Neighborly cubical polytopes and spheres

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Neighborly cubical polytopes and spheres

Diplomarbeit bei
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Contents

Introduction

This work focusses on two closely related objects of discrete geometry: polytopes and PL spheres. We are particularly interested in simplicially or cubically neighborly polytopes and spheres. The exact definitions of the objects and the references are postponed to the next chapters. Here we will only give an intuition of the objects dealt with later.

Polytopes have been studied for a very long time. Probably the most famous historic work on polytopes was done by the Greek about 400 BC who described the Platonic Solids – the most famous 3-polytopes.

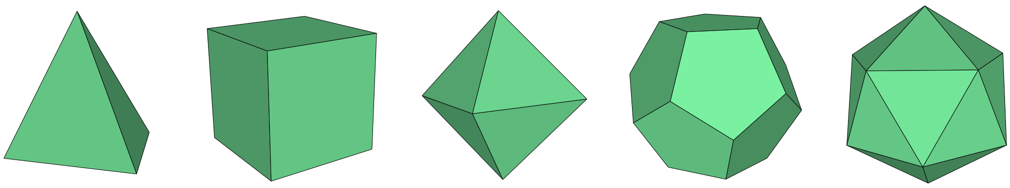


Figure 1: The Platonic Solids

Two dimensional polytopes are nothing but convex n-gons. In higher dimension polytopes may be defined as the convex hull of finitely many points.

PL spheres are a family of topological objects which are **P**iecewise **L**inearly homeomorphic to the boundary of a polytope. They are closely related to polytopes since the boundary of a polytope is always a PL sphere. The converse is not true. Hence the class of PL spheres is bigger than the class of polytopes. The PL spheres having a realization as the boundary of a polytope are called polytopal or realizable.

Polytopes as well as spheres may be looked at in two different ways. The *combinatorial* point of view is only interested in the incidence structure. This may be encoded in partially ordered sets, called posets. It is completely described by the vertex-facet incidences. The *geometric* point of view considers a polytope or sphere together with coordinates usually in some $\mathbb{R}^d$.

Simplicial and cubical polytopes are one special family of polytopes. Since the boundary of a d-dimensional polytope is a $(d-1)$-dimensional sphere the following may be rephrased in terms of spheres. Simplicial d-dimensional polytopes are polytopes whose $(d-1)$-dimensional faces are simplices. A simplex is a higher dimensional analog of a triangle which is a 2-dimensional simplex. Thus a three dimensional simplicial polytope is bounded by triangles. Similarly, cubical d-dimensional polytopes are bounded by hypercubes or $(d-1)$-cubes, which are a natural generalization of the usual 3-dimensional cube. Hence three dimensional cubical polytopes are bounded by quadrilaterals (2-cubes).

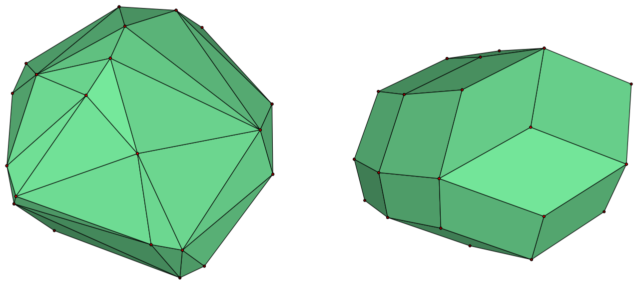


Figure 2: A simplicial (left) and a cubical (right) 3-polytope

We investigate simplicially neighborly and cubically neighborly polytopes and spheres. Simplicially or cubically neighborly d-polytopes (resp. $(d-1)$-spheres) have the $(\lfloor d/2 \rfloor - 1)$-skeleton of high dimensional simplices or cubes, respectively. The k-skeleton of a polytope or sphere P is the complex made up of all faces of P of dimension at most k . As $\lfloor d/2 \rfloor - 1 = 0$ for $d = 2, 3$, the first interesting examples are 4-polytopes and 3-spheres with the 1-skeleton or graph of high dimensional simplices or cubes.

Simplicially neighborly polytopes are a well known family of polytopes often only called neighborly polytopes. Besides having the skeleta of high dimensional simplices, they are especially interesting because they maximize the f-vector of all polytopes with a given number of vertices, i.e. they satisfy the inequalities of the Upper Bound Theorem with equality. Further all even dimensional simplicially neighborly polytopes are simplicial. The cyclic polytopes are one family of simplicially neighborly polytopes that have a member for any dimension d and any number of vertices $n > d$. Their facets are obtained from Gale's Evenness Condition.

Cubically neighborly spheres are a very recent class of PL spheres. They were first constructed by Babson, Billera, and Chan [3] in 1997. Babson, Billera, and Chan used techniques called 'fissuring' and 'mirroring' and sequences of particular triangulations of cyclic polytopes to prove the existence of neighborly cubical spheres. Their proof does not yield a direct formula for the combinatorial structure of the spheres.

In 2000, Joswig and Ziegler [11] constructed neighborly cubical d-polytopes as projections of deformed n-cubes preserving the $(\lfloor d/2 \rfloor - 1)$-skeleton. They also gave an analog of Gale's Evenness Condition describing the combinatorics of the neighborly cubical polytopes.

The comparison of these two constructions was the starting point of our investigation.

In Chapter 1 we start with some general facts on polytopes, polytopal complexes and their combinatorial structure encoded in posets and automorphism groups. Then we present PL spheres and combinatorial manifold to provide the basis for the later analysis.

In the second chapter we concentrate on simplicial complexes. We define and discuss simplicially neighborly polytopes and spheres because they serve as a model for the cubically neighborly polytopes and spheres. Since the construction of Babson, Billera, and Chan is based on sequences of pulling triangulations of cyclic polytopes we present a thorough analysis of the cyclic polytopes and their automorphism groups. This leads to a characterization of pulling triangulations of cyclic polytopes depending on dimension and number of vertices.

The main part focuses on cubically neighborly polytopes and spheres. At the beginning of Chapter 3 we present a purely combinatorial description of cubes and the cubical analog of the definition of simplicially neighborly polytopes and spheres. Then we analyze 'mirroring' and 'fissuring' which are the two contruction techniques used by Babson, Billera, and Chan to build their neighborly cubical PL spheres.

Based on the inductive construction of Babson, Billera, and Chan, we develop the more general framework of *neighborly increasing sequences* which abstract sequences of pulling triangulations of cyclic polytopes. Those sequences comprise sequences of pulling triangulations of arbitrary neighborly simplicial polytopes which are already a generalization of sequences of pulling triangulations of the cyclic polytopes. In dimension two and three all neighborly increasing sequences are sequences of pulling triangulations of polytopes, but in dimension four the concept really is more general: We construct a neighborly increasing sequence for the non-polytopal neighborly simplicial Altshuler sphere N^{10}_{425}.

Similar to the inductive construction of Babson, Billera, and Chan, we construct neighborly cubical PL spheres from an arbitrary neighborly increasing sequence. Furthermore, we give a direct formula for the combinatorics of the spheres depending on the combinatorics of the sequence. For the 'canonical' sequence of pulling triangulations of cyclic polytopes the neighborly cubical PL sphere is combinatorially isomorphic to the boundary of a neighborly cubical polytope. Further the BBC-construction of *odd* dimensional spheres from cyclic polytopes is essentially unique.

We end with an easy way to realize a small family of equivelar surfaces (cf. McMullen, Schulz and Wills [14]). These polyhedral surfaces are particularly interesting because of their ‘unusually large genus’, meaning that they have ‘more holes than vertices’. Some of those surfaces were already investigated by Coxeter [6] and Ringel [17] in different contexts. It turns out that these surfaces are indeed mirror complexes of n-gons and thus lie in some neighborly cubical 4-polytopes. Hence they may be realized as a subcomplex of a Schlegel diagram in $\mathbb{R}^3$.

I want to thank Michael Joswig for giving me the opportunity to work on this subject and for supervising this thesis. Further thanks to Carsten Lange, Dagmar Timmreck, and Nikolaus Witte who carefully read this work and were always open for questions and discussion.

Chapter 1

Polytopal complexes and topology

In this chapter we will present some basic facts and definitions about the objects we will deal with later.

For facts concerning polytopes, polytopal complexes and posets we refer to Ziegler [19] and Grünbaum [9].

1.1 Polytopes

In the introduction we already presented some pictures to illustrate 3-dimensional polytopes. The following is the definition in arbitrary dimension.

Definition 1.1. A *(convex) polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

$$P = \operatorname{conv}(\{p_1, \ldots, p_n\}) \subset \mathbb{R}^d.$$

The *dimension* of a polytope is the dimension of its affine hull. A polytope is *full dimensional* if its affine hull is $\mathbb{R}^d$. In the following **all polytopes are full dimensional**. A *valid* inequality for P is a linear inequality $c^T x \leq c_0$ with $c \in \mathbb{R}^d$, $c_0 \in \mathbb{R}$ satisfied by all $x \in P$. A *face* F of a polytope P is the intersection

$$F := P \cap \{x \in \mathbb{R}^d : c^T x = c_0\}$$

where $c^T x \leq c_0$ is a valid inequality. P and the empty set are both faces of P since $\{x \in \mathbb{R}^d : 0^T x = 0\} = \mathbb{R}^d$ and $\{x \in \mathbb{R}^d : 0^T x = 1\} = \emptyset$. All faces of a polytope are polytopes themselves. A k-dimensional face or polytope is called k-face or k-polytope, respectively. The *facets* of a d-polytope are the $(d-1)$-faces. The 0-faces are called *vertices.* A d-polytope is called *simple* if every vertex lies in d facets or equivalently d edges. It is *simplicial* if all facets are simplices.

The *vertex figure* of a polytope P at a vertex v is the intersection of P with a hyperplane intersecting all edges incident to v. If the polytope is simple then the vertex figure is a simplex.

There are two essentially unique descriptions of a full dimensional polytope P

- the inner description or vertex description: P is the convex hull of its vertices,

- the outer description or facet description: P is the intersection of the halfspaces supporting its facets.

The two descriptions are equivalent by the Main Theorem of polytope theory. The inner description yields a purely combinatorial description of a polytope as a collection of vertex sets of its faces. This information is encoded in posets defined in Section 1.3.

The first interesting examples of neighborly polytopes come up in dimension four. Neighborliness is defined in Sections 2.1 and 3.2 for simplicial resp. cubical complexes. A tool to visualize 4-dimensional polytopes are Schlegel diagrams. A Schlegel diagram is a projection of the polytope onto one of its facets preserving all the faces except the one projected onto. The definition of the projection is given in Ziegler [19, Ch. 5]. Since a facet of a d-polytope is a $(d-1)$-polytope this technique is often used to visualize polytopes of dimension at most four. Figure 1.1 shows the construction of a Schlegel diagram of a 2- and a 3-polytope.

