Optimal Control of Semilinear Parabolic Equations with State-Constraints of Bottleneck Type

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Abstract

We consider optimal distributed and boundary control problems for semilinear parabolic equations, where pointwise constraints on the control and pointwise mixed control-state constraints of bottleneck type are given. Our main result states the existence of regular Lagrange multipliers for the state-constraints. Under natural assumptions, we are able to show the existence of bounded and measurable Lagrange multipliers. The method is based on results from the theory of continuous linear programming problems.

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1 Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary Γ and let $T > 0$ be given fixed. We consider the following semilinear partial differential equation:

$$\left\{ \begin{aligned} y_t + A\,y + d\,(x,t,y) &= u && \text{in } Q = \Omega \times]0,T[\ , \\ \partial_{\nu_A} y + b\,(x,t,y) &= v && \text{on } \Sigma\ , \\ y(x,0) &= y_o(x) && \text{in } \Omega\ , \end{aligned} \right. \tag{1.1}$$

where A is a uniformly elliptic differential operator defined below, Σ denotes the lateral boundary of Q and $\partial_{\nu_A} y$ denotes the outward conormal derivative of y associated with A. Let us specify the assumptions on the data:

(A1) The boundary Γ is of class C^2. The operator A is defined by

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} y) \quad \text{with}$$
$$a_{ij} = a_{ji} \in \mathcal{C}^{1,\beta}(\overline{\Omega}),\ for\ i,j = 1 \cdots n,$$
$$\forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n, \quad m\sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le M\sum_{i=1}^{n}\xi_i^2 \text{ with } m > 0 \ .$$

(A2)

(i) The function $d = d(x,t,y) : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. It is supposed to satisfy the following assumption of smoothness :

- For almost every $(x,t) \in Q$, d is continuously differentiable with respect to y.
- There exists a continuous monotone nondecreasing function $\eta : [0,\infty) \to \mathbb{R}^+$ with $\eta(0) = 0$ such that, for all $M > 0$, there is a constant $c_M > 0$ satisfying

  $$\|d_y(\cdot, y_1) - d_y(\cdot, y_2)\|_{L^\infty(Q)} \le c_M\, \eta(|y_1 - y_2|)$$

  for all real y_i with $|y_i| \le M$, $i = 1, 2$. Moreover,

  $$\|d_y(\cdot, 0, 0)\|_{L^\infty(Q)} + \|d(\cdot, \cdot, 0, 0)\|_{L^q(Q)} \le c_B$$

  holds with some constant c_B and some $q > N/2 + 1$.

(ii) Analogous conditions are imposed on $b = b(x,t,y)$ on $\Sigma \times \mathbb{R}$ with the same constants, where $L^s(\Sigma)$ is substituted for $L^q(Q)$ with some $s > N + 1$.

Let us fix $q > N/2 + 1$ and $s > N + 1$ throughout this paper.

(iii) y_o belongs to $C(\overline{\Omega})$.

We shall denote the real function d and the associated Nemytskii operator $d : y(\cdot) \mapsto d(\cdot, y(\cdot))$ by the same symbol. In other words, we write $d(x,t,y(x,t)) := d(y)(x,t),\ (x,t) \in Q$, since this will not cause confusion.

We consider the following optimal control problem

$$(\mathcal{P}) \qquad \begin{cases} \min\ J(y,u,v) \\ (y,u,v) \text{ satisfies } (1.1) \\ (y,u,v) \in C\ , \end{cases}$$

where

$$J(y,u,v) = \frac{\alpha_Q}{2}\|y - z_Q\|_Q^2 + \frac{\alpha_\Omega}{2}\|y(T) - z_\Omega\|_\Omega^2 + \frac{\alpha_\Sigma}{2}\|y - z_\Sigma\|_\Sigma^2 + \frac{\alpha_u}{2}\|u\|_Q^2 + \frac{\alpha_v}{2}\|v\|_\Sigma^2. \qquad (1.2)$$

$\|\cdot\|_S$ denotes the natural norm of $L^2(S)$, and α_Q, α_Ω, α_Σ, α_u, α_v are nonnegative real constants such that $\alpha_Q + \alpha_\Omega + \alpha_\Sigma + \alpha_u + \alpha_v \neq 0$. Moreover, $z_Q \in L^q(Q)$, $z_\Sigma \in L^s(\Sigma)$ and $z_\Omega \in L^\infty(\Omega)$ are given fixed.

C is a convex set of constraints of bottleneck type. We consider two kinds of such sets.

- C_1 is the set of all $(y,u,v) \in C(\overline{Q}) \times L^\infty(Q) \times L^\infty(\Sigma)$ satisfying the inequalities

  $$(\mathbf{I}) \qquad \begin{cases} 0 \le u \ , \ 0 \le v & (\mathbf{Ia}) \\ u \le c_Q + \beta_Q\, y & \\ v \le c_\Sigma + \beta_\Sigma\, y & (\mathbf{Ib}) \end{cases}$$

  a.e. on Q and Σ, respectively, where c_Q, $\beta_Q \in L^\infty(Q)$ and c_Σ, $\beta_\Sigma \in L^\infty(\Sigma)$ are given nonnegative functions. We assume the existence of $\delta > 0$ such that $c_\Sigma, c_Q \ge \delta > 0$ holds a.e. on Q and Σ, respectively.

- C_2 is the set of all $(y,u,v) \in C(\overline{Q}) \times L^\infty(Q) \times L^\infty(\Sigma)$ satisfying

  $$(\mathbf{II}) \qquad \begin{cases} 0 \le u \le u_b \ , \ 0 \le v \le v_b & (\mathbf{IIa}) \\ \text{and} & (\mathbf{Ib}) \end{cases}$$

  a.e. on Q and Σ, respectively, where $u_b \in L^\infty(Q)$ and $v_b \in L^\infty(\Sigma)$ are given.

