Obstructions to Polytope Projection

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Outline

Deformed Products of even Polygons

Theorem (Sanyal, Ziegler, S. '04/'05)

Let C_{2p} be an even 2p-gon. There exists a realization of $(C_{2p})^r$ in $\mathbb{R}^{2r}$ such that its projection to $\mathbb{R}^4$ has fatness arbitrarily close to 9.

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- f-vectors of 4-polytopes: Fatness.
- Dimensional ambiguity: The graph survives the projection.
- Polyhedral Surfaces in $\mathbb{R}^3$: Two interesting surfaces of genus $\mathcal{O}(n \log n)$ survive the projection.

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Proof

- Orthogonal product of special $2p$-gons: $Ax \leq b$
- Controlled deformation: $Ax \leq b \rightsquigarrow A'x \leq b'$
- Gale duality yields combinatorial description of the projection

Deformed Products of odd Polygons

Theorem (Sanyal, Ziegler, S. '06)

Let C_p be an odd p-gon and $r \geq 4$. There is no realization of $(C_p)^r$ such that the projection $\mathbb{R}^{2r} \to \mathbb{R}^4$ preserves the graph.

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Let C_p be an odd p-gon and $r \geq 2$. There is no realization of $(C_p)^r$ such that the projection $\mathbb{R}^{2r} \to \mathbb{R}^r$ preserves all vertices.

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Proof

- Gale Duality
- Borsuk-Ulam and Sarkaria Inequality

Wedge Product

Definition (Wedge Product)

$W_{p,q}$ of a p-gon $Ax \leq \mathbb{1}$ and a $(q-1)$-simplex $Bx \leq \mathbb{1}$:

$$\begin{pmatrix} B & & & & \mathbb{1}a_1 \\ & B & & & \mathbb{1}a_2 \\ & & \ddots & & \vdots \\ & & & B & \mathbb{1}a_p \end{pmatrix} x \leq \mathbb{1}$$

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- $W_{p,q}$ is a simple $(p(q-1)+2)$-polytope with pq facets.
- $W_{p,q}$ contains q^p p-gons.

Theorem (Sanyal, Ziegler, S. '06)

The wedge product $W_{p,q}$ contains a regular surface of type $\{p, 2q\}$ with

- *f-vector $q^{p-3}(pq, pq^2, 2q^2)$, and*
- *genus $g = 1 - pq^{p-2} + (p-2)q^{p-1}$*

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- For $p = 3$ the surface has $\mathcal{O}(q)$ vertices and $\mathcal{O}(q^2)$ genus.
- A surface of type $\{p, 2q\}$ embedded in $\mathbb{R}^{p(q-1)+2}$.

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Lemma (Perles)

Every 2-dimensional polytopal complex in $\mathbb{R}^d$ may be projected to $\mathbb{R}^5$.

Theorem (Sanyal, Ziegler, S. '06)

For $p \geq 3$ and $q = 2$ there exists a realization of $W_{p,2}$ in $\mathbb{R}^{p+2}$ such that the surface of type $\{p, 4\}$ survives the projection to $\mathbb{R}^4$.

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- Surfaces of McMullen, Schulz, Wills 1983 of genus $\mathcal{O}(n \log n)$.

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Proof

- Orthogonal wedge product $Wx \leq \mathbb{1}$
- Deformation $Wx \leq \mathbb{1} \rightsquigarrow W'x \leq b'$
- Gale Duality

Theorem (Sanyal, Ziegler, S. '06)

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Proof

- Gale Duality
- Borsuk-Ulam and connectivity lower bound for $\mathbb{Z}_2$-index.

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Corollary

No realization of surfaces of type $\{p, 2q\}$ in $\mathbb{R}^3$ via projection of $W_{p,q}$.

Outline

Idea

Projection Problem

- Simple Polytope $P \subset \mathbb{R}^d$
- Subcomplex $S \subset P$
- Projection $\pi : \mathbb{R}^d \to \mathbb{R}^k$
- Does S survive the projection?