1.2 Polytopal complexes

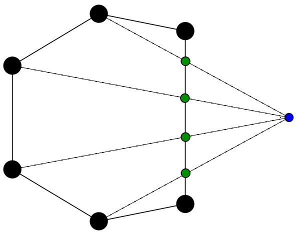
Definition 1.2. A *polytopal complex* is a collection $\mathcal{C}$ of polytopes in $\mathbb{R}^d$ satisfying the following conditions

(i) the empty set is in $\mathcal{C}$,

(ii) for every $P \in \mathcal{C}$ all its faces are in $\mathcal{C}$ and

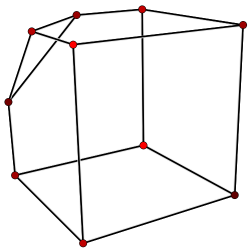
(iii) for every $P, Q \in \mathcal{C}$: $P \cap Q$ is a face of both P and Q.

The elements of a polytopal complex are called *faces*.

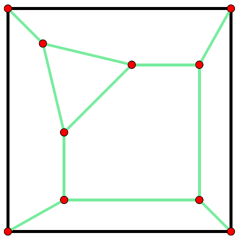


(a) The construction of the Schlegel diagram of a hexagon

(b) The 1-dimensional Schlegel diagram of a hexagon



(c) A 3-cube with one vertex cut off



(d) The Schlegel diagram of the truncated cube

Figure 1.1: The top figures show a projection performed to construct a Schlegel diagram of a 2-dimensional polytope. The bottom figures show a 3-polytope and its Schlegel diagram.

Since every polytope is the convex hull of its vertices and all faces of a polytope are polytopes, a polytopal complex may also be considered in a more abstract way, as a collection of vertex sets of the faces.

An inclusion maximal face of a polytopal complex is called a *facet*. A polytopal complex is *finite* if $|\mathcal{C}|$ is finite. The complex is *pure* if all facets have the same dimension. The *dimension* of a polytopal complex is the maximal dimension of its facets. The *k-skeleton* of $\mathcal{C}$ is the polytopal complex given by all faces of $\mathcal{C}$ of dimension at most k, denoted

$$(\mathcal{C})_k := \{F \in \mathcal{C} : \dim F \leq k\}.$$

The graph $\Gamma(\mathcal{C})$ of $\mathcal{C}$ is its 1-skeleton. The *f-vector* of a k-dimensional polytopal complex is the vector $(f_0, \ldots, f_k) \in \mathbb{N}^{k+1}$ where f_i denotes the number of i-faces. In the following **all complexes are finite**.

The *underlying space* of a polytopal complex $\mathcal{C}$ is the union of all polytopes of the complex, denoted $\|\mathcal{C}\| := \bigcup\{P \in \mathcal{C}\}$. The topology on $\mathcal{C}$ is the induced subspace topology on $\|\mathcal{C}\|$ of the usual topology on $\mathbb{R}^d$. Note that in the non-finite case the topology has to be defined differently.

Two polytopal complexes are *combinatorially isomorphic* or *equivalent* if there exists a bijection between their faces preserving inclusion. A *cubical* resp. *simplicial complex* is a polytopal complex whose facets are combinatorially equivalent to cubes resp. simplices.

Definition 1.3. Let $\mathcal{C}$ be a polytopal complex and $F \in \mathcal{C}$ a face. The *deletion* of F in $\mathcal{C}$ is the subcomplex of faces which do not contain F:

$$\operatorname{del}(C, F) := \{G \in \mathcal{C} : F \nsubseteq G\}.$$

1.3 Posets

In this section we introduce the notion of a **p**artially **o**rdered **set** which represents a very general combinatorial structure. In particular we give basic properties of posets of polytopal complexes and polytopes.

Definition 1.4. A *partially ordered set* or *poset* is a pair $(\mathcal{P}, \preceq)$ of a set $\mathcal{P}$ and a partial order $\preceq$ which is a binary relation $\preceq$ such that

- $\preceq$ is reflexive; $x \preceq x$ for all $x \in \mathcal{P}$,
- $\preceq$ is transitive; if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ and
- $\preceq$ is antisymmetric; if $x \preceq y$ and $y \preceq x$ then $x = y$.

An *interval* of a poset is a subposet of $(\mathcal{P}, \preceq)$ containing all elements between x and y: $[x, y] := \{z \in \mathcal{P} : x \preceq z \preceq y\}$ for $x, y \in \mathcal{P}$ with $x \preceq y$. For $x, y \in \mathcal{P}$ we write $x \dot{\preceq} y$ if y covers x, i.e. the interval $[x, y]$ consists of x and y only. A sequence $x_1 \preceq x_2 \preceq \ldots \preceq x_n$ with $x_i \in \mathcal{P}$ for $i = 1, \ldots, n$ is a *chain*. The *upper ideal* or *principal filter* of an element $x \in \mathcal{P}$ is the subposet of all elements bigger than x, denoted $\bigvee_{\mathcal{P}}(x) := \{y \in \mathcal{P} : x \preceq y\}$.

A poset is *bounded* if it has a unique maximal element, denoted $\hat{1}$, and a unique minimal element, denoted $\hat{0}$. A bounded poset is *graded* if all maximal chains have the same length. To each element x of a graded poset, one can assign the *rank* $r(x)$ of the element which is the length of a maximal chain in the interval $[\hat{0}, x]$ minus 1.

A poset is a *lattice* if it is bounded and for all $x, y \in \mathcal{P}$ join and meet exist, i.e.

- there exists a unique minimal element $z \in \mathcal{P}$ such that $x \preceq z$ and $y \preceq z$, the *join* $x \vee y$, and

- there exists a unique maximal element $z' \in \mathcal{P}$ such that $z' \preceq x$ and $z' \preceq y$, the *meet* $x \wedge y$.

If $\mathcal{P}$ is a graded lattice, the elements of rank 1 are the *atoms*. The *coatoms* are the elements of rank $r(\widehat{1}) - 1$. A graded lattice is *atomic* if each element except $\widehat{0}$ is a join of atoms. It is *coatomic* if each element except $\widehat{1}$ is a meet of coatoms.

The typical way to visualize posets is the *Hasse diagram* which is a directed graph whose nodes are the elements of the posets. A directed edge (x, y) is contained in the Hasse diagram if $x \dot{\preceq} y$.

Posets arise naturally from various geometric objects. For example, we obtain the face poset $(\mathcal{F}(\mathcal{C}), \subseteq)$ of a polytopal complex $\mathcal{C}$ by taking all faces (including the empty set) ordered by inclusion. For ease of notation we use $\mathcal{C}$ to denote the complex as well as the face poset. The face poset $\mathcal{L}(P)$ of a polytope P is a graded lattice called the face lattice. It is the poset of all faces of P, including P and the empty set, ordered by inclusion: $\mathcal{L}(P) := (\mathcal{F}(P), \subseteq)$.

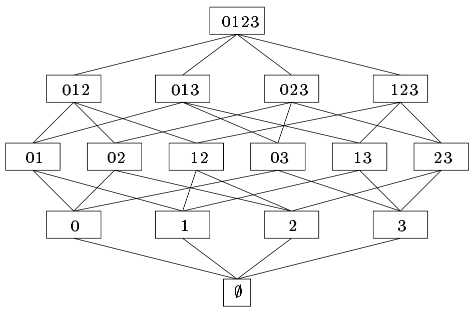
Since each face of a polytope is the convex hull of its vertices, the face lattice is atomic. Further the face lattice is coatomic as every face F is the intersection of the facets containing F.

Example. Let's review some definitions in case of a tetrahedron. The geometry is given either by the inner or the outer description

Vertices	Facets	
$v_0 = (0,0,0)$	F_0 :	$-x \leq 0$
$v_1 = (1,0,0)$	F_1 :	$-y \leq 0$
$v_2 = (0,1,0)$	F_2 :	$-z \leq 0$
$v_3 = (0,0,1)$	F_3 :	$x + y + z \leq 1$

The combinatorial structure is given by the vertex-facet incidences or the Hasse diagram encoding the face lattice (the arrows in the Hasse diagram are omitted). The partial order is the usual inclusion $\subseteq$.

Facet	Vertex set
F_0	$\{v_0, v_2, v_3\}$
F_1	$\{v_0, v_1, v_3\}$
F_2	$\{v_0, v_1, v_2\}$
F_3	$\{v_1, v_2, v_3\}$



The link of a vertex of a polytopal complex is defined differently by Ziegler [19] and Babson, Billera, and Chan [3]. Ziegler defines the link as a subcomplex whereas Babson, Billera, and Chan define it as a subposet of the face poset. We call the link defined by Babson, Billera, and Chan the vertex figure. The difference is shown in Figure 1.2.

Definition 1.5 (Ziegler). Let $\mathcal{P}$ be a polytopal complex and $v \in \mathcal{P}$ a vertex. The *(closed) star* of v in $\mathcal{P}$ is the polytopal complex of all faces containing v, and all their faces

$$\text{star}(\mathcal{P}, v) = \{G \in \mathcal{P} : \exists H \in \mathcal{P} \text{ with } G \subseteq H \text{ and } v \subseteq H\}$$

The *link* of v in $\mathcal{P}$ is the subcomplex

$$\text{link}(\mathcal{P}, v) := \{G \in \text{star}(\mathcal{P}, v) : v \notin G\}.$$

Definition 1.6 (Babson, Billera, and Chan). Let $v \in \mathcal{C}$ be a vertex of a cubical or simplicial complex $\mathcal{C}$ and $(\mathcal{F}(\mathcal{C}), \subseteq)$ its face poset. The *vertex figure* $\mathcal{C}/v$ (called link by Babson, Billera, and Chan) is a polytopal complex combinatorially isomorphic to the upper ideal $\bigvee_{\mathcal{C}}(v)$, i.e. the poset of all faces containing v.

It is not clear that the upper ideal of a vertex v of an arbitrary polytopal complex always has a realization as a polytopal complex. In the case of finite cubical and simplicial complexes we obtain a spherical vertex figure by taking the intersection of the complex with a suitably small sphere. A polytopal complex is obtained by 'straightening' the spherical vertex figure. This is possible because simplices as well as cubes are simple polytopes and hence all faces of the vertex figure are simplices.

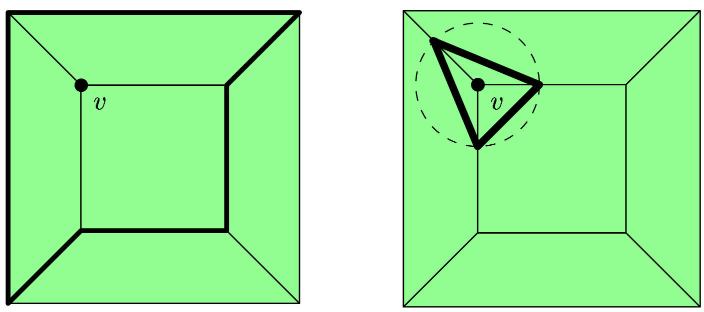


Figure 1.2: The difference between the link (left) and the vertex figure (right) of a vertex v in a cubical 2-complex

1.4 Algebra and automorphism groups

To analyze the automorphism group of a poset and later of the face lattice of polytopes we need some basic algebra.