Remark 1.1 *The simple choice for J was taken for convenience. It is quite standard in the context of least squares problems. Any other sufficiently smooth integral functional can be chosen as well. For instance, we refer to the objectives defined in Casas [3] or Raymond and Zidani [5].*

We shall denote by $(\mathcal{P}_i)$ the problem $(\mathcal{P})$ under the choice $C = C_i$, $i = 1, 2$.
Let us fix the state-space for y as follows:

$$\mathcal{Y} = \{\ y \in W(0,T) \ \mid \ y_t + A\,y \in L^q(Q),\ \partial_{\nu_A} y \in L^s(\Sigma),\ y(0) \in \mathcal{C}(\overline{\Omega})\ \}\ ,$$

where $\partial_{\nu_A} y$ is defined according to [5], and $W(0,T) = \{y \in L^2(0,T;H^1(\Omega)) \mid y_t \in L^2(0,T;H^1(\Omega)')\ \}$.
$\mathcal{Y}$ is endowed with the norm

$$\|y\|_{\mathcal{Y}} = \|y_t + A\,y\|_{L^q(Q)} + \|\partial_{\nu_A} y\|_{L^s(\Sigma)} + \|y(0)\|_{\mathcal{C}(\overline{\Omega})}.$$

Then $\mathcal{Y}$ is a Banach space continuously embedded into $\mathcal{C}(\overline{Q})$ (see [1] for example).

Our main task is to show the existence of regular Lagrange multipliers for the state-constraints (Ib). The space for these multipliers depends on how the inequalities are considered. We might define the inequalities in the space $L^\infty(Q) \times L^\infty(\Sigma)$, since u and v are bounded and measurable. In this case, the multipliers have to be defined in the associated dual space. We aim to avoid this space of nonregular multipliers for several well known reasons. Let us briefly sketch the main idea we have in mind. Introduce the Lagrange function $\mathcal{L} : Y \times L^\infty(Q) \times L^\infty(\Sigma) \times W(0,T) \times L^\infty(Q) \times L^\infty(\Sigma) \to \mathbb{R}$

$$\begin{aligned} \mathcal{L}(y,u,v,p,\mu_1,\mu_2) = \; & J(y,u,v) - \int_Q [y_t + A\,y + d(y) - u]\,p\,dx\,dt - \int_\Sigma [\partial_{\nu_A} y + b(y) - v]\,p\,d\sigma\,dt \\ & + \int_Q (u - c_Q - \beta_Q\,y)\,\mu_1\,dx\,dt + \int_\Sigma (v - c_\Sigma - \beta_\Sigma\,y)\,\mu_2\,d\sigma\,dt \end{aligned} \tag{1.3}$$

($d\sigma = d\sigma(x)$ denotes the surface measure on Γ). In this way, the mixed pointwise control-state-constraints are eliminated by multipliers μ_1, μ_2, while the control constraints remain unchanged. Let $(\bar{y},\bar{u},\bar{v})$ be a locally optimal triplet for $(\mathcal{P})$. Tacitly assuming that we shall be able to find regular multipliers μ_1, μ_2, the first order necessary optimality conditions for $(\bar{y},\bar{u},\bar{v})$ read

$$\begin{array}{lcll} \mathcal{L}_y(\bar{y},\bar{u},\bar{v},p,\mu_1,\mu_2)\,y & = & 0 & \forall y \in \{\, y \in \mathcal{Y} \mid y(0) = 0 \,\} \\ \mathcal{L}_u(\bar{y},\bar{u},\bar{v},p,\mu_1,\mu_2)(u-\bar{u}) & \geq & 0 & \forall u \in U_{ad} \\ \mathcal{L}_v(\bar{y},\bar{u},\bar{v},p,\mu_1,\mu_2)(v-\bar{v}) & \geq & 0 & \forall v \in V_{ad}. \end{array} \tag{1.4}$$

Moreover, the **complementary slackness conditions**

$$\mu_1\,(\bar{u} - c_Q - \beta_Q\,\bar{y}) = 0 \text{ a.e. on } Q \qquad \text{and} \qquad \mu_2\,(\bar{v} - c_\Sigma - \beta_\Sigma\,\bar{y}) = 0 \text{ a.e. on } \Sigma \tag{1.5}$$

must be satisfied. From these relations we obtain the first order optimality system consisting of the state-equation, all inequality constraints, the **adjoint equation**

$$\left\{ \begin{array}{rcll} -p_t + A\,p + d_y(\bar{y})p & = & \alpha_Q\,(\bar{y} - z_Q) - \beta_Q\mu_1 & \text{in } Q \\ \partial_{\nu_A} p + d_y(\bar{y})p & = & \alpha_\Sigma\,(\bar{y} - z_\Sigma) - \beta_\Sigma\mu_2 & \text{on } \Sigma\,, \\ p(T) & = & \alpha_\Omega\,(\bar{y}(T) - z_\Omega) & \text{in } \Omega\,, \end{array} \right. \tag{1.6}$$

the **variational inequalities**

$$\int_Q (\alpha_u\,\bar{u} + p + \mu_1)(u - \bar{u})\,dx\,dt \;\geq\; 0 \quad \forall u \in U_{ad} \tag{1.7}$$

$$\int_\Sigma (\alpha_v\,\bar{v} + p + \mu_2)(v - \bar{v})\,d\sigma\,dt \;\geq\; 0 \quad \forall v \in V_{ad}, \tag{1.8}$$

the complementary slackness conditions, and the **nonnegativity conditions**

$$\mu_1(x,t) \geq 0,\ \mu_2(x,t) \geq 0$$

to be fulfilled for almost every (x,t). Let $(p,\mu_1,\mu_2) \in W(0,T) \times L^\infty(Q) \times L^\infty(\Sigma)$ be any triplet satisfying, together with $(\bar{y},\bar{u},\bar{v})$, the optimality conditions formulated above . Then the function p is said to be an **adjoint state**, and the μ_i are called **Lagrange multipliers** associated with $(\bar{y},\bar{u},\bar{v})$.