←

Examples

- Vertices in PoP's
- Surface in $W_{p,q}$

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Associated Polytope

- Polytope $Q \subset \mathbb{R}^e$
- Subcomplex $T \subset Q$

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Gale Duality

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Obstructions

- Borsuk-Ulam
- Sarkaria Inequality
- $\mathbb{Z}_2$-index

Faces surviving Projection

Lemma

Consider the projection of a polytope $P \subset \mathbb{R}^d$ to $\mathbb{R}^k$.

$$\left(\begin{array}{c|c} \hline \hline \hline A_{d-k} & A_k \\ \hline \hline \hline \end{array}\right) x \leq b$$

Faces surviving Projection

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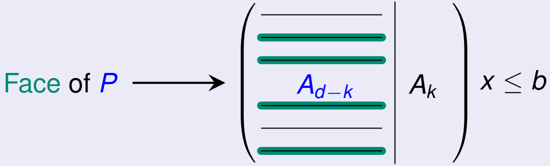

A face of P survives the projection $\Leftrightarrow$ rows of A_{d-k} positively span.

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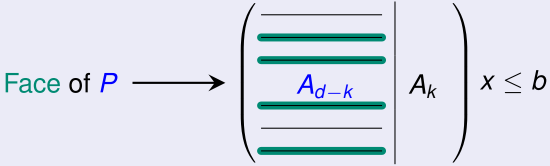

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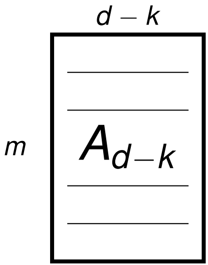
Proof

- Linear Algebra calculation

Gale Duality

Vector Configuration

- m vectors in $\mathbb{R}^{d-k}$



Gale Duality

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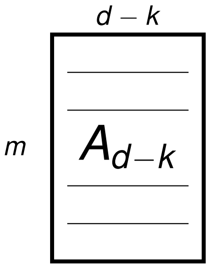
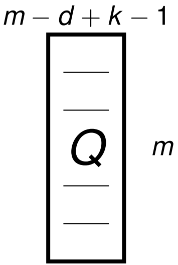
Polytope

- m points in $\mathbb{R}^{m-d+k-1}$





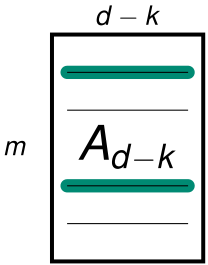
Gale Duality

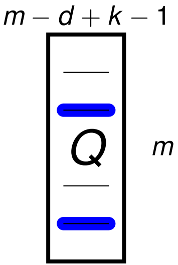
Vector Configuration

- m vectors in $\mathbb{R}^{d-k}$
- Positive dependences of A_{d-k}

Polytope

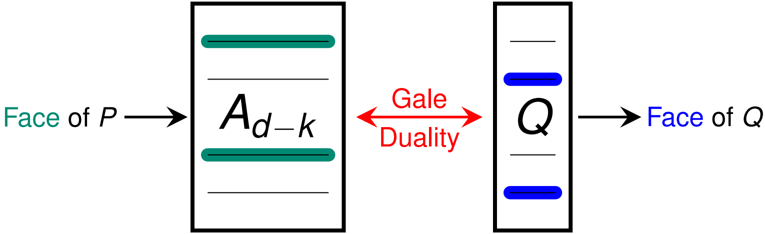
- m points in $\mathbb{R}^{m-d+k-1}$
- Faces of Gale Dual Q





Projection

Does there exist a realization of P in $\mathbb{R}^d$ such that the subcomplex S survives the projection to $\mathbb{R}^k$?



Embeddability

Does there exist a polytope Q in with a certain subcomplex T?

Example

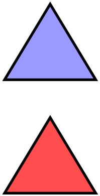
Project Triangle $\times$ Triangle to $\mathbb{R}^2$ preserving all vertices.

Example

Project Triangle × Triangle to $\mathbb{R}^2$ preserving all vertices.

Triangle × Triangle

- 4-dimensional
- 6 facets
- S is the set of vertices



$$\left(\begin{array}{cc|cc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{array}\right)$$

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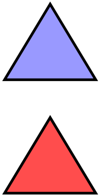
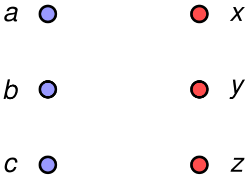


$$\left(\begin{array}{cc|cc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ \hline x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{array}\right)$$



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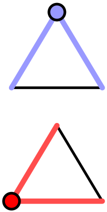
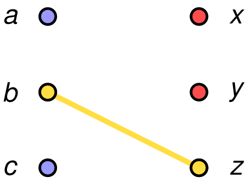


a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
x_1	x_2	x_3	x_4
y_1	y_2	y_3	y_4
z_1	z_2	z_3	z_4



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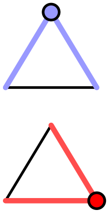

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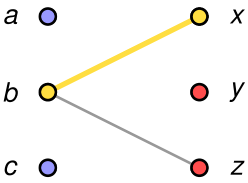

Example

Project Triangle × Triangle to $\mathbb{R}^2$ preserving all vertices.