Definition 1.7. A *group action* of a group G on a set Ω is a mapping $\Omega \times G \to \Omega$ denoted $(a, g) \mapsto ag$ satisfying

1. $a1 = a$ for $1 \in G$ and all $a \in \Omega$,
2. $a(gh) = (ag)h$ for all $g, h \in G$ and $a \in \Omega$.

The *stabilizer* of an element $a \in \Omega$ is a subgroup of G denoted by $G_a = \{g \in G : ag = a\}$.

Let H be a group. A *representation* of G as a group of group automorphism of H is a homomorphism $\varphi : G \to \mathrm{Aut}(H)$.

Definition 1.8. A subgroup $H \leq G$ of a group G is *normal* if $g^{-1}Hg = H$ for all $g \in G$. It is denoted by $H \trianglelefteq G$.

A group G is called *regular* with respect to a group action $\Omega \times G \to \Omega$ on a set Ω, if for all $a, a' \in \Omega$ there exists a unique $g \in G$ such that $ag = a'$.

Definition 1.9. Let H, G be groups and $\varphi : G \to \mathrm{Aut}(H)$ a representation of G as a group of automorphisms of H mapping $g \in G$ to $\varphi_g \in \mathrm{Aut}(H)$. Define a multiplication on the set product $H \times G$ by

$$(h_1, g_1)(h_2, g_2) := (\varphi_{g_2}(h_1)h_2, g_1g_2) \qquad h_1, h_2 \in H \text{ and } g_1, g_2 \in G.$$

This defines the *semidirect product* of H and G with respect to φ denoted by $H \rtimes_\varphi G$.

Now let G act on the set Ω. This defines a canonical action of G on the set product H^Ω by $(h_\omega)g = (h_{(\omega g^{-1})})$ for any $(h_\omega)_{\omega\in\Omega} \in H^\Omega$ and $g \in G$. It also induces a representation $\psi : G \to \mathrm{Aut}(H^\Omega)$. The *wreath product* of H and G according to the action of G on Ω, denoted $H \,\mathrm{wr}_\Omega\, G$, is the semidirect product of groups $H^\Omega \rtimes_\psi G$.

Definition 1.10. The *automorphism group* $\mathrm{Aut}(\mathcal{P})$ of a poset $(\mathcal{P}, \preceq)$ is the group of all poset isomorphisms from $(\mathcal{P}, \preceq)$ to $(\mathcal{P}, \preceq)$, i.e. all bijections of the elements of $\mathcal{P}$ onto themselves preserving the order $\preceq$.

Additional structure of a poset simplifies the description of the automorphism group. If the graded lattice is atomic then it suffices to take a look at all permutations of the atoms preserving the faces. If the graded lattice is atomic and coatomic it suffices to consider permutations of the atoms preserving the atom-coatom inclusions.

Thus to describe the automorphism group of the face lattice of a polytope (which is atomic and coatomic) it suffices to consider vertex permutations preserving facets. Hence the automorphism group is isomorphic to a subgroup of the symmetric group $\mathbb{S}_{f_0}$.

The automorphism group of a polytopal complex also depends strongly on its skeleta. Since the length of the maximal chains (corresponding to the dimension plus one of the faces) is preserved by an automorphism of the complex, any such automorphism restricted to the k-skeleton of the complex is an automorphism of the k-skeleton. As a consequence we obtain

Lemma 1.11. *Let $\mathcal{C}$ be a polytopal complex. Then the automorphism group of $(\mathcal{C}, \subseteq)$ is isomorphic to a subgroup of the automorphism group of $((\mathcal{C})_k, \subseteq)$:*

$$\mathrm{Aut}(\mathcal{C}) \cong H \leq \mathrm{Aut}((\mathcal{C})_k).$$

The automorphism group of a polytope is sometimes also called symmetry group. We make a distinction between the two groups: the symmetry group of a polytopal complex or polytope is the group of euclidean motions mapping the polytopal complex or polytope bijectively onto itself. The automorphism group is the purely combinatorial group defined above. It is clear, that the symmetry group is isomorphic to a subgroup of the automorphism group but they need not be equal: The symmetry group of a triangle with three sides of different length consists of the identity only, but the automorphism group is the symmetric group $\mathbb{S}_3$.

1.5 Spheres and manifolds

Since we only deal with finite polytopal complexes the topology is the induced subspace topology of $\mathbb{R}^d$ on the underlying space. The underlying spaces of finite polytopal complexes are compact. In this section we give some basic notions of piecewise linear topology. We refer to Hudson [10] and Rourke and Sanderson [18] for further study.

A polytopal complex $\mathcal{C}$ is a *k-sphere* or *k-ball* if its underlying space $\|\mathcal{C}\| \subseteq \mathbb{R}^d$ is homeomorphic to the standard k-sphere $\{x \in \mathbb{R}^{k+1} : \sum_{i=1}^{k} x_i^2 = 1\}$ or k-ball $\{x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i^2 \leq 1\}$, respectively.

We define PL spheres for polytopal complexes. This is connected to the definition of a PL sphere as a simplicial complex, since the barycentric subdivision of a polytopal complex is a simplicial complex. The underlying space of the barycentric subdivision of a polytopal complex is the underlying space of the complex itself and they are trivially homeomorphic. The manifolds we consider may have a boundary.

Definition 1.12. A *PL k-sphere* is a polytopal complex whose underlying space is piecewise linearly homeomorphic to the boundary of the $(k+1)$-simplex. A *combinatorial k-manifold* (or *PL k-manifold*) is a polytopal complex such that each vertex has a neighborhood PL-homeomorphic to a k-simplex, or equivalently, every vertex figure of the barycentric subdivision is a PL $(k-1)$-sphere or PL $(k-1)$-ball.

A cubical or simplicial complex is a combinatorial k-manifold if every vertex figure is either a PL $(k-1)$-sphere or PL $(k-1)$-ball.

A PL k-sphere is *polytopal* (or *realizable*) if there exists a $(k+1)$-polytope whose boundary complex is combinatorially isomorphic to $\mathcal{C}$. Here polytopal means that there exists a convex realization of the polytopal complex in dimension $k+1$, whereas our definition of polytopal complex only implies a (not necessarily convex) realization in arbitrary dimension.

For example, every k-dimensional simplicial complex on n vertices may be realized in $\mathbb{R}^{2k+1}$ as a subcomplex of the Schlegel diagram of the $(2k+2)$-dimensional cyclic polytope on n vertices. But there exist non-polytopal simplicial PL k-spheres meaning that there is no realization as the boundary of a $(k+1)$-polytope. For example, one of the Altshuler spheres N^{10}_{425} is a non-polytopal simplicial PL 3-sphere (cf. Altshuler [1] and Bokowski and Garms [5] where N^{10}_{425} is denoted M^{10}_{425}).

We paraphrase Lutz [13, p. 9] to explain a subtlety working with PL spheres: For all $d \neq 4$ a combinatorial d-manifold which is a d-sphere is a PL d-sphere. In dimension $d \leq 3$ this follows from work by Moise and for $d \geq 5$ from work by Kirby and Siebenmann. In dimension $d = 4$ this is an open problem.

Chapter 2

Simplicial complexes

In this chapter we deal with simplicial complexes which may be looked at in two different ways: geometric and abstract.

A *geometric* simplicial complex is a polytopal complex whose faces are simplices, i.e. the faces are usually given as the convex hull of vertices in some $\mathbb{R}^d$.

An *abstract* simplicial complex on n vertices is a non-empty collection of subsets of $[n]$ closed under taking subsets. Thus the smallest abstract simplicial complex contains the empty set only.

It is obvious that every geometric simplicial complex yields an abstract simplicial complex by forgetting about the coordinates of the vertices. The converse is also true since a simplicial complex Σ on n vertices may be interpreted as a subcomplex of the $(n-1)$-dimensional simplex. Therefore we do not distinguish between abstract and geometric simplicial complexes.

In the following faces of Σ are represented either by vertex(-index) sets $F = \{i_1, \dots, i_k\} \subseteq [n]$ or by vectors $\widetilde{F} = (b_1, \dots, b_n) \in \{0,1\}^n$ with

$$b_i = \begin{cases} 0 & \text{if } i \in F, \\ 1 & \text{if } i \notin F. \end{cases} \tag{2.1}$$

Note that this is exactly the opposite of the usual incidence matrix notation. Thus, if a vector contains $d+1$ zeros, then the corresponding face is a d-simplex. For ease of notation we call both representations F. We use the above notation which is the same as in the article on neighborly cubical spheres by Babson, Billera and Chan [3], because it simplifies the combinatorial definition of the mirror complex in Section 3.3.

The faces of a simplicial complex form a poset. The partial ordering on the vector representation is $1 < 0$ extended componentwise. This is induced

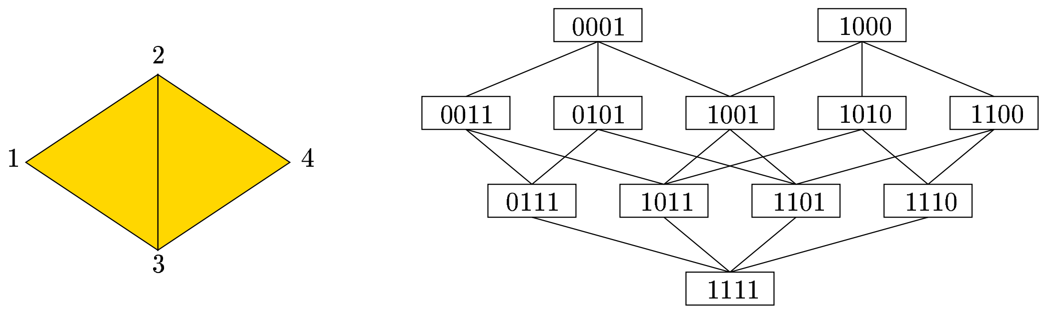


Figure 2.1: A simplicial complex and the corresponding poset

by the usual inclusion on the vertex sets. As in Figure 2.1 the minimal element is the empty set, represented by $(1, \dots, 1)$.

2.1 Simplicially neighborly spheres and polytopes

We start off with the definition of simplicial neighborliness and then present some facts motivating this definition.

Definition 2.1. A polytopal complex $\mathcal{P}$ is *simplicially k-neighborly* if every k-subset of its vertices is a face: its $(k-1)$-skeleton is combinatorially isomorphic to the $(k-1)$-skeleton of a simplex.

A PL $(d-1)$-sphere is *simplicially neighborly* if it is simplicially $\lfloor d/2 \rfloor$-neighborly.

A d-polytope is *simplicially k-neighborly* resp. *simplicially neighborly* if its boundary complex is simplicially k-neighborly resp. simplicially neighborly.