We shall derive conditions ensuring the existence of Lagrange multipliers μ_1, μ_2 in $L^\infty(Q)\times L^\infty(\Sigma)$. The idea is as follows: A constraint qualification is formulated for the state-constraints in L^∞, where the natural cone of nonnegative functions has a nonempty interior. Therefore, this condition has a good chance to be satisfied. Next a known result by Zowe and Kurcyusz [7] yields that $(\bar{y},\bar{u},\bar{v})$ is a solution of the associated **linearized problem**

$$(\mathcal{P}^\ell) \qquad \min \ J'(\bar{y},\bar{u},\bar{v})(y,u,v)$$

subject to the linearized equation

$$\left\{ \begin{array}{rcl} y_t + A\,y + d(\bar{y}) + d_y(\bar{y})\,(y-\bar{y}) & = & u \\ \partial_{\nu_A} y + b(\bar{y}) + b_y(\bar{y})\,(y-\bar{y}) & = & v \\ y(x,0) & = & y_o(x) \end{array} \right. \tag{1.9}$$

and to the (linear) constraints $(y,u,v) \in C$.
$(\mathcal{P}^\ell)$ is a linear continuous programming problem of bottleneck type, for which the duality theory of linear programming holds. In [2], we have proven existence of bounded and measurable solutions $\mu_i \geq 0$ for the associated dual problem. These solutions are Lagrange multipliers for $(\mathcal{P}^\ell)$. Then it is easy to verify that they are multipliers for $(\mathcal{P})$, too.

2 Existence of optimal solutions to $(\mathcal{P}_i)$

2.1 Existence of a solution to the state-equation

The following theorem ensures the existence and uniqueness of the state y associated with (u,v).

Theorem 2.1 *For each pair $(u,v) \in L^q(Q)\times L^s(\Sigma)$, equation (1.1) has a unique solution $y = y(u,v) \in \mathcal{Y}$. Moreover*

$$\|y(u,v)\|_{W(0,T)} + \|y(u,v)\|_{\mathcal{C}(\overline{Q})} \leq C(1 + \|u\|_{L^q(Q)} + \|v\|_{L^s(\Sigma)} + \|y_o\|_{\mathcal{C}(\overline{\Omega})}) \ . \tag{2.1}$$

Proof - See [4]-Theorem 4.13 or [5]-Theorem 3.1. ■

Thanks to this result, the operator $\mathcal{T}$,

$$\begin{array}{llll} \mathcal{T} \ : & \mathcal{Y} \times L^\infty(Q) \times L^\infty(\Sigma) & \to & L^\infty(Q) \times L^\infty(\Sigma) \times \mathcal{C}(\overline{\Omega}) \\ & (y, u, v) & \mapsto & \begin{bmatrix} y_t + A\, y + d(y) - u \\ \partial_{\nu_A} y + b(y) - v \\ y(0) \end{bmatrix} \end{array} \tag{2.2}$$

is of class $\mathcal{C}^1$.

2.2 Boundedness properties for feasible elements

In the sequel, the set of $(y, u, v) \in \mathcal{Y} \times L^\infty(Q) \times L^\infty(\Sigma)$ satisfying all constraints of $(\mathcal{P})$ including the state equation, is said to be the *feasible set*. We shall prove in this section that the feasible set is uniformly bounded for both the problems $(\mathcal{P}_i)$, $i = 1, 2$. Let us first state some auxiliary results.

Proposition 2.1 (Comparison principle for the linear equation) *Suppose that functions a, $u \in L^q(Q)$, $\alpha, v \in L^s(\Sigma)$, and $y_o \in \mathcal{C}(\overline{\Omega})$ are given, such that*

$$a(x,t) \geq c_o \text{ a.e. in } Q, \ \ \alpha(x,t) \geq c_o \text{ a.e. on } \Sigma$$

holds with some constant $c_o \in \mathbb{R}$. Let $y \in \mathcal{C}(\overline{Q})$ be the unique solution of the linear parabolic equation

$$\left\{ \begin{aligned} y_t + A\, y + a\, y \ &= u \\ \partial_{\nu_A} y + \alpha\, y \ &= v \\ y(x, 0) \ &= y_o(x). \end{aligned} \right. \tag{2.3}$$

If u, v and y_o are nonpositive a.e. on their domains of definition, then $y \leq 0$ holds everywhere in $\overline{Q}$.

This result can be found in [5], Proposition 3.2. It holds even for $y_o \in L^2(\Omega)$ in the sense that y and its trace are almost everywhere nonpositive.

Now we are able to derive a comparison principle for the **nonlinear** state-equation. If u is a bounded and measurable function being nonnegative almost everywhere on its domain of definition, then we shall write for convenience $u \geq 0$.

Theorem 2.2 (Comparison principle for the state-equation)
*Assume **(A1)-(A2)** and suppose $u_i \in L^q(Q)$ and $v_i \in L^s(\Sigma)$, $i = 1, 2$. Let y_i be the corresponding states, that is $\mathcal{T}(y_1, u_1, v_1) = 0$ and $\mathcal{T}(y_2, u_2, v_2) = 0$. If $u_1 \geq u_2$ and $v_1 \geq v_2$, then $y_1 \geq y_2$.*

Proof - We know that the states y_i belong to $\mathcal{C}(\overline{Q})$. Since d is of class $\mathcal{C}^1$ with respect to y, we may use the mean value theorem so that

$$d(x,t,y_2(x,t)) - d(x,t,y_1(x,t)) = d_y(x,t,y_\theta(x,t))(y_2(x,t) - y_1(x,t)) \; ,$$

where $y_\theta(x,t) = y_1(x,t) + \theta(x,t)\,(y_2(x,t) - y_1(x,t))$, and the function $\theta \in (0,1)$ can be taken measurable. As y_i is continuous and d_y is a Carathéodory function, it is easy to conclude that $a(x,t) = d_y(x,t,y_\theta(x,t))$ is bounded and measurable in Q. In particular, a is bounded from below. The same argument applies to b, setting $\alpha(x,t) = b_y(x,t,y_\theta(x,t))$ on Σ. In view of this, $y_2 - y_1$ is the solution of the linear equation

$$\left\{ \begin{aligned} (y_2 - y_1)_t + A\,(y_2 - y_1) + a\,(y_2 - y_1) &= u_2 - u_1 \\ \partial_{\nu_A}\,(y_2 - y_1) + \alpha\,(y_2 - y_1) &= v_2 - v_1 \\ (y_2 - y_1)(x,0) &= 0. \end{aligned} \right.$$

