) exists, such that $u \geq 0$, y solves the linearized equation (2.8), and*

$$\mathcal{L}''(\bar{y}, \bar{u}, \bar{p}, \bar{\lambda})[y,u]^2 < 0. \tag{3.3}$$

Proof: We take an arbitrary but fixed $b < 1/2$ and put

$$u(t) = \begin{cases} 1 & \text{on } [0, b] \\ 0 & \text{on } (b, 1]. \end{cases}$$

Then $\int_0^\pi \int_0^b y^2 \, dx dt$ is positive. Hence

$$\alpha_0 \int_0^\pi \int_0^b y^2 \, dx dt \to -\infty$$

as $\alpha_0 \to -\infty$. Therefore, the expression (3.2) becomes negative for sufficiently small α_0, if y^2 is substituted there for 0 in the first integral. □

For the numerical verification we need a rough estimate on how small α_0 should be chosen. To get a negative value of $\mathcal{L}''(\bar{y},\bar{u},\bar{p},\bar{\lambda})[y,u]^2$, we must have

$$\alpha_0 \int_0^\pi \int_0^b y^2\,dxdt + \int_0^\pi \int_b^1 y^2\,dxdt + \int_0^1 2(1-t)y^2(\pi,t)\,dt + \nu \int_0^1 u^2 dt < 0,$$

hence

$$|\alpha_0| > \frac{\int_0^\pi \int_b^1 y^2\,dxdt + \int_0^1 2(1-t)y^2(\pi,t)\,dt + \nu \int_0^1 u^2 dt}{\int_0^\pi \int_0^b y^2\,dxdt} = \frac{I_1 + I_2 + I_3}{I_0} \tag{3.4}$$

must hold. Here, $b \in [0, 1/2)$ can be chosen arbitrarily. We take the value $b = 1/4$. Thus we evaluate the integrals I_j for

$$u(t) = \begin{cases} 1 & \text{on } [0,1/4] \\ 0 & \text{on } (1/4,1] \end{cases}$$

and the associated state y. The state y solves the homogeneous heat equation subject to homogeneous initial condition, homogeneous boundary condition at $x = 0$ and

$$y_x(\pi,t) = \begin{cases} 1 & \text{on } [0,1/4] \\ -2\bar{y}(\pi,t) & \text{on } (1/4,1]. \end{cases}$$

To avoid tedious estimates, which might be performed by means of a Fourier series representation of y, we have evaluated the integrals I_j, $j = 0,..,3$ numerically. The result is

$$I_0 = .0103271, \quad I_1 = .0401844, \quad I_2 = .0708107, \quad I_3 = .001$$

$$\frac{I_1 + I_2 + I_3}{I_0} = 10.845. \tag{3.5}$$

4. Numerical Verification

In this section it is our goal to first demonstrate that problem (P) can be solved to good accuracy using a finite difference method. Next, as was done in [7; 8] we will, through an eigenvalue computation, verify that the computed solution satisfies the SSC and is thus a local minimizer.

Then, in order to check the properties of the specific example shown analytically above, we will compute an additional eigenvalue. We start by presenting the finite-dimensional analogue of (P) and then outline how the algebraic problems are solved. We define the following discretization of problem (P).

$$
\begin{aligned}
&\text{minimize} \quad f_h(y_h, u_h) = \\
&\frac{dxdt}{2} \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{j,i} \beta_j \gamma_i (y_{j,i} - y_d(x_j, t_i))^2) + \frac{\nu dt}{2} \Big(\sum_{i=0}^{m} \gamma_i u_i^2 \Big) \\
&\quad + \frac{dt}{2} \Big(\sum_{i=0}^{m} \gamma_i (a_y(t_i) y_{n,i} + a_u(t_i) u_i) \Big)
\end{aligned}
$$

subject to $\qquad (\mathrm{P}_h)$

$$
\begin{aligned}
&\frac{y_{j,i+1} - y_{j,i}}{dt} = \frac{1}{2}(y_{j-1,i} - 2y_{j,i} + y_{j+1,i} \\
&+ y_{j-1,i+1} - 2y_{j,i+1} + y_{j+1,i+1})/dx^2 + e_Q(x_j, t_{i+\frac{1}{2}}) \\
&\qquad i = 0, \dots, m-1, \quad j = 1, \dots, n-1 \\
&y_{j,0} = 0, \quad j = 0, \dots, n \\
&y_{2,i} - 4y_{1,i} + 3y_{0,i} = 0, \quad i = 1, \dots, m \\
&(y_{n-2,i} - 4y_{n-1,i} + 3y_{n,i})/(2dx) + y_{n,i}^2 \\
&= u_i + e_\Sigma(t_i), \quad i = 1, \dots, m \\
&u_a \le u_i \le u_b, \quad i = 0, \dots, m \\
&dxdt \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_j \gamma_i y_{j,i} \le 0.
\end{aligned}
$$

Here $x_j = jdx$, $dx = \pi/n$, $t_i = idt$, $dt = T/m, \beta_0 = \beta_n = \frac{1}{2}, \beta_j = 1$ otherwise; analogously for γ.