The simplicially neighborly polytopes are often only called neighborly polytopes. As we introduce cubically neighborly polytopes in Section 3.2 we add simplicially or cubically in case of ambiguity.

The above definition is motivated by a result of Flores and van Kampen, see Grünbaum [9, Theorem 11.1.3]. It is a topological obstruction to the realizability of the skeleton of a simplex.

Theorem 2.2 (Flores-van Kampen). *The k-skeleton of the $(2k+2)$-simplex is not homeomorphic to any subset of $\mathbb{R}^{2k}$.*

The following corollary explains the choice of parameters in the definition of neighborliness for PL spheres.

Corollary 2.3. *A PL $(d-1)$-sphere S cannot have the $\lfloor d/2 \rfloor$-skeleton of the n-simplex for $n \geq d+1$.*

Proof. Assume S has the $\lfloor d/2 \rfloor$-skeleton of the n-simplex for some $n \geq d+1$. Since the $\lfloor d/2 \rfloor$-skeleton of the $(d+1)$-simplex is included in the $\lfloor d/2 \rfloor$-skeleton of the n-simplex, S contains the $\lfloor d/2 \rfloor$-skeleton of the $(d+1)$-simplex.

Let $p \in S$ be a point in S that is not contained in the $\lfloor d/2 \rfloor$-skeleton of S. Then $S \setminus \{p\}$ is homeomorphic to $\mathbb{R}^{d-1}$ and contains a subset which is homeomorphic to the $\lfloor d/2 \rfloor$-skeleton of the $(d+1)$-simplex.

If $d = 2k$ is even then $\lfloor d/2 \rfloor = k$ and we obtain a subset of $\mathbb{R}^{2k-1}$ which is homeomorphic to the k-skeleton of the $(2k+1)$-simplex. By taking the cone over this subset we get a subset of $\mathbb{R}^{2k}$ which is homeomorphic to the k-skeleton of the $(2k+2)$-simplex. This contradicts the Flores-van Kampen theorem.

If $d = 2k+1$ is odd then $\lfloor d/2 \rfloor = k$ and we get a subset of $\mathbb{R}^{2k}$ which is homeomorphic to the k-skeleton of the $(2k+2)$-simplex. This is also a contradiction to the Flores-van Kampen theorem. $\square$

Simplicially neighborly vs. neighborly simplicial

There exists a difference between simplicially neighborly and neighborly simplicial polytopes. As we show in the following a simplicially neighborly polytope need not be simplicial itself. Obviously a simplicial polytope need not be neighborly at all. The next proposition shows that in even dimension all simplicially neighborly polytopes are indeed simplicial.

Proposition 2.4. *If P is a simplicially k-neighborly d-polytope then every r-face for $r \leq 2k-1$ is a simplex.*

Proof. Let F be an r-face for $r \leq 2k-1$. Assume F is not a simplex and thus has at least $r+2$ vertices $v_1, \ldots, v_{r+2}$. Since F has dimension r there exists an affine dependence, i.e. $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, r+2$ not all zero such that

$$\sum_{i=1}^{r+2} \lambda_i v_i = 0 \quad \text{with} \quad \sum_{i=1}^{r+2} \lambda_i = 0.$$

Now partition the index set into $\Lambda^0 = \{i : \lambda_i = 0\}$, $\Lambda^+ = \{i : \lambda_i > 0\}$ and $\Lambda^- = \{i : \lambda_i < 0\}$ with $L := \sum_{i \in \Lambda^+} \lambda_i = \sum_{i \in \Lambda^-} -\lambda_i$. Then the intersection $\text{conv}(\{v_i : i \in \Lambda^+\}) \cap \text{conv}(\{v_i : i \in \Lambda^-\})$ is not empty because

$$\sum_{i \in \Lambda^+} \frac{\lambda_i}{L} v_i = \sum_{i \in \Lambda^-} -\frac{\lambda_i}{L} v_i.$$

Thus neither $\{v_i : i \in \Lambda^+\}$ nor $\{v_i : i \in \Lambda^-\}$ is a proper face of P. But either Λ^+ or Λ^- has cardinality at most $\lfloor (r+2)/2 \rfloor \leq k$ which contradicts the fact that P is k-neighborly. □

The above proposition implies that $\lfloor d/2 \rfloor$ is the right threshold for the neighborliness of polytopes.

Corollary 2.5. *If P is a $(\lfloor d/2 \rfloor + 1)$-neighborly d-polytope then P is a simplex.*

In odd dimension not all simplicially neighborly polytopes are simplicial: Let P be a $2k$-dimensional neighborly simplicial polytope, i.e. with the $(k-1)$-skeleton of the n-simplex for some $n > 2k$. Then the pyramid over P has the $(k-1)$-skeleton of the $(n+1)$-simplex, since all $(k-1)$-faces that do not contain the apex are in $P \subseteq \text{pyr}(P)$ and the join of all $(k-2)$-faces with the apex are in $\text{pyr}(P)$. The resulting polytope is still neighborly because $\lfloor (2k+1)/2 \rfloor - 1 = k-1$ but it has a face isomorphic to P. Hence if P is not a simplex then $\text{pyr}(P)$ is a neighborly polytope that is not simplicial.

In dimensions two and three being neighborly is having the 0-skeleton of a simplex. So any n-gon or 3-polytope with n vertices is simplicially neighborly, but of course not all 3-polytopes are simplicial. In dimension higher than three to construct neighborly polytopes is not that obvious.

One special family of neighborly simplicial polytopes, the cyclic polytopes, is discussed in Section 2.2. A non-cyclic neighborly simplicial polytope in dimension 4 on 8 vertices was constructed by Grünbaum [9, Thm. 7.2.4]. These are the smallest interesting parameters because every neighborly simplicial $2n$-polytope on at most $2n+3$ vertices is combinatorially isomorphic to the corresponding cyclic polytope (cf. Gale [7]).

In the construction of Section 3.6.2 we need polytopes obtained by taking subsets of the vertices of neighborly simplicial polytopes. These are described by the next proposition.

Proposition 2.6. *The convex hull of a subset of the vertices of a k-neighborly d-polytope of cardinality at least $k \geq d+1$ is k-neighborly.*

Further, for every k-neighborly simplicial d-polytope P there exists a realization such that every subset of cardinality at least $k \geq d+1$ of the vertices of P are the vertices of a k-neighborly simplicial polytope.

The first part of the proposition is due to the fact that the hyperplanes supporting the $(k-1)$-faces of the original polytope are also supporting hyperplanes of the $(k-1)$-faces of the polytope obtained from the subset of the vertices.

In the second part of the proposition the d-polytope is simplicial. Then the vertices may be perturbed in a way that they are in general position, i.e. no $d+1$ vertices lie on one hyperplane. Thus the convex hull of a subset of the vertices of this perturbed realization is a simplicial polytope.

2.2 Cyclic polytopes

Boundary complexes and triangulations of cyclic polytopes are important classes of simplicial complexes needed in the later construction.

Definition 2.7. Let $t_1 < t_2 < \ldots < t_n$ be a sequence of n increasing real numbers and the moment curve given by

$$\begin{aligned} \gamma_d : \mathbb{R} &\to \mathbb{R}^d, \\ t &\mapsto \gamma_d(t) = (t, t^2, \ldots, t^d). \end{aligned}$$

The *d-dimensional cyclic polytope* $\mathrm{cyc}_d(n)$ on n vertices is any polytope combinatorially isomorphic to the convex hull of the vertices $v_i = \gamma_d(t_i)$ for $i = 1, \ldots, n$.

In the following the vertices of the cyclic polytope are in 'increasing' order if no other ordering is explicitly mentioned. In even dimension there is a more symmetric curve yielding the same polytope due to Carathéodory

$$\begin{aligned} \gamma'_d : \mathbb{R} &\to \mathbb{R}^d \qquad (d = 2k, k \in \mathbb{N}) \\ t &\mapsto \gamma'_d(t) = (\sin(t), \cos(t), \sin(2t), \cos(2t), \ldots, \sin(\frac{d}{2}t), \cos(\frac{d}{2}t)) \end{aligned}$$

To get a 'most' symmetric cylic polytope the vertices are $v_i = \gamma'_d((i-1)\frac{2\pi}{n})$ for $i = 1, \ldots, n$, where most symmetric means, that in this realization all combinatorial automorphism correspond to congruences, see Kaibel and Waßmer [12]. In particular, this gives the regular n-gons $\mathrm{cyc}_2(n)$ in dimension two.

The combinatorial structure of the cyclic polytope is given by its facets described by the following theorem due to Gale [7]. A very nice proof is found in Ziegler [19].

Theorem 2.8 (Simplicial Gale Evenness Condition). *Let the vertices of* $\mathrm{cyc}_d(n)$ *be denoted by their indices* $i \in [n] = \{1, \ldots, n\}$. *A d-subset* $F \subseteq [n]$ *is a facet of* $\mathrm{cyc}_d(n)$ *if and only if*

$$2 \,\big|\, \#\{k : k \in F, i < k < j\} \qquad \text{for } i, j \in [n] \setminus F, \text{ with } i < j,$$

that is, for any $i, j \in [n] \setminus F$ *with* $i < j$ *the number of* $k \in F$ *between* i *and* j *is even.*

In the notation of Equation (2.1) this corresponds to an even number of zeros between any two ones in the vector representation of the facets of the cyclic polytope. This combinatorial description depends only on the ordering of the t_i's and not on the exact value. Thus the cyclic polytope is often defined as the convex hull of the vertices $v_i' = \gamma_d(i)$ for $i = 1, \ldots, n$. The proof of Theorem 2.8 also yields the neighborliness of cyclic polytopes.

Corollary 2.9. *The cyclic polytope* $\mathrm{cyc}_d(n)$ *is a neighborly simplicial polytopes, that is, any subset* $S \subseteq [n]$ *of* $|S| \leq \lfloor \frac{d}{2} \rfloor$ *vertices forms a face.*

Automorphism group

In the construction of Section 3.6.2 we need the pulling triangulation with respect to any given vertex of the cyclic polytope. The tool we use to analyze the triangulations is the automorphism group of a polytope.

The automorphism group of the cyclic polytopes is well understood and is characterized by the following theorem taken from Kaibel and Waßmer [12, Theorem 8.3].

Theorem 2.10. *The automorphism groups of the cyclic polytope of dimension d on n vertices for* $2 \leq d < n$ *are*

	$n = d+1$	$n = d+2$	$n \geq d+3$
d even	$\mathbb{S}_n$	$\mathbb{S}_{n/2} \operatorname{wr}_{[2]} \mathbb{Z}_2$	$\mathbb{D}_n$
d odd	$\mathbb{S}_n$	$\mathbb{S}_{\lceil n/2 \rceil} \times \mathbb{S}_{\lfloor n/2 \rfloor}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

In particular, for even dimension the automorphism group acts transitive by the vertices and thus has one orbit.