We complete the proof by Proposition 2.1 ■

Corollary 2.1 *Let all assumptions of the previous Theorem be satisfied. Then its conclusion remains true, if* (d,b) *is replaced by* $(d - \beta_Q y, b - \beta_\Sigma y)$ *in the nonlinear state-equation.*

Proof - This is obvious, since $(d - \beta_Q y)_y = d_y - \beta_Q$ belongs to $L^\infty(Q)$ and therefore is bounded from below. The same argument holds for b. ■

To prove the next theorem, we consider the state function $y = \hat{y}$, which corresponds to controls u and v acting at their upper limits, that is, $u = c_Q + \beta_Q\, y$ and $v = c_\Sigma + \beta_\Sigma\, y$. Inserting these controls in the equation of state, we arrive at the system

$$\left\{ \begin{aligned} \hat{y}_t + A\hat{y} + d\,(\hat{y}) - \beta_Q\,\hat{y} &= c_Q \\ \partial_{\nu_A}\hat{y} + b\,(\hat{y}) - \beta_\Sigma\,\hat{y} &= c_\Sigma \\ \hat{y}(x,0) &= y_o(x). \end{aligned} \right. \tag{2.4}$$

According to Theorem 2.1, this system admits a unique solution $\hat{y} \in \mathcal{C}(\overline{Q})$.

Theorem 2.3 *The feasible sets of* $(\mathcal{P}_i)$, $i = 1,2$, *are bounded in* $\mathcal{C}(\bar{Q}) \times L^\infty(Q) \times L^\infty(\Sigma)$.

Proof - (i) For C_2, the result is obvious: Here, the admissible controls $u(x,t)$ and $v(x,t)$ belong to the uniformly bounded sets $[0,u_b]$ and $[0,v_b]$, respectively. Therefore, relation (2.1) yields that all associated states $y = y(u,v)$ are uniformly bounded as well.

(i) The situation is more interesting for C_1 : Let $u \geq 0$ and $v \geq 0$ satisfy, together with y, the state constraints. Then $u = c_Q - \delta_Q + \beta_Q\, y$, $v = c_\Sigma - \delta_\Sigma + \beta_\Sigma\, y$, where $\delta_Q \geq 0$, $\delta_\Sigma \geq 0$. Therefore,

$$\left\{ \begin{array}{rll} y_t + A\, y + d\,(y) - \beta_Q\, y & = c_Q - \delta_Q & \leq c_Q \\ \partial_{\nu_A} y + b\,(y) - \beta_\Sigma\, y & = c_\Sigma - \delta_\Sigma & \leq c_\Sigma \\ y(0) & = y_o. & \end{array} \right. \tag{2.5}$$

Corollary 2.1 yields $y \leq \hat{y}$, hence $u \leq c_Q + \beta_Q\, y \leq c_Q + \beta_Q\, \hat{y}$ (notice that β_Q is nonnegative). This shows that all admissible controls u are uniformly bounded. The same holds for v. Thanks to the estimate (2.1), all y are uniformly bounded, too. ∎

Remark 2.1 *A study of the proof reveals that the property $u \in L^\infty(Q)$, $v \in L^\infty(\Sigma)$ was not needed to apply the comparison principle. To apply this principcle, it is sufficient to have $u \in L^q(Q)$ and $v \in L^s(\Sigma)$. In this case, the uniform boundedness $u \leq c_Q + \beta_Q\, \hat{y}$ still holds. Therefore, we are justified to regard the sets C_i in $L^q(Q) \times L^s(\Sigma)$. This will not change them in comparison with their former definition in $L^\infty(Q) \times L^\infty(\Sigma)$, and the statement of the Theorem remains valid.*

We are mainly interested in necessary optimality conditions for optimal controls. This refers to **locally** optimal controls as well. Therefore, we are justified to assume that a pair of locally optimal controls is given. Nevertheless, we briefly discuss the existence of (globally) optimal controls, since this is an important information on the well-posedness of the problem.

Theorem 2.4 *Problem $(\mathcal{P}_i)$ (i=1,2) has at least one optimal solution $(\bar{y}, \bar{u}, \bar{v}) \in \mathcal{Y} \times L^\infty(Q) \times L^\infty(\Sigma)$.*

Proof - We shall only briefly sketch the idea of the proof. Thanks to Remark 2.1, we are justified to view C_i as a bounded set of $\mathcal{Y} \times L^q(Q) \times L^s(\Sigma)$. If $\{(y_n, u_n, v_n)\}$ is a minimizing sequence, we can therefore assume that $\{u_n - d(y_n)\}$ and $\{v_n - b(y_n)\}$ are weakly converging in L^q and L^s, respectively. By compactness of the linear solution mapping, $\{y_n\}$ tends strongly to $\bar{y}$ in $\mathcal{C}(\bar{Q})$, hence also $d(y_n)$ to $d(\bar{y})$ and $b(y_n)$ to $b(\bar{y})$. Moreover, $u_n \to \bar{u}$ and $v_n \to \bar{v}$ (weakly). We obtain by the continuity of the linear solution mapping that $(\bar{y}, \bar{u}, \bar{v})$ solves the nonlinear state-equation. Moreover, the lower semicontinuity of J with respect to (u, v) permits to show in a standard way the optimality of $(\bar{y}, \bar{u}, \bar{v})$. ∎

3 Linearization of $(\mathcal{P})$

Let us write for short $w := (y, v, u)$. With this notation, problem $(\mathcal{P})$ admits the abstract form

$$\min \; \{ \; J(w) \;\; | \;\; \mathcal{T}(w) = 0, \;\; w \in C \; \},$$

where $\mathcal{T}$ is differentiable and the convex and closed set C stands for C_1 or C_2. To formulate a corresponding regularity condition, we define the **linearized cone** of C at $\bar{w}$,

$$C(\bar{w}) = \{ \ \lambda(w - \bar{w}) \mid \lambda \geq 0, \ w = \ (y, u, v) \in C \ \}.$$

The following general result is known for the linearization of $(\mathcal{P})$ at the optimal point [7, 6] :

Theorem 3.1 *Assume that the regularity assumption*

$$\mathcal{T}'(\bar{w}) \cdot C(\bar{w}) = L^\infty(Q) \times L^\infty(Q) \times L^\infty(\Sigma) \ , \tag{3.1}$$

is satisfied at $\bar{w}$. Then $\bar{w}$ is solution of the linearized problem

$$\begin{cases} \min \quad J'(\bar{w}) \cdot w \\ \mathcal{T}'(\bar{w}) \cdot (w - \bar{w}) = 0 \ , \\ w \in C \ . \end{cases}$$

Remark 3.1 *The original result of [7] states that*

$$J'(\bar{w}) \cdot h \geq 0 \quad \forall h \in C(\bar{w}) : \mathcal{T}'(\bar{w})\, h = 0.$$

We have $h = \lambda\,(w - \bar{w})$. Inserting this for $\lambda = 1$, $J'(\bar{w}) \cdot (w - \bar{w}) \geq 0$ follows, so that $\bar{w}$ attains the minimal value.