Triangle × Triangle

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- 6 facets
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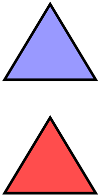
Associated Polytope

- 3-dimensional
- 6 vertices



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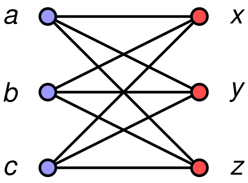

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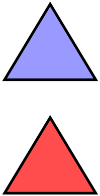
Associated Polytope

- 3-dimensional
- 6 vertices
- $T = K_{3,3}$



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Topological Obstruction

$K_{3,3}$ is not planar.

Products of Polygons

Problem

Project the product $(C_p)^r$ of odd p-gons from $\mathbb{R}^{2r}$ to $\mathbb{R}^r$ preserving its vertices S.

- Associated polytope Q has dimension $r(p-1)-1$ and rp vertices.
- Vertices $S \subset (C_p)^r \rightsquigarrow K^{*r} = T \subset \partial Q$

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- Vertices $S \subset (C_p)^r \rightsquigarrow K^{*r} = T \subset \partial Q$

Use topological combinatorics to show that $T \not\subset \partial Q$ using $\mathbb{Z}_2$-index:

- Borsuk-Ulam:

$$\operatorname{ind}_{\mathbb{Z}_2}(\partial Q) = r(p-1)-2.$$

- Sarkaria Coloring/Embedding Theorem (Matoušek Version):

$$\operatorname{ind}_{\mathbb{Z}_2}\left((T)^{*2}_{\Delta}\right) \geq rp - \chi(\mathrm{KG}(\mathcal{F}(T))) - 1 = r(p-1)-1$$

- p odd implies that the chromatic number is r.

Wedge Product Surfaces

Problem

Project the wedge product $W_{p,q}$ for $p \geq 4$ and $q \geq 3$ from $\mathbb{R}^{p(q-1)+2}$ to $\mathbb{R}^p$ preserving the regular surface S.

- Associated polytope Q has dimension $p + 1$ and pq vertices.
- Surface $S \subset W_{p,q} \rightsquigarrow (D_q)^{*(p-1)} = T \subset \partial Q$

Wedge Product Surfaces

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Project the wedge product $W_{p,q}$ for $p \geq 4$ and $q \geq 3$ from $\mathbb{R}^{p(q-1)+2}$ to $\mathbb{R}^p$ preserving the regular surface S.

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- Surface $S \subset W_{p,q} \rightsquigarrow (D_q)^{*(p-1)} = T \subset \partial Q$

Show that $T \not\subset \partial Q$ using $\mathbb{Z}_2$-index:

- Borsuk-Ulam: $\operatorname{ind}_{\mathbb{Z}_2}(\partial Q) = p$.
- Connectivity lower bound for $\mathbb{Z}_2$-index

$$\operatorname{ind}_{\mathbb{Z}_2}\left((T)^{*2}_{\Delta}\right) \geq \operatorname{conn}\left((T)^{*2}_{\Delta}\right) + 1 \geq 2p - 3$$

- $q \geq 3$ implies that $(T)^{*2}_{\Delta}$ is $(2p-4)$-connected.
- $p \geq 4$ implies that $2p - 3 > p$.

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Projection Problem

- Simple Polytope $P \subset \mathbb{R}^d$
- Projection $\pi : \mathbb{R}^d \to \mathbb{R}^k$
- Subcomplex $S \subset P$

↓ Preserved faces

Vector Configuration

- Positive dependences

↓ Gale Duality

Associated Polytope

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Problems

- f-vectors of 4-polytopes
- Dimensional Ambiguity
- Polyhedral Surfaces

Idea

Projection Problem

- Simple Polytope $P \subset \mathbb{R}^d$
- Projection $\pi : \mathbb{R}^d \to \mathbb{R}^k$
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No!

↑ ↓ Preserved faces

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Obstructions

- Borsuk-Ulam
- Sarkaria Inequality
- $\mathbb{Z}_2$-index