For odd dimension the automorphism group does not act transitive by the vertices, except for the simplex. For $n = d+2$ *the automorphism group action has two orbits; for* $n \geq d+3$ *it has* $\lceil n/2 \rceil$ *orbits.*

Kaibel and Waßmer show even more, namely that the automorphism group is isomorphic to the symmetry group, i.e. the cyclic polytopes may be realized in a way that for each combinatorial isomorphism there exists an euclidean motion mapping the polytope bijectively onto itself.

Pulling triangulations

The pulling triangulation is a special triangulation of a simplicial polytope with respect to one of its vertices. The geometric intuition is as follows (cf. Figure 2.2): Let P be a simplicial d-polytope embedded in $\mathbb{R}^d$ and v one of

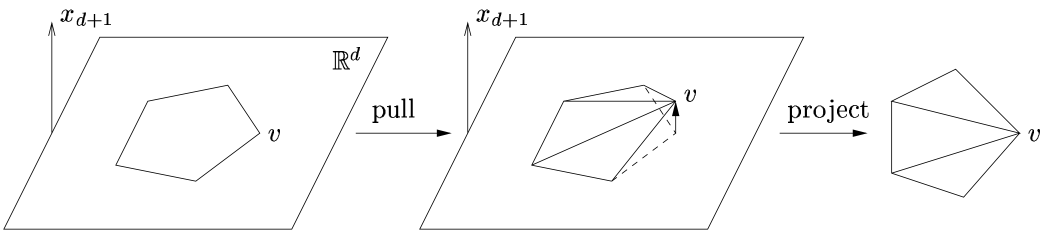


Figure 2.2: Geometric description of the pulling triangulation: First embed the d-polytope into $\mathbb{R}^{d+1}$, then *pull* a vertex and take the projection of the upper convex hull.

its vertices. Now put P into the hyperplane $x_{d+1} = 0$ in $\mathbb{R}^{d+1}$, *pull* the vertex v a little bit in positive x_{d+1} direction and take the convex hull. The pulling triangulation corresponds to the orthogonal projection of the upper hull back into $x_{d+1} = 0$, which is a subdivision of P.

The simplices of this triangulation are pyramids with apex v over the facets F of $\operatorname{conv}(\operatorname{vert}(P) \setminus \{v\})$ that 'cannot be seen' from v, i.e. v lies in the interior of the halfspace supporting the facet F.

It may also be described in a purely combinatorial way. Let B be a ball whose boundary is a simplicial PL sphere S with a vertex v. The simplices of the pulling triangulation of B with respect to v are $\{v * F : F \in \operatorname{del}(S, v)\}$.

For ease of notation we refer to the vertices of the standard cyclic polytope $v_i = \gamma_d(i)$ by their index. One special case we need later is the pulling triangulation of the cyclic d-polytope on n vertices with respect to the vertex n. The facets of $\operatorname{cyc}_d(n-1)$ which cannot be seen by the vertex n are facets of $\operatorname{cyc}_d(n-1)$ that are also facets of $\operatorname{cyc}_d(n)$. Combinatorially speaking, we take the facets of $\operatorname{cyc}_d(n-1)$ with an even number of final zeros, because they are facets of $\operatorname{cyc}_d(n)$ too, and will therefore not be covered. In other words the facets of the pulling triangulation of the cyclic polytope $\operatorname{cyc}_d(n)$ are the facets of $\operatorname{cyc}_{d+1}(n)$ ending with an odd number of zeros.

Example. Take a look at the cyclic polytope $\operatorname{cyc}_2(5)$ – a pentagon. Its facets are

$$\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}.$$

Denoted as a subcomplex of the 4-simplex the facets correspond to the following vectors in $\{0,1\}^5$:

$$(00111), (10011), (11001), (11100), (01110).$$

The simplices of the pulling triangulation are (see Figure 2.3):

$$(0011\,0), (1001\,0), (1100\,0).$$

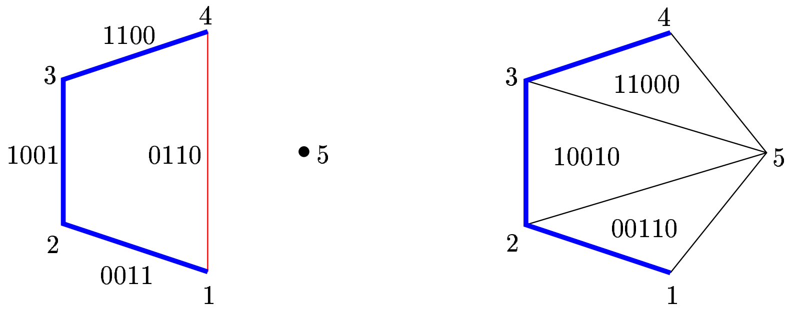


Figure 2.3: Pulling triangulation of the cyclic polytope $\text{cyc}_2(5)$ with 0/1-incidence vectors encoded as in Equation (2.1). The faces of $\text{cyc}_2(4)$ which cannot be seen from vertex number 5 are the thick lines.

These are exactly the facets of $\text{cyc}_2(4)$ with an even number of final zeros, with a zero appended, that is the facets of $\text{cyc}_3(5)$ ending with an odd number of zeros.

The next corollary is a consequence of the fact that by Theorem 2.10 the automorphism group of the cyclic polytopes in even dimension acts transitive by the vertices.

Corollary 2.11. *All pulling triangulations of the* ***even*** *dimensional cyclic d-polytope on n vertices are combinatorially equivalent for $d \geq 2$.*

Thus in *even* dimension the pulling triangulation does not depend on the choice of the vertex you pull. For the cyclic polytope of *odd* dimension d on $d+2$ vertices there exist exactly two different pulling triangulations. For $n \geq d+3$ vertices the orbit of each vertex with respect to the symmetry group action contains exactly two elements if n is even. If n is odd there exists one fixed vertex.

Corollary 2.12. *The number of pulling triangulations of the* ***odd*** *dimensional cyclic d-polytope on n vertices for $d \geq 3$ is*

number of vertices	$n = d+1$	$n = d+2$	$n \geq d+3$
number of pulling triangulations	1	2	$\lceil n/2 \rceil$

This may already be seen in dimension three on the cyclic polytope on five vertices which we analyze in the next example.

Example. Let us have a close look at the pulling triangulations of the cyclic 3-polytope on 5 vertices. First of all notice that this cyclic polytope is combinatorially equivalent to a bipyramid over a triangle shown in Figure 2.4.

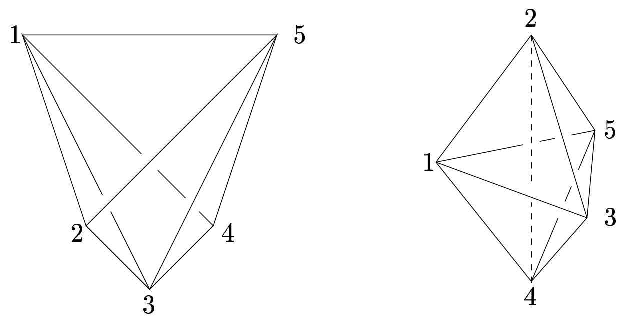


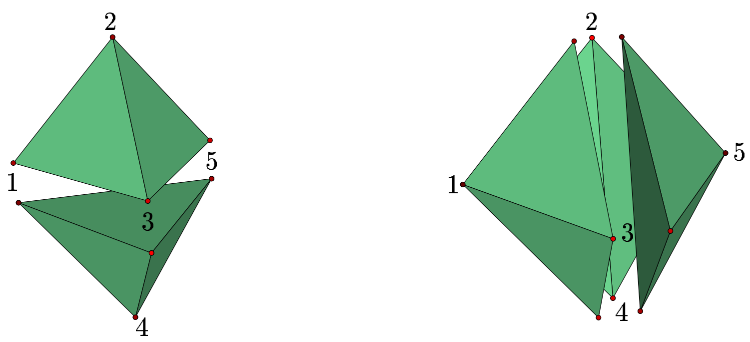
Figure 2.4: Two realizations of the cyclic 3-polytope on five vertices

By Theorem 2.10 the symmetry group of $\mathrm{cyc}_3(5)$ is $\mathrm{Aut}(\mathrm{cyc}_3(5)) = \mathbb{S}_3 \times \mathbb{S}_2$. The $\mathbb{S}_3$ component of the product permutes the vertices labelled with odd numbers, not changing the even vertices. The $\mathbb{S}_2$ component permutes the vertices labelled with even numbers, not changing the odd vertices.

It follows that the five different pulling triangulations which were a priori possible by taking each of the five vertices may be partitioned into two classes. The two classes correspond to the pulling triangulations with respect to an even or an odd vertex. The simplices of the corresponding triangulations are

- **apex** 1 **(odd case):** $\{1,2,3,5\},\{1,3,4,5\}$
- **apex** 2 **(even case):** $\{1,2,3,4\},\{2,3,4,5\},\{1,2,4,5\}$

So there exist two combinatorially different triangulations shown in Figure 2.5.



(a) odd case (b) even case

Figure 2.5: The two different pulling triangulations of the cyclic 3-polytope on 5 vertices

Chapter 3

Cubical complexes

In this chapter we start with a combinatorial description of cubes. Then the definition of cubically neighborly spheres and polytopes is given and discussed. Further we establish cubical analogs of the simplicial case studied in Section 2.1. In Sections 3.3 and 3.4 two construction techniques are discussed:

- *mirror complexes* or *mirroring* which produces cubical subcomplexes of high dimensional cubes for a given simplicial complex, and
- *fissuring* which modifies cubical complexes.

The main part of this chapter is an analysis of the construction of neighborly cubical spheres due to Babson, Billera, and Chan [3]. First we present the neighborly cubical polytopes constructed by Joswig and Ziegler [11]. Then we extend the construction of Babson, Billera, and Chan to neighborly increasing sequences defined in Section 3.6.1 and deduce a closed formula for the combinatorics of the spheres.

At the end of this chapter an easy way to realize surfaces of 'unusually large genus', as they were called by McMullen, Schulz and Wills [14], is presented.

3.1 Cubes

We have a look at cubes in a mostly combinatorial way. It is explicitly mentioned if a special realization of the cube is meant, e.g. the standard cube as defined in Definition 3.1. Otherwise we refer to a cube as all polytopes that are combinatorially equivalent to the standard cube. Figure 3.1 shows two different realizations of a cube: One is a standard 3-cube and the other is a very special realization with all opposite facets orthogonal two each other constructed by Ziegler [20] known as the Sharir Cube.

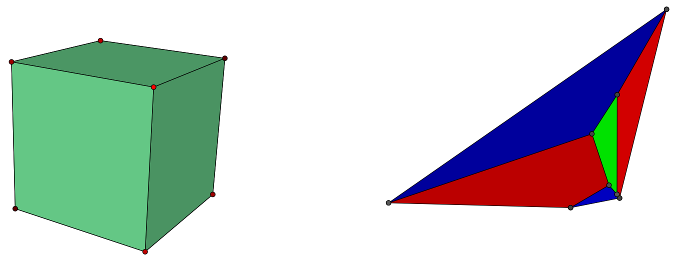


Figure 3.1: Two geometric realizations of the 3-cube. The standard cube (left) and the Sharir cube (right)

In our case the following definition is the most convenient.