In our case, the sets $C = C_i$, $i = 1, 2$, have obviously a nonempty interior in the space $\mathcal{C}(\bar{Q}) \times L^\infty(Q) \times L^\infty(\Sigma)$. Therefore it makes sense to assume the following stronger regularity condition, which is known to be sufficient for (3.1) to hold ([6]):

$$\begin{cases} \quad \exists(\tilde{y}, \tilde{u}, \tilde{v}) \in \operatorname{int} C \text{ such that} \\ \mathcal{T}'(\bar{y}, \bar{u}, \bar{v})\,(\tilde{y} - \bar{y}, \tilde{u} - \bar{u}, \tilde{v} - \bar{v}) = 0. \end{cases} \tag{3.2}$$

Here, int C denotes the L^∞-interior and $\tilde{y}$ belongs to $\mathcal{Y}$. In other words, we have to find a triplet $(\tilde{y}, \tilde{u}, \tilde{v})$ of $\mathcal{Y} \times L^\infty(Q) \times L^\infty(\Sigma)$ such that

$$\begin{cases} (\tilde{y} - \bar{y})_t + A\,(\tilde{y} - \bar{y}) + d_y(\bar{y})(\tilde{y} - \bar{y}) &= \tilde{u} - \bar{u} \\ \partial_{\nu_A}(\tilde{y} - \bar{y}) + b_y(\bar{y})(\tilde{y} - \bar{y}) &= \tilde{v} - \bar{v} \\ \tilde{y}(0) - \bar{y}(0) &= 0 \end{cases} \tag{3.3}$$

and

$$\begin{aligned} \varepsilon &\leq \tilde{u}(x,t) \leq c_Q(x,t) - \varepsilon + \beta_Q(x,t)\,\tilde{y}(x,t) \\ \varepsilon &\leq \tilde{v}(x,t) \leq c_\Sigma(x,t) - \varepsilon + \beta_\Sigma(x,t)\,\tilde{y}(x,t) \end{aligned} \tag{3.4}$$

holds a.e. on Q and Σ, respectively, where ε is positive. We shall verify this condition under the following very natural assumption:

(A3) There is a positive constant δ such that

$$\begin{aligned} c_Q(x,t) + \beta_Q(x,t)\,\bar{y}(x,t) &\geq \delta \\ c_\Sigma(x,t) + \beta_\Sigma(x,t)\,\bar{y}(x,t) &\geq \delta\,. \end{aligned}$$

In other words, the sets $\{\, u \in L^\infty(Q) |\, 0 \leq u \leq c_Q + \beta_Q\,\bar{y}\,\}$ and $\{\, v \in L^\infty(\Sigma) |\, 0 \leq v \leq c_\Sigma + \beta_\Sigma\,\bar{y}\,\}$ (defined upon $\bar{y}$) are assumed to have a nonempty interior. In particular, this assumption is satisfied, if c_Q and c_Σ are bounded from below by a positive constant (a condition being assumed in this paper) and $\bar{y}$ is known to be nonnegative.

To verify that **(A3)** implies (3.2), we introduce the "maximal linearized solution" $\hat{y}$ by inserting $\hat{u} := c_Q + \beta_Q\hat{y}$, $\hat{v} := c_\Sigma + \beta_\Sigma\hat{y}$ in the linearized equation (1.9). In this way, $\hat{y}$ is defined by

$$\left\{ \begin{aligned} \hat{y}_t + A\,\hat{y} + d(\bar{y}) + d_y(\bar{y})(\hat{y} - \bar{y}) &= c_Q + \beta_Q\hat{y} \quad (= \hat{u}) \\ \partial_{\nu_A}\hat{y} + b(\bar{y}) + b_y(\bar{y})(\hat{y} - \bar{y}) &= c_\Sigma + \beta_\Sigma\hat{y} \quad (= \hat{v}) \\ \hat{y}(0) &= y_o \end{aligned} \right. . \tag{3.5}$$

Lemma 3.1 *The relation $\hat{y}(x,t) \geq \bar{y}(x,t)$ holds for all $(x,t) \in \overline{Q}$.*

Proof - The function $\bar{y}(x,t)$ satisfies

$$\left\{ \begin{aligned} \bar{y}_t + A\,\bar{y} + d(\bar{y}) &= \bar{u} = c_Q + \beta_Q\bar{y} - \delta_Q \\ \partial_{\nu_A}\bar{y} + b(\bar{y}) &= \bar{v} = c_\Sigma + \beta_\Sigma\bar{y} - \delta_\Sigma \\ \bar{y}(0) &= y_o, \end{aligned} \right. \tag{3.6}$$

where δ_Q and δ_Q are nonnegative. Subtracting (3.6) from (3.5) yields

$$\left\{ \begin{aligned} (\hat{y} - \bar{y})_t + A\,(\hat{y} - \bar{y}) + d_y(\bar{y})(\hat{y} - \bar{y}) - \beta_Q(\hat{y} - \bar{y}) &= \delta_Q \\ \partial_{\nu_A}(\hat{y} - \bar{y}) + b_y(\bar{y})(\hat{y} - \bar{y}) - \beta_\Sigma(\hat{y} - \bar{y}) &= \delta_\Sigma \\ \hat{y}(0) - \bar{y}(0) &= 0. \end{aligned} \right.$$