The discrete control problem (P_h) is essentially a generic nonlinear optimization problem of the form

$$
\min F^h(z) \quad \text{subject to} \quad G^h(z) = 0, \quad H^h(z) \le 0 \tag{4.1}
$$

where z comprises the discretized control and state variables. $G^h(z)$ symbolizes the state equation and boundary conditions while $H^h(z)$ stands for both pointwise control bounds and the integral state constraint, the only constraints of inequality type prescribed above. We state the well-known SSC for (4.1), assuming $z \in \mathbf{R}^{N_h}$, $G^h : \mathbf{R}^{N_h} \to \mathbf{R}^{M_h}$, $M_h < N_h$. Let z^* be an admissible point satisfying the first-order necessary optimality conditions with associated Lagrange multipliers μ^* and λ^*. Let further

$$
N(z^*) = (\nabla G^h(z^*), \nabla H_a(z^*))
$$

be a column-regular $N_h \times (M_h + P_h)$ matrix where $M_h + P_h < N_h$ and $\nabla H_a(z^*)$ denotes the gradients of the P_h active inequality constraints with positive Lagrange multipliers. For (4.2) we have $N_h = (m+1)(n+2)$ and $M_h = (m+1)(n+1)$ resulting in $m+1$ degrees of freedom which are further reduced by one through the active integral nonnegativity constraint on y and by any active bounds on u. Let finally $N = QR$ be a QR decomposition and $Q = (Q_1, Q_2)$ a splitting into the first $M_h + P_h$ and the remaining columns. The point z^* is a strict local minimizer if a $\gamma > 0$ exists such that, see, for example, [13]

$$\lambda_{\min}(L_2(z^*)) = \gamma > 0. \tag{4.2}$$

Here $L_2(z^*)$ is the projected Hessian of the Lagrangian

$$L_2(z^*) = Q_2^T(\nabla^2 F^h(z^*) - \mu^{*T}\nabla^2 G^h(z^*))Q_2.$$

No Hessian of H^h appears on the right due to its linearity. To clarify the relationship between the way we proceed here and which is standard in optimization and the analysis of the previous section we add the following explanations. In order to verify the SSC in the discrete case we have to determine the smallest eigenvalue of the Hessian on the tangent space of the active constraints. We do this by explicitly computing the orthogonal projection matrix Q_2 onto the tangent space and forming $L_2(z^*)$. Due to the verified regularity of the computed solution or the nondegeneracy of the active constraints the tangent space is equal to the nullspace of the active constraint gradients and thus the smallest eigenvalue of $L_2(z^*)$ corresponds to the minimal value of its quadratic form on the space of all (y, u) satisfying the linearized equation as well as having vanishing u components corresponding to indices i for which the solution is at the bound which coincidentally also is zero. These components do include the ones corresponding to the interval $[0, 1/4]$. Next, we will detail how condition (4.2) will be checked.

As was already done in [7; 8] the control problems are written in the form of AMPL [6] scripts. This way, a number of nonlinear optimization codes can be utilized for their solution. It had been an observation in our previous work that from the then available codes only LOQO [14] was able to solve all the problems effectively and for sufficiently fine discretizations. This has changed. As recent comparisons [9] have shown, the trust region interior point method KNITRO [2] which became available only recently, may outperform LOQO on such problems. It was used for the computations reported below. The following procedure is independent of the solver used.

After computing a solution an AMPL *stub* (or $*.nl$) file is written as well as a file with the computed Lagrange multipliers. This allows to

$1/dx$	$1/dt$	$\|u-\bar{u}\|_\infty$	$\|u-\bar{u}\|_2$	$\|y-\bar{y}\|_\infty$	$\|y-\bar{y}\|_2$
127	41	3.739e-2	4.590e-5	6.096e-3	2.730e-7
192	61	1.331e-2	1.152e-5	1.770e-3	1.850e-7

Table 1. Solution errors for problem 4.2

check the SSC (4.2) with the help of a Fortran, alternatively, a C or Matlab, program. This program reads the files and verifies first the necessary first-order optimality conditions, the column regularity of $N(z^*)$ and the strict complementarity. For this, it utilizes routines provided by AMPL which permit evaluation of the objective and constraint gradients. Next, the the QR decomposition of $N(z^*)$ is computed by one of the methods exploiting sparsity. We have utilized the algorithm described in [12]. AMPL also provides a routine to multiply the Hessian of the Lagrangian times a vector. This is called with the columns of Q_2 and thus $L_2(z^*)$ can be formed. Its eigenvalues are computed with LAPACK routine DSYEV and the smallest eigenvalue $\gamma = \gamma_h$ is determined. The use of this eigenvalue routine is possible since the order of the matrices corresponding to the "free" control variables is moderate. In case of distributed control problems when this number may be on the order of the state variables, a sparse solver, preferably just for finding the minimal eigenvalue, will have to be used.

With the procedure described above the SSC for problem (P_h) can be checked for constant or variable α_0. In the nonconstant case and for α_0 below the bound given above an additional eigenvalue problem is solved. Let Q in the previous section be split into $Q = (Q_1, Q_2)$ where now Q_1 corresponds to the equality constraints only and thus has M_h columns. Then, in analogy to (4.2) we define $L_2(z^*)$ and call its smallest (leftmost on the real line) eigenvalue δ_h. We will have to obtain a negative δ_h for sufficiently negative α_0. As described above, this Q_2 projects onto the nullspace of the equality constraints only and thus $L_2(z^*)$ is the projection of the Hessian onto the larger subspace of pairs (y, u) satisfying the linearized equation only and for which u may be nonzero everywhere on $[0, T]$.

Problem (4.2) was solved as described above for two discretizations which were chosen to be about equidistant in both coordinates. In Table 1 are the errors of both state and control listed in two different norms. In Table 2 we list the eigenvalues for both discretizations and various values of α_0. As can be seen the sign change for δ_h occurs in both cases between -10.5 and -11 while the estimate (3.5) above yielded a bound of -10.845. The computed state is shown in the same figure.