Definition 3.1. The *d-dimensional standard cube* C_d is defined as the intersection of the following $2d$ halfspaces:

$$H_i^+ = \{x \in \mathbb{R}^d : +x_i \leq 1\}$$
$$H_i^- = \{x \in \mathbb{R}^d : -x_i \leq 1\} \qquad \text{for } i = 1, \dots, d$$

The facets of the d-dimensional cube are $F_i^+ = \{x \in \mathbb{R}^d : x_i = +1\}$ and $F_i^- = \{x \in \mathbb{R}^d : x_i = -1\}$ for $i = 1, \dots, d$. They may also be represented by vectors $F_i^\pm = (h_1, \dots, h_d) \in \{0, \pm 1\}^d$ with:

$$h_j = \begin{cases} \pm 1 & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

As the face lattice is coatomic, any non empty face of the d-cube may also be written as vector in $\{0, \pm 1\}^d$ representing the facets in which the face is included (cf. Figure 3.2). For ease of notation we often omit the 1 of ± 1. Since opposite facets are parallel, a face cannot lie in two opposite facets and thus have a $+1$ and a -1 at the same position of the vector. The dimension of a face is the number of zeros in its vector representation.

The labelling of the faces of the n-cube with vectors in $\{0, \pm 1\}^n$ induces a labelling of the vertices contained in a face. An easy computation yields a binary representation of the vertices of a face. A vertex of the d-cube $v \in \{0, \dots, 2^d - 1\}$ given as a binary number $v = \sum_{i=0}^{d-1} b_i 2^i$ lies in a face $G = (h_1, \dots, h_d)$ if and only if:

$$b_i = \begin{cases} 1 & \text{if } h_{d-i} = +1 \\ 0 & \text{if } h_{d-i} = -1 \\ 0 \text{ or } 1 & \text{if } h_{d-i} = 0. \end{cases}$$

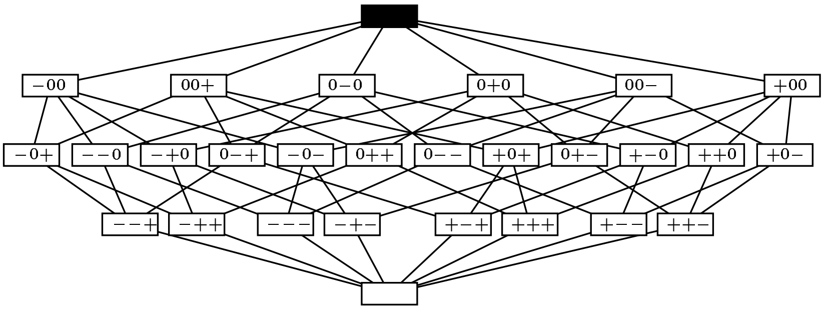


Figure 3.2: Face lattice of a 3-cube

In other words, the binary representation of a vertex must have a 1 resp. 0 where the vector representation of the face has a +1 resp. −1. At the positions of the 0's in the label of the face, the vertex may have either a 1 or a 0. Take a look at the combinatorics of an easy example.

Example. Let $d = 3$. The vector representation of the facets of the standard 3-cube with the 1's omitted is:

$$(-00), (+00), (0-0), (0+0), (00-), (00+).$$

The vertices of these facets are (cf. Figure 3.3):

$$\{0, 1, 2, 3\}, \{4, 5, 6, 7\}, \{0, 1, 4, 5\}, \{2, 3, 6, 7\}, \{0, 2, 4, 6\}, \{1, 3, 5, 7\}.$$

The edge $(-+0)$ contains the vertices 2 and 3:

face	−	+	0
vertex 2	0	1	0
vertex 3	0	1	1

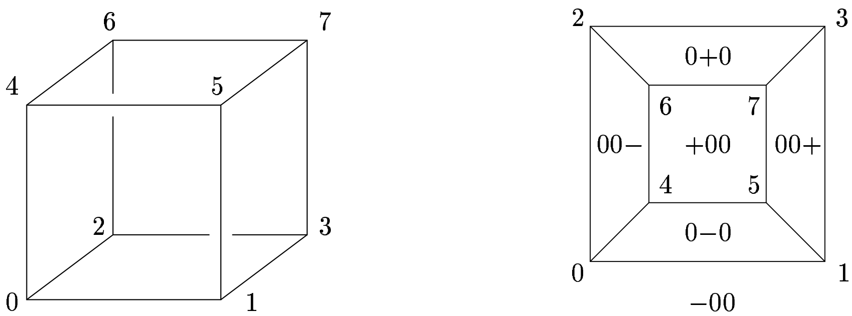


Figure 3.3: The combinatorics of the 3-cube and its Schlegel diagram with facet labels. The label of the projection facet is −00.

An edge $\{1,7\}$ does not exist since $1 = 001_2$ and $7 = 111_2$ differ in more than one bit and thus do not match any $\{0,\pm 1\}$-mask with only one zero representing an edge. In this encoding of the cube an edge corresponds to one bit-flip in the binary representation of the vertices. Similarly, a k-face is produced by k bit-flips at the positions of the zeros in the vector representation.

Automorphism group

In Section 3.3 we analyze the automorphism group of subcomplexes of the cube which have the same graph as the cube. The automorphism group of the graph of the cube is the semidirect product of a vertex transitive part and a vertex stabilizer.

In the combinatorial description, the vertices of the n-cube are given by vectors in $\{\pm 1\}^n$. The operation of the two parts of the automorphism group on the vertices is very easy to describe:

- the elements of $\mathbb{Z}_2^n$ correspond to flipping the ± 1 at a given position;
- the elements of $\mathbb{S}_n$ permute the entries of the vector representation.

These also characterize the automorphism group of the graph of the n-cube.

Proposition 3.2. *The automorphism group of the graph of the n-cube is*

$$\mathrm{Aut}(\Gamma(\mathrm{C}_n)) \cong \mathbb{Z}_2^n \rtimes \mathbb{S}_n = \mathbb{Z}_2 \,\mathrm{wr}_{[n]}\, \mathbb{S}_n.$$

With Lemma 1.11 we obtain that the automorphism group of the cube is isomorphic to a subgroup of the automorphism group of the graph of the cube. But an easy computation yields that all automorphisms of the graph of the cube are also automorphisms of the cube itself since they bijectively map the facets of the cube onto themselves.

Corollary 3.3. *The symmetry group of the n-dimensional cube C_n is*

$$\mathrm{Aut}(\mathrm{C}_n) \cong \mathbb{Z}_2 \,\mathrm{wr}_{[n]}\, \mathbb{S}_n.$$

3.2 Neighborliness

In this section we try to establish results for cubical polytopes and sphere similar to the ones presented in Section 2.1 for simplicially neighborly polytopes and spheres.

Similar to simplicially neighborly complexes there is a notion of cubically neighborly complexes. As in the simplicial case, the cubically neighborly polytopes have a skeleton of a higher dimensional cube.

Definition 3.4. A polytopal complex is *cubically k-neighborly* if its $(k-1)$-skeleton is combinatorially isomorphic to the $(k-1)$-skeleton of a cube.

A PL $(d-1)$-sphere is *cubically neighborly* if it is $\lfloor \frac{d}{2} \rfloor$-neighborly.

A d-polytope is *cubically k-neighborly* if its boundary complex is k-neighborly. It is *cubically neighborly* if it has the $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton of the n-cube for some $n \geq d$.

The topological obstruction to the realizability of skeleta of simplices given by the Flores-van Kampen theorem can be translated to the cubical case with the following theorem [9, Theorem 11.1.1 & 11.1.2]

Theorem 3.5. *The boundary complex of every polytope is a refinement of the boundary complex of the simplex. In particular, the k-skeleton of a d-polytope contains a refinement of the k-skeleton of the d-simplex.*

This implies an obstruction for the realizability of the skeleta of cubes.

Proposition 3.6. *The k-skeleton of the $(2k+2)$-dimensional cube C_{2k+2} cannot be realized in $\mathbb{R}^{2k}$.*

As for simplicial spheres this gives rise to the choice of parameters in the definition of neighborly cubical spheres by the following corollary.

Corollary 3.7. *A PL $(d-1)$-sphere which is not combinatorially equivalent to the boundary of the d-cube cannot have the $\lfloor d/2 \rfloor$-skeleton of the n-cube for $n \geq d+1$.*

By using the prism construction instead of the cone the proof is exactly the same as in the simplicial case.

Cubically neighborly vs. neighborly cubical

As in the simplicial case, there is a difference between cubically neighborly and neighborly cubical polytopes. To establish the analogies we need some basic facts on cubical polytopes and cubes.

Proposition 3.8 (Blind & Blind [4]). *Every cubical d-polytope has at least 2^d vertices. If it has exactly 2^d vertices, then it is a cube.*

This implies by induction

Corollary 3.9. *If all k-faces of a d-polytope P have 2^k vertices for all $1 \leq k \leq d-1$ then P is cubical. If further P has 2^d vertices then P is a d-cube.*

The following proposition is a cubical analog of Proposition 2.4 which describes the faces up to a certain dimension of simplicially k-neighborly polytopes.

Proposition 3.10. *If P is a cubically k-neighborly d-polytope with $k < d$ then every r-face for $r \leq 2k-2$ is a cube.*

To prove this proposition we need the following two Lemmas.

Lemma 3.11. *Every simple cubical d-polytope is a polygon or a d-cube.*

Lemma 3.12. *Let P be a cubically k-neighborly polytope. Then for each vertex v the vectors from v to any $k-1$ neighbors span a $(k-1)$-face of P.*

Proof. The vertex figure of a cubically k-neighborly polytope is a simplicially $(k-1)$-neighborly polytope. Let v be a vertex of P. Then the vectors from v to any $k-1$ of its neighbors match a $k-1$ subset of the vertices in the vertex figure P/v. Since P/v is $(k-1)$-neighborly, these form a $(k-2)$-face of P/v which corresponds to a $(k-1)$-face of P. □

Proof of Proposition 3.10. The proof proceeds by induction on r. First note that for $r = 0$ all r-faces (vertices) are 0-cubes. Now let F be a r-face of P with $r \leq 2k-2$. By induction hypothesis it follows that F is cubical. If F is simple, then F is a cube. So assume that F is not simple, i.e. contains a vertex v of degree larger than r. Then choose $r+1$ neighbors $v_1, \ldots, v_{r+1}$ of v in F. Since they all lie in F, the vector $\{v - v_1, \ldots, v - v_{r+1}\}$ must be linearly dependend. Thus there exists a subset of $\{v - v_1, \ldots, v - v_{r+1}\}$ of size at most $\lfloor (r+1)/2 \rfloor$ which does not span a face of F. But since $\lfloor (r+1)/2 \rfloor \leq \lfloor (2k-1)/2 \rfloor = k-1$ this contradicts the assumption that P is cubical k-neighborly. □

Comparing Propositions 2.4 and 3.10 shows that the result in the cubical case is a little bit weaker then in the simplicial case.