The comparison principle yields $\hat{y} - \bar{y} \geq 0$. ∎

Theorem 3.2 *Assumption **(A3)** implies the regularity condition (3.1).*

Proof - To construct $(\tilde{y}, \tilde{u}, \tilde{v})$, we take a constant $\lambda < 1$ close to 1 and define $\tilde{y}$ by

$$\left\{ \begin{aligned} \tilde{y}_t + A\,\tilde{y} + d(\bar{y}) + d_y(\bar{y})(\tilde{y} - \bar{y}) &= \lambda c_Q + \beta_Q\tilde{y} \quad (=: \tilde{u}) \\ \partial_{\nu_A}\tilde{y} + b(\bar{y}) + b_y(\bar{y})(\tilde{y} - \bar{y}) &= \lambda c_\Sigma + \beta_\Sigma\tilde{y} \quad (=: \tilde{v}) \\ \tilde{y}(0) &= y_o. \end{aligned} \right. \tag{3.7}$$

Then

$$\begin{cases} (\tilde y - \bar y)_t + A\,(\tilde y - \bar y) + d_y(\bar y)(\tilde y - \bar y) &= \tilde u - \bar u \\ \partial_{\nu_A}(\tilde y - \bar y) + b_y(\bar y)(\tilde y - \bar y) &= \tilde v - \bar v \\ \tilde y(0) - \bar y(0) &= 0, \end{cases} \tag{3.8}$$

hence the equation $\mathcal{T}'(\bar w)(\tilde w - \bar w) = 0$ holds. Moreover, by continuity, $\tilde y$ tends uniformly in $\bar Q$ towards $\hat y$, as λ tends to 1. Therefore,

$$\tilde u = \lambda c_Q + \beta_Q \tilde y \geq \frac{\delta}{2} \text{ and } \tilde v = \lambda c_\Sigma + \beta_\Sigma \tilde y \geq \frac{\delta}{2}$$

follows from assumption **(A3)** and Lemma 3.1 for λ sufficiently close to 1. Moreover,

$$\tilde u = \lambda c_Q + \beta_Q \tilde y < c_Q + \beta_Q \tilde y \text{ and } \tilde v = \lambda c_\Sigma + \beta_\Sigma \tilde y < c_\Sigma + \beta_\Sigma \tilde y\ .$$

Altogether, the last relations show that $(\tilde y, \tilde u, \tilde v)$ belongs to the L^∞-interior of C. ∎

From now on, we assume that (A3) is fulfilled so that Theorem 3.1 is applicable.

The linearized problem has been defined in an abstract setting. Inserting the concrete expressions for J' and $\mathcal{T}'$, it admits the following explicit form:

$$(\mathcal{P}_i^\ell) \qquad \begin{array}{ll} \min & \alpha_Q \displaystyle\int_Q (\bar y - z_d)\cdot y\ dx\ dt + \alpha_\Omega \int_\Omega (\bar y(T) - z_\Omega)\cdot y(T)\ dx + \\ & \alpha_\Sigma \displaystyle\int_\Sigma (\bar y - z_\Sigma)\cdot y\ d\sigma\ dt + \alpha_u \int_Q \bar u \cdot u\ dx\ dt + \alpha_v \int_\Sigma \bar v \cdot v\ d\sigma\ dt \\ & \text{subject to} \end{array}$$

$$\begin{cases} y_t + A\,y + d_y(\bar y)\,y &= u + d_y(\bar y)\bar y - d(\bar y) \\ \partial_{\nu_A} y + b_y(\bar y)\,y &= v + b_y(\bar y)\bar y - b(\bar y) \\ y(0) &= y_o, \end{cases} \tag{3.9}$$

$$(y, u, v) \in C_i\ .$$

For convenience, we set

$$\varpi_Q := d(\bar y) - d_y(\bar y)\cdot \bar y \in L^\infty(Q),\ \varpi_\Sigma := b(\bar y) - b_y(\bar y)\cdot \bar y \in L^\infty(\Sigma)\ ,$$

and

$$a_Q := -[\bar y - z_d]\ \in L^\infty(Q),\ a_\Omega := -\alpha_\Omega\,[\bar y(T) - z_\Omega]\ \in L^\infty(\Omega),\ a_\Sigma := -\alpha_\Sigma\,[\bar y - z_\Sigma]\ \in L^\infty(\Sigma),$$

$$a_u := -\alpha_u \bar u\ \in L^\infty(Q),\ a_v := -\alpha_v \bar v\ \in L^\infty(\Sigma)\ ,$$

so that $(\mathcal{P}_i^\ell)$ can be written as

$$(\mathcal{P}_i^\ell)\quad \left\{\begin{array}{l} \max \quad \displaystyle\int_Q a_Q\, y\; dx\; dt + \int_\Omega a_\Omega\, y(T)\; dx + \int_\Sigma a_\Sigma\, y\; d\sigma\; dt + \int_Q a_u\, u\; dx\; dt + \int_\Sigma a_v\, v\; d\sigma\; dt \\ \quad\quad \text{subject to} \\ \quad\quad \left\{\begin{array}{rl} y_t + A\,y + d_y(\bar y)\,y & = u - \overline{\omega}_Q \\ \partial_{\nu_A} y + b_y(\bar y)\,y & = v - \overline{\omega}_\Sigma \\ y(0) & = y_o \end{array}\right. \\ \quad\quad (u,v) \text{ satisfies } \mathbf{Ia} \text{ or } \mathbf{IIa} \text{ and} \\ \quad\quad u \le c_Q + \beta_Q\, y,\;\; v \le c_\Sigma + \beta_\Sigma\, y\ . \end{array}\right.$$

This kind of linear control problems has been studied in our paper [2].

