Polyhedral Surfaces in Wedge Products

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We introduce the wedge product of two polytopes which is dual to the wreath product of Joswig and Lutz [6]. The wedge product of a p-gon and a $(q-1)$-simplex contains many p-gon faces of which we select a subcomplex corresponding to a surface. This surface is regular of type $\{p, 2q\}$, that is, all faces are p-gons, all vertices have degree $2q$, and the combinatorial automorphism group acts transitively on the flags of the surface. We show that for certain choices of parameters p and q there exists a realization of the wedge product such that the surface survives the projection to $\mathbb{R}^4$. For a different choice of parameters such a realization does not exist.

1. Wedge Product

Consider a d_1-dimensional polytope P_1 with m_1 facets and a d_2-dimensional polytope P_2 with m_2 facets. The *wedge product* $P_1 \lhd P_2$ of P_1 and P_2 is a $(m_1 d_2 + d_1)$-dimensional polytope with $m_1 m_2$ facets. The construction is dual to the wreath product of Joswig and Lutz, i.e. $P_1 \lhd P_2 = (P_2^{\triangle} \wr P_1^{\triangle})^{\triangle}$. In the following we consider the wedge product $W_{p,q-1} = C_p \lhd \Delta_{q-1}$ of a p-gon C_p and a $(q-1)$-simplex Δ_{q-1}. The polytope $W_{p,q-1}$ is a simple $(p(q-1)+2)$-polytope with pq facets. It contains q^p p-gon faces which may be identified with vectors $(i_1, \ldots, i_p) \in \{1, \ldots, q\}^p$.

2. Polyhedral Surfaces

We know little about the realization of arbitrary polyhedral surfaces. Triangulated surfaces can always be realized as a subcomplex of the six dimensional cyclic polytope and thus in $\mathbb{R}^5$ via Schlegel projection. By a lemma of Perles (cf. Grünbaum [5, p. 204]) we know that every two dimensional polyhedral complex embedded in any $\mathbb{R}^d$ also has a realization in $\mathbb{R}^5$. There is also a negative result by Betke and Gritzmann [2] showing that if the number of vertices of odd degree and the number of faces containing odd vertices of a surface satisfy a certain inequality then this surface cannot be embedded in any $\mathbb{R}^d$.

The 2-skeleton of the wedge product $W_{p,q-1}$ contains many p-gon faces. Each p-gon edge is contained in exactly q p-gons. We consider the following subset of the p-gon faces:

$$S_{p,2q} = \left\{ (i_1, \ldots, i_p) \in \{1, \ldots, q\}^p \, : \, \sum_{j=1}^{p} i_j \equiv 0, 1 \mod q \right\}.$$

By abuse of notation $S_{p,2q}$ denotes the set of p-gons as well as the subcomplex obtained by taking the p-gons, their edges and their vertices.

Theorem 1. *The subcomplex $S_{p,2q}$ of the wedge product $W_{p,q-1}$ is a closed, orientable, regular polyhedral surface with f-vector $(pq^{p-2}, pq^{p-1}, 2q^{p-1})$ and genus $1 + \frac{1}{2}q^{p-2}(pq - 2q - p)$. In particular, all faces of the surface are p-gons and every vertex has degree $2q$.*

This family of surfaces is a generalization of well known surfaces. The surface $S_{3,6}$ is the classical Dyck's Regular Map [4]. For $p = 3$, the surfaces $S_{3,2q}$ are triangulated surfaces which occur in Coxeter and Moser [3] in the study of discrete groups. For $q = 2$ we obtain a subfamily of the surfaces of McMullen, Schulz, and Wills [7] of "unusually large genus." By generalizing the above construction we obtain a bigger family of surfaces containing all the surfaces of type $\{p, 4\}$ of McMullen, Schulz, and Wills. In the following we consider projections of polytopes. We say that a subcomplex of the polytope *survives* the projection if it is a subcomplex of the projected polytope as well.

Theorem 2. *For $q = 2$ there exists a polytope combinatorially equivalent to the wedge product $W_{p,1}$ such that the surface $S_{p,4}$ survives the projection to $\mathbb{R}^4$.*

We prove this theorem by starting with the standard wedge product of a p-gon and an interval which contains the surface $S_{p,4}$ as a subcomplex. Then we deform the polytope to obtain a realization which has the property that the projection to $\mathbb{R}^4$ preserves the surface. A simpler for of this deformation technique was introduced by Amenta and Ziegler [1] and also successfully used by Ziegler [10] to construct four dimensional polytopes of high fatness. The most recent account on deformed products and their applications can be found in Sanyal [9] and [8].Via Schlegel diagrams we obtain a new way to realize some of the surfaces of McMullen, Schulz, and Wills of "unusually large genus" in $\mathbb{R}^3$. The same scheme of projecting high dimensional polytopes containing a surface can also be applied to surfaces of high complexity contained in products of polygons or products of simplices. The following theorem shows that it is not always possible to obtain a realization of the surfaces via projection.

Theorem 3. *For $q \geq 3$ and $p \geq 4$ there exists* no *polytope combinatorially equivalent to the wedge product $W_{p,q-1}$ such that the surface $S_{p,2q}$ survives the projection to $\mathbb{R}^4$.*

The proof proceeds in two steps. First we use Gale duality to associate an embeddability problem to the projection problem, where the surface which is supposed to survive the projection corresponds to a subcomplex of an associated polytope. In a second step, we use methods from topological combinatorics to show that the subcomplex cannot be embedded in the associated polytope. Hence the surface cannot survive the projection to $\mathbb{R}^4$.

References

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