In *even* dimension all simplicially neighborly polytopes were simplicial but Proposition 3.10 only implies, that cubically neighborly $2k$-polytopes are $(2k-2)$-cubical. In dimension two any polytope is 1-cubical, but in dimension four there exists a cubically neighborly 4-polytope with the graph of the 5-cube which is not cubical. It was constructed by Joswig and Ziegler [11, Section 4.4], who gave explicit coordinates for such a polytope: it has the graph of the 5-cube but a facet with 12 vertices which is not a cube.

In *odd* dimension the construction similar to the simplicial case is taking the prism over a cubically neighborly polytope: Let P be a cubically neighborly $2k$-polytope for $k > 0$ and P' be the prism over P. Then the k-skeleton of P may be encoded by its vector representation in $\{0, \pm 1\}^n$ and contains all vectors $v \in \{0, \pm 1\}^n$ with at most k zeros. The k-skeleton of the prism P' may be represented by vectors $v' \in \{0, \pm 1\}^{n+1}$ with

$$v' \in \underbrace{\{(v, -1) : v \in (P)_k\}}_{\text{bottom}} \cup \underbrace{\{(v, 0) : v \in (P)_{k-1}\}}_{\text{middle}} \cup \underbrace{\{(v, +1) : v \in (P)_k\}}_{\text{top}}.$$

Thus the prism P' is also cubically neighborly, but it contains two facets that are combinatorially isomorphic to P. Consequently, P' is cubical if and only if P is a cube.

As in the simplicial case, the first interesting dimension is four, since being cubically 1-neighborly is equivalent to having 2^n vertices, i.e. the 0-skeleton of the n-cube for some $n \in \mathbb{N}$. So in dimension two or three every polytope on 2^n vertices is cubically neighborly.

3.3 Mirror complexes

The *mirroring* operation is a technique to get cubical complexes from simplicial complexes by prescribing vertex figures. It was already used by Coxeter [6] in 1936 who constructed regular skew polyhedra with skew polygons as vertex figures. This was generalized to a technique to construct a cubical complex with a given simplicial complex as vertex figure. Our combinatorial approach is taken from Babson, Billera, and Chan [3].

As mentioned in Chapter 2 a simplicial complex on n vertices may always be interpreted as a subcomplex of a $(n-1)$-dimensional simplex. The corresponding mirror complex is a subcomplex of the n-dimensional cube.

In general it is not true that any cubical complex is a subcomplex of a high dimensional cube. The cubical complex corresponding to a triangular prism without top and bottom triangle in Figure 3.4 is not the subcomplex of any cube since its graph is not bipartite. The graph of the cube is bipartite because in the labelling introduced above, every edge corresponds to a bit-flip. Thus it connects a vertex with an odd number of 1's to a vertex with an even number of 1's in their binary representation.

Geometric description

Since the vertex figure of the n-cube is an $(n-1)$-simplex, a simplicial complex on n vertices can be seen as a subcomplex of the vertex figure of the n-cube. If we inscribe the simplicial complex into one vertex figure of the standard cube, it may be 'mirrored' to all other vertices by reflecting it in all coordinate

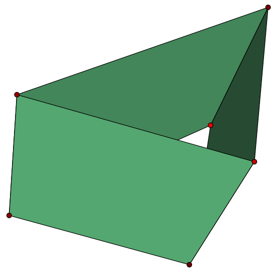


Figure 3.4: A cubical complex which is not the subcomplex of any cube.

hyperplanes. All k-faces of the simplicial complex correspond to $(k+1)$-faces of the cube. These induce a subcomplex of the cube which is the *mirror complex.*

Example. Take a look at the mirror complex of two edges on three vertices. In Figure 3.5 you can follow the construction of the mirror complex:

(a) the simplicial complex,

(b) the simplicial complex embedded in one vertex figure,

(c) the simplicial complex mirrored to all vertex figures, and

(d) its mirror complex.

Combinatorial description

The mirror complex also has a very nice description in a purely combinatorial way.

Definition 3.13. Let $\Sigma \subseteq \{0,1\}^n$ be a simplicial complex on n vertices encoded as in Equation (2.1). The mirror complex $\mathrm{M}(\Sigma)$ of Σ is defined by:

$$\mathrm{M}(\Sigma) = \{(a_1,\dots,a_n) \in \{0,\pm 1\}^n : (|a_1|,\dots,|a_n|) \in \Sigma\} \cup \{\emptyset\}.$$

The facets of the mirror complex form a connected subcomplex of a cube. Thus the mirror complex is a cubical complex. It is connected since all edges of the cube are contained in the mirror complex since they correspond to the vertices of the simplicial complex.

The ordering of the poset of the mirror complex is induced by the ordering of the simplicial poset, i.e. $-1 < 0$ and $+1 < 0$ extended componentwise.

The vertex figure at a vertex of a cubical complex is the interval lying above it in the poset. Each vertex of the mirror complex matches a vector

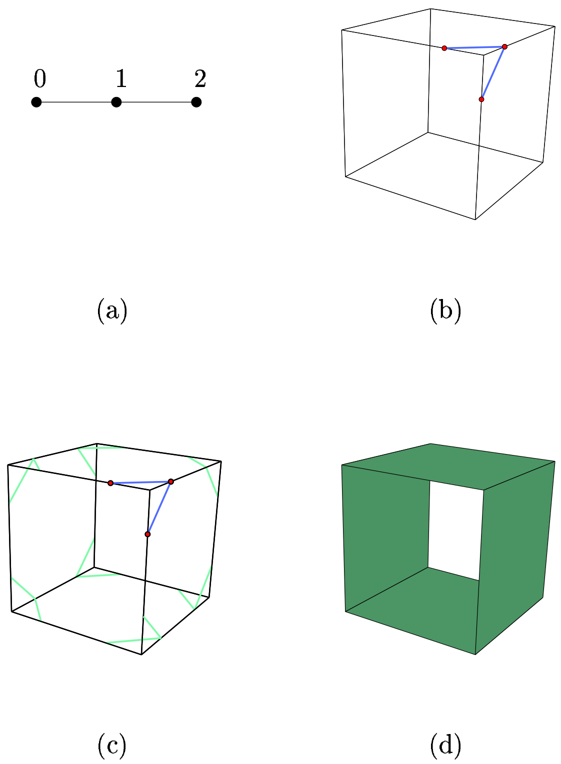


Figure 3.5: The mirror complex of two edges in a 3-cube

in $\{\pm 1\}^n$. All faces containing this vertex must therefore have the same signs as the vertex at all positions. Thus the vertex figure is combinatorially isomorphic to the original simplicial complex, up to signs.

It follows that if Σ is a simplicial PL $(d-1)$-sphere then $\mathrm{M}(\Sigma)$ is a closed connected cubical combinatorial d-manifold. Furthermore, the mirror complex of the boundary of a simplicial combinatorial manifold $\mathrm{M}(\partial\Sigma)$ is the boundary of its mirror complex $\partial\,\mathrm{M}(\Sigma)$ as shown in Figure 3.6.

The f-vector of the mirror complex $f(\mathrm{M}(\Sigma))$ may easily be computed from the f-vector $(f_0, \dots, f_{n-1})$ of Σ. A k-face of Σ contains $k+1$ vertices. Its vector representation thus has $k+1$ zeros and $n-k-1$ ones. By choosing a sign for every one in the vector, a k-face of Σ produces 2^{n-k-1} faces of dimension $k+1$ in the mirror complex

$$f_{k+1}(\mathrm{M}(\Sigma)) = 2^{n-k-1} f_k(\Sigma).$$

The correspondence between k-faces of the simplicial complex Σ and the

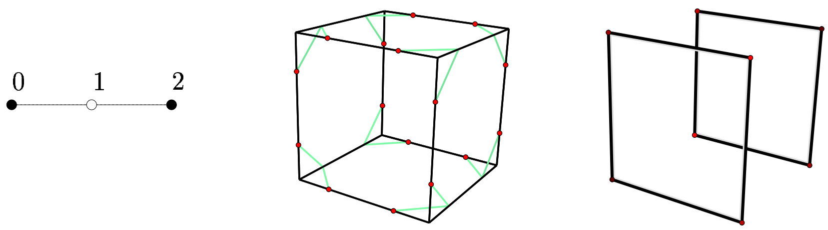


Figure 3.6: The boundary of the mirror complex is the mirror complex of the boundary.

$(k+1)$-faces of $\mathrm{M}(\Sigma)$ implies that if Σ has the k-skeleton of the $(n-1)$-dimensional simplex, then $\mathrm{M}(\Sigma)$ has the $(k+1)$-skeleton of the n-cube. This means that simplicial neighborliness of the simplicial complex becomes cubical neighborliness in the mirror complex. In particular, the mirror complex of the $(n-1)$-simplex and the boundary of the $(n-1)$-simplex are the n-cube and the boundary of the n-cube, respectively.

We summarize facts about the mirror complex in the following proposition.

Proposition 3.14. *The mirror complex* $\mathrm{M}(\Sigma)$ *of the simplicial complex* Σ *has the following properties:*

1. $\mathrm{M}(\Sigma)$ *is a cubical complex.*
2. *If* Σ *is a combinatorial manifold then mirroring commutes with the boundary operator:* $\partial\,\mathrm{M}(\Sigma) = \mathrm{M}(\partial\Sigma)$.
3. *The vertex figure of every vertex of the mirror complex is isomorphic to* Σ*. Hence the mirror complex of a simplicial PL* $(d-1)$*-sphere is a closed connected cubical combinatorial* d*-manifold.*
4. *The* f*-vector is* $f(\mathrm{M}(\Sigma)) = (2^n, 2^{n-1}f_0, 2^{n-2}f_1, \ldots, 2^1 f_{n-2}, f_{n-1})$.
5. *If* Σ *is simplicially* k*-neighborly, then its mirror complex* $\mathrm{M}(\Sigma)$ *is cubically* $(k+1)$*-neighborly.*

Example. Now take a look at the combinatorics of the mirror complex obtained from the simplicial complex consisting of two edges on three vertices. The vector representation of the complex is

$$\Sigma = \{(001), (100), (011), (101), (110), (111)\}.$$

The corresponding mirror complex is a subcomplex of the 3-cube. The mirror complex of Σ is

$$\begin{aligned}\mathrm{M}(\Sigma) = \{&(00-), (00+), (-00), (+00),\\ &(0--), (0-+), (0+-), (0++), (-0-), (-0+),\\ &(+0-), (+0+), (--0), (-+0), (+-0), (++0)\\ &(---), (--+), (-+-), (-++), (+--),\\ &(+-+), (++-), (+++), \emptyset\}.\end{aligned}$$

The posets of the two complexes are

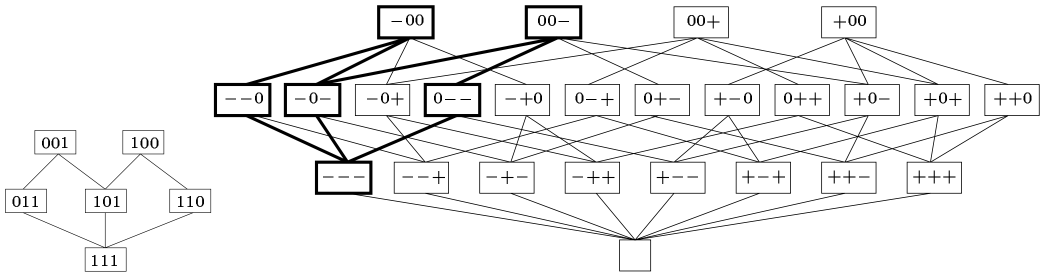


Figure 3.7: Posets of a simplicial complex and its mirror complex. The bold elements of the poset of the mirror complex form the vertex figure of the vertex 0. The induced poset is isomorphic to the poset of the simplicial complex.