4 Existence of regular Lagrange multipliers

Let us assume from now on that Γ and the coefficients of A are sufficiently smooth to ensure the existence of a Green's function $G = G(x,\xi,t)$ associated with the linearized partial differential equation. This assumption can certainly be avoided. However, we want to directly apply our results of [2], where all main theorems were proved on using Green's functions. A study of our technique in [2] reveals that the theory can be developed also for weak solutions satisfying comparison principles. We shall not discuss this generalization here. The solution y of (3.9) has the integral representation

$$\begin{aligned} y(x,t) = & \int_\Omega G(x,\xi,t)\, y_o(\xi)\, d\xi + \int_0^t \int_\Omega G(x,\xi,t-s)\,(u(\xi,s) - \bar w_Q(\xi,s))\, d\xi\, ds \\ & + \int_0^t \int_\Gamma G(x,\xi,t-s)\,(v(\xi,s) - \bar w_\Sigma(\xi,s))\, d\sigma(\xi)\, ds \\ = & \;\; y_c(x,t) + \int_0^t \int_\Omega G(x,\xi,t-s)\, u(\xi,s)\, d\xi\, ds + \int_0^t \int_\Gamma G(x,\xi,t-s)\, v(\xi,s)\, d\sigma(\xi)\, ds, \end{aligned} \tag{4.1}$$

where y_c denotes the constant part of y corresponding to $(y_o, -\bar w_Q, -\bar w_\Sigma)$. Inserting the expression (4.1) in $(\mathcal{P}_i^\ell)$, and using the Fubini theorem in the objective, we arrive at the following **linear continuous programming problem** of bottleneck type:

$$\max \int_Q a_1(x,t) u(x,t)\, dx\, dt + \int_\Sigma a_2(x,t) v(x,t)\, d\sigma\, dt$$

subject to

$$
\begin{aligned}
u(x,t) \leq\ & \bar{c}_Q(x,t) && + \int_0^t \int_\Omega \beta_Q(x,t) G(x,\xi,t-s)\, u(\xi,s)\, d\xi\, ds \\
& && + \int_0^t \int_\Gamma \beta_Q(x,t) G(x,\xi,t-s)\, v(\xi,s)\, d\sigma(\xi)\, ds \\
v(x,t) \leq\ & \bar{c}_\Sigma(x,t) && + \int_0^t \int_\Omega \beta_\Sigma(x,t) G(x,\xi,t-s)\, u(\xi,s)\, d\xi\, ds \\
& && + \int_0^t \int_\Gamma \beta_\Sigma(x,t) G(x,\xi,t-s)\, v(\xi,s)\, d\sigma(\xi)\, ds, \\
u(x,t) \geq\ & 0, && (u(x,t) \leq u_b) \\
v(x,t) \geq\ & 0, && (v(x,t) \leq v_b),
\end{aligned}
\tag{4.2}
$$

where the upper bounds for the controls are only required in $(\mathcal{P}_1^\ell)$. The function $a_1 \in L^\infty(Q)$ is defined by

$$
\begin{aligned}
a_1(x,t) = & \int_\Omega G(x,\xi,T-t)\, a_\Omega(\xi) d\xi + \int_t^T \int_\Omega G(x,\xi,s-t)\, a_Q(\xi,s)\, d\xi\, ds \\
& + \int_t^T \int_\Gamma G(x,\xi,s-t)\, a_\Sigma(\xi,s)\, d\sigma(\xi)\, ds + a_u(x,t)
\end{aligned}
$$

while $a_2 \in L^\infty(\Sigma)$ has the same form with $a_v(x,t)$ substituted for the last item. Moreover, $\bar{c}_Q := c_Q + \beta_Q y_c$, $\bar{c}_\Sigma := c_\Sigma + \beta_\Sigma y_c$ were introduced. Let us define another auxiliary function $\psi \in \mathcal{Y}$ by

$$
\left\{
\begin{aligned}
-\psi_t + A\,\psi + d_y(\bar{y})\psi &= a_Q \\
\partial_{\nu_A}\psi + b_y(\bar{y})\psi &= a_\Sigma \\
\psi(T) = &= a_\Omega.
\end{aligned}
\right.
\tag{4.3}
$$

Then we have $a_1 = \psi + a_u$ and $a_2 = \psi|_\Sigma + a_v$, and the relation

$$
\begin{aligned}
& \int_Q a_1(x,t)\, u(x,t)\, dx\, dt + \int_Q a_2(x,t)\, v(x,t)\, d\sigma\, dt = \\
& \int_\Omega a_\Omega\, y(T)\, dx + \int_Q a_Q\, y\, dx\, dt + \int_\Sigma a_\Sigma\, y\, d\sigma\, dt + \int_\Omega a_u\, u\, dx\, dt + \int_\Sigma a_v\, v\, d\sigma\, dt
\end{aligned}
\tag{4.4}
$$

is obtained integrating by parts on using ψ. In this formula, y is defined by the linearized equation (3.9) with zero constant parts $\bar{w}_Q$, $\bar{w}_\Sigma$, y_o (in other words, y_c is substracted from the solution of the system linearized at $(\bar{y}, \bar{u}, \bar{v})$).

The associated **dual problem** has the form

$(\mathcal{D}^\ell)$ $$\min \int_Q (\bar{c}_Q\, \mu_1 + u_b\, \mu_3)\, dx\, dt + \int_\Sigma (\bar{c}_\Sigma\, \mu_2 + v_b\, \mu_4)\, d\sigma\ dt$$

subject to

$$
\begin{aligned}
\mu_1(x,t)+\mu_3(x,t) \geq\ & a_1(x,t) + \int_t^T \int_\Omega G(x,\xi,s-t)\,\beta_Q(\xi,s)\,\mu_1(\xi,s)\,d\xi\,ds \\
& + \int_t^T \int_\Gamma G(x,\xi,s-t)\,\beta_\Sigma(\xi,s)\,\mu_2(\xi,s)\,d\sigma(\xi)\,ds \\
\mu_2(x,t)+\mu_4(x,t) \geq\ & a_2(x,t) + \int_t^T \int_\Omega G(x,\xi,s-t)\,\beta_Q(\xi,s)\,\mu_1(\xi,s)\,d\xi\,ds \\
& + \int_t^T \int_\Gamma G(x,\xi,s-t)\,\beta_\Sigma(\xi,s)\,\mu_2(\xi,s)\,d\sigma(\xi)\,ds, \\
\mu_i(x,t) \geq\ & 0\ , \qquad i=1,...,4.
\end{aligned}
\tag{4.5}
$$

In the case of $(\mathcal{P}_1)$, the functions μ_3, μ_4 do not appear: The associated dual problem is obtained by setting $\mu_3=\mu_4=0$. If we denote by φ the solution of the system

$$
\left\{
\begin{aligned}
-\varphi_t + A\varphi + d_y(\bar{y})\,\varphi &= \beta_Q\,\mu_1 \\
\partial_{\nu_A}\varphi + b_y(\bar{y})\,\varphi &= \beta_\Sigma\,\mu_2 \\
\varphi(T) &= 0,
\end{aligned}
\right.
\tag{4.6}
$$

then the constraints of the dual problem admit the simpler form

$$
\begin{aligned}
\mu_1+\mu_3 &\geq a_1+\varphi \\
\mu_2+\mu_4 &\geq a_2+\varphi \\
\mu_i &\geq 0, i=1,\ldots,4\ .
\end{aligned}
$$

In this way, the dual problem is seen to be a linear parabolic control problem with state-constraints. The theory in [2] yields the following theorem on the existence of at least one **bounded and measurable** optimal solution $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3,\bar{\mu}_4)$ for the dual problem:

Theorem 4.1 *Suppose that (A3) is satisfied. Then the dual problem* $(\mathcal{D}^\ell)$ *admits at least one optimal solution* $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3,\bar{\mu}_4)$ *that belongs to* $L^\infty(Q)\times L^\infty(\Sigma)\times L^\infty(Q)\times L^\infty(\Sigma)$.

The proof of the theorem was performed separately for the cases of boundary control and distributed control. However, it is clear that it can be extended to the general problem above (see the remarks in the last section of [2]). The idea of [2] is as follows: First of all, the feasible set of $(\mathcal{D}^\ell)$ is nonempty. This is a consequence of the duality relation $\max(\mathcal{P}^\ell)=\inf(\mathcal{D}^\ell)$, which follows from the simple structure of the constraints of $(\mathcal{P}^\ell)$. If a feasible $(\mu_1,\mu_2,\mu_3,\mu_4)$ is given such that $\mu_1(x,t)+\mu_3(x,t)$ is positive on some subset of Q, and $\mu_1(x,t)+\mu_3(x,t) > a_1(x,t)+\varphi(x,t)$ holds there, then we are able to diminish $\mu_1+\mu_3$ on this set, until the equality $\mu_1+\mu_3=\max\{a_1+\varphi,0\}$

is achieved. In the same way, $\mu_2 + \mu_4$ is handled. Here, we essentially use the positivity of $G(x, \xi, t)$. The value of the dual objective decreases, since $\bar{c}_Q$, $\bar{c}_\Sigma$, u_b, v_b are positive. Finally, one is able to show that the infimum of $(\mathcal{D}^\ell)$ is attained in a uniformly bounded set. The solution is found by weak compactness.

Remark 4.1 *Assumption (A3) implies that $\bar{c}_Q$ and $\bar{c}_\Sigma$ are strictly positive. This can be seen as follows: The difference $y_c - \bar{y}$ satisfies a linear system with homogeneous initial condition , where the nonnegative functions $\bar{u}$ and $\bar{v}$ stand on the right hand side. The comparison principle yields $y_c - \bar{y} \geq 0$, hence $\bar{c}_Q = c_Q + \beta_Q y_c \geq c_Q + \beta_Q \bar{y} > 0$.*

We recall the **complementary slackness conditions**,

$$(\bar{\mu}_1 + \bar{\mu}_3 - a_1 - \varphi)\, \bar{u} = 0 \ , \ \bar{\mu}_3(\bar{u} - u_b) = 0 \ \text{ a.e in } Q \ , \tag{4.7}$$

$$(\bar{\mu}_2 + \bar{\mu}_4 - a_2 - \varphi)\, \bar{v} = 0 \ , \ \bar{\mu}_4(\bar{v} - v_b) = 0 \ \text{ a.e on } \Sigma \ , \tag{4.8}$$

which are well known from the theory of linear continuous programming.

Theorem 4.2 *If $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4)$ is a bounded and measurable optimal solution of the dual problem $(\mathcal{D}^\ell)$, then $\bar{\mu}_1$ and $\bar{\mu}_2$ are Lagrange multipliers for the (nonlinear) optimal control problem $(\mathcal{P})$, associated with the state-constraints $u \leq c_Q + \beta_Q y$ and $v \leq c_\Sigma + \beta_\Sigma y$, respectively.*

Proof - We define $p := -(\psi + \varphi)$ and verify the optimality conditions (1.6), (1.7), (1.8): The adjoint equation (1.6) is easily obtained by adding the systems for ψ and φ with negative sign. Let us show the variational inequality (1.7). We find

$$\int_Q (\alpha \bar{u} + p + \bar{\mu}_1)(u - \bar{u})\, dx\, dt = \int_Q (-a_u + p + \bar{\mu}_1)(u - \bar{u})\, dx\, dt$$

$$= \int_Q (\psi - a_1 + p + \bar{\mu}_1)(u - \bar{u})\, dx\, dt = -\int_Q (a_1 + \varphi - \bar{\mu}_1)(u - \bar{u})\, dx\, dt =$$

$$\int_Q (a_1 + \varphi - \bar{\mu}_1 - \bar{\mu}_3)\, \bar{u}\, dx\, dt - \int_Q (a_1 + \varphi - \bar{\mu}_1 - \bar{\mu}_3)\, u\, dx\, dt + \int_Q \bar{\mu}_3(\bar{u} - u_b)\, dx\, dt - \int_Q \bar{\mu}_3(u - u_b)\, dx\, dt \geq 0.$$

The last inequality follows from the complementary slackness conditions (4.7), (4.8) for the terms containing $\bar{u}$, the nonnegativity of $\bar{\mu}_1$, $\bar{\mu}_3$, and from the inequality constraints $u \geq 0$, $u \leq u_b$. We have shown (1.4). The variational inequality (1.8) is verified in the same way. In view of the complementary slackness conditions, $\bar{\mu}_1$, $\bar{\mu}_2$ satisfy all properties of Lagrange multipliers.

We have performed the proof for problem $(\mathcal{P}_2)$, the case of $(\mathcal{P}_1)$ is treated by $\bar{\mu}_3 = 0$, $\bar{\mu}_4 = 0$. ■

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