The f-vector of Σ is $f(\Sigma) = (3, 2)$. The f-vector of the mirror complex is $f(\mathrm{M}(\Sigma)) = (8, 12, 4) = (2^3, 3 \cdot 2^2, 2 \cdot 2^1)$.

The simplicial complex Σ has the 0-skeleton of the 2-simplex: it contains all vectors in $\{0, 1\}^3$ with exactly one zero. Thus its mirror complex $\mathrm{M}(\Sigma)$ has the 1-skeleton of the 3-cube because it contains all vectors in $\{0, \pm 1\}^3$ with exactly one zero.

Automorphism group of the mirror complex

To analyze the automorphism group of the mirror complex we need the following purely algebraic lemma.

Lemma 3.15. *Let Ω be a set, G a group and H a subgroup of G acting transitively on Ω. Then G is generated by H and the stabilizer G_a of any element $a \in \Omega$, i.e. $G = \langle H, G_a \rangle$.*

Further, if H is a regular normal subgroup, then G is isomorphic to the semidirect product of H and G_a for any $a \in \Omega$ with respect to an appropriate representation $\varphi : G_a \to \mathrm{Aut}(H)$: $G = H \rtimes_\varphi G_a$.

Proof. Choose $a \in \Omega$. Let $g \in G$ and G_a be the stabilizer of a. Define $a' = ag$. Since H acts transitively on Ω there exists $h \in H$ such that $a'h = a$. Thus $agh = a'h = a$, i.e. gh lies in the stabilizer of a. Consequencely, G is generated by H and G_a with $g = (gh)h^{-1}$ and $gh \in G_a$ and $h^{-1} \in H$.

Now let H be a regular normal subgroup of G. Since H is a normal subgroup of G the following is a representation of G_a as a group of group automorphisms of H

$$\begin{aligned} \varphi : G_a &\to \mathrm{Aut}(H), \\ g &\mapsto \varphi_g \text{ with } \varphi_g(h) = g^{-1}hg. \end{aligned}$$

The semidirect product $H \rtimes_\varphi G_a$ has the following multiplication

$$(h_1, g_1)(h_2, g_2) = (\varphi_{g_2}(h_1)h_2, g_1g_2) = (g_2^{-1}h_1g_2h_2, g_1g_2)$$

for $h_1, h_2 \in H$ and $g_1, g_2 \in G_a$. Define the map ψ from the set product $H \times G_a$ to G

$$\begin{aligned} \psi : H \times G_a &\to G, \\ (h, g) &\mapsto gh. \end{aligned}$$

It remains to show that this map is indeed a group isomophism, i.e. ψ is injective and surjective and preserves multiplication.

Let $(h, g) \in H \times G_a$ with $ag = a$ and $\psi(h, g) = 1$. Then $gh = 1$ if and only if $g = h^{-1} \in H$. So $g \in G_a \cap H$. This implies that $g = 1$ since H is a regular normal subgroup and by definition there exists a unique element $h' \in H$ such that $ah' = a$ which must be equal to 1. Thus $h = g = 1$ and ψ is injective.

The map ψ is also surjective because for all $g \in G$:

$$\psi(h^{-1}, gh) = (gh)h^{-1} = g.$$

Finally, if $(h_1, g_1), (h_2, g_2) \in H \times G_a$ then

$$\begin{aligned} \psi\left((h_1, g_1)(h_2, g_2)\right) &= \psi\left(g_2^{-1}h_1g_2h_2, g_1g_2\right) \\ &= g_1g_2g_2^{-1}h_1g_2h_2 \\ &= g_1h_1g_2h_2 \\ &= \psi(h_1, g_1)\psi(h_2, g_2). \end{aligned}$$

Thus G is isomorphic to $H \rtimes_\varphi G_a$ for any $a \in \Omega$. □

As the mirror complex is produced using the reflection symmetry of the cube it has the following automorphism group.

Proposition 3.16. *Let* Σ *be a simplicial complex on* n *vertices. Then the automorphism group of its mirror complex is* $\mathrm{Aut}(\mathrm{M}(\Sigma)) \cong \mathbb{Z}_2 \operatorname{wr}_{[n]} \mathrm{Aut}(\Sigma)$.

Proof. The graph of the n-cube is the 1-skeleton of the mirror complex of Σ. Thus by Lemma 1.11 and Proposition 3.2 the automorphism group of the mirror complex is isomorphic to a subgroup H of the automorphism group of the graph of the cube

$$\mathrm{Aut}(\mathrm{M}(\Sigma)) \cong H \leq \mathbb{Z}_2 \operatorname{wr}_{[n]} \mathbb{S}_n \cong \mathrm{Aut}(\Gamma(\mathrm{C}_n)).$$

Further, the simplicial complex Σ is a subcomplex of the $(n-1)$-simplex on all vertices and thus its automorphism group is isomorphic to a subgroup of $\mathbb{S}_n$.

The action of $\mathrm{Aut}(\Sigma)$ on the vertices of the mirror complex corresponds to a permutation of entries of the vertices, i.e. for $s \in \mathrm{Aut}(\Sigma)$ and $v \in \{\pm 1\}^n$:

$$(v_1, \ldots, v_n)s = (v_{s^{-1}(1)}, \ldots, v_{s^{-1}(n)}).$$

An element $g \in G_v$ is fully determined by its action on the vertex figure at a vertex. Thus the vertex stabilizer G_v is isomorphic to the automorphism group $\mathrm{Aut}(\Sigma)$.

By construction, the group $\mathbb{Z}_2^n$ is a subgroup of G which acts transitively on the vertices by flipping the (± 1)-entries, i.e. for $r = (r_1, \ldots, r_n) \in \mathbb{Z}_2^n$ and $v \in \{\pm 1\}^n$:

$$(v_1, \ldots, v_n)r = ((-1)^{r_1} v_1, \ldots, (-1)^{r_n} v_n).$$

This gives a regular normal subgroup of G, since any two vertices determine a unique vector by its $\{\pm 1\}$-flips and for all $g \in \mathbb{Z}_2 \operatorname{wr}_{[n]} \mathbb{S}_n \geq G$ and $r \in \mathbb{Z}_2^n$: $g^{-1} r g \in \mathbb{Z}_2^n$.

Thus G is isomorphic to the semidirect product of $\mathbb{Z}_2^n$ and $\mathrm{Aut}(\Sigma)$ by Lemma 3.15. This is the wreath product $\mathbb{Z}_2 \operatorname{wr}_{[n]} \mathrm{Aut}(\Sigma)$ with the canonical action of the group $\mathrm{Aut}(\Sigma)$ on $[n]$. □

The automorphism group of the mirror complex decomposes into two parts. The first part arises from the reflection due to the mirroring construction that results in $\{\pm 1\}$-flips in the vector representation of the cubical faces. The second part comes from the symmetries of the simplicial complex and corresponds to a permutation of the entries in the vector representation.

Example. Let us have a look at the automorphism group of the previous example, the mirror complex of two lines with three vertices. The simplicial complex is $\{(001), (100)\}$. Its only non-trivial automorphism permutes the end vertices. This is a subgroup of $\mathbb{S}_3$, namely $\langle (02)(1) \rangle$, isomorphic to $\mathbb{Z}_2$. So the automorphism group of the mirror complex is isomorphic to $\mathbb{Z}_2^3 \rtimes_\varphi \mathbb{Z}_2$ with $\varphi_0(z_1, z_2, z_3) = (z_1, z_2, z_3)$ and $\varphi_1(z_1, z_2, z_3) = (z_3, z_2, z_1)$ for $0, 1 \in \mathbb{Z}_2$.

3.4 Cubical fissures

The *cubical fissure* or *fissuring* is an operation that produces a new cubical complex from a given one. Let C be a pure cubical d-dimensional complex, C_1 and C_2 facet-disjoint d-dimensional subcomplexes of C such that $C_1 \cup C_2 = C$. The cubical fissure $\text{fis}_C(C_1, C_2)$ of C between C_1 and C_2 is defined by lifting C_1 to height one, dropping C_2 to minus one and filling the fissure with $C_1 \cap C_2 \times [-1, 1]$. Combinatorially this is equivalent to the poset

$$\text{fis}_C(C_1, C_2) = (C_1 \times \{+1\}) \cup (C_1 \cap C_2 \times \{0\}) \cup (C_2 \times \{-1\}).$$

with $+1 < 0$ and $-1 < 0$ in the last component. An example is given in Figure 3.8.

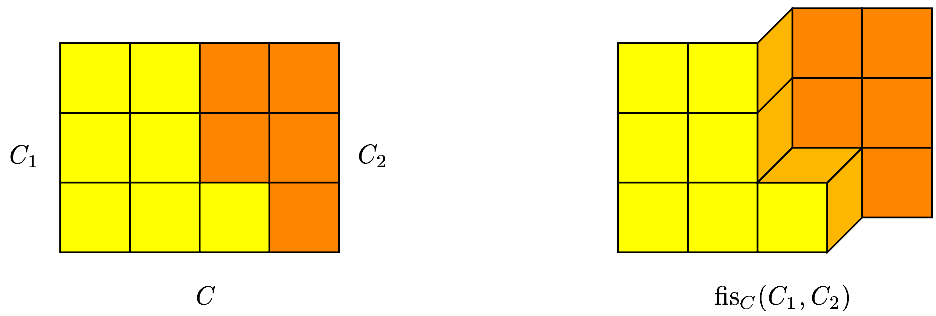


Figure 3.8: Cubical fissure of C between C_1 and C_2

When C is a subcomplex of the boundary complex of an n-cube then the cubical fissure gives a subcomplex of the $(n+1)$-cube. The vectors obtained by appending plus one, zero or minus one to all facets in C_1, $C_1 \cap C_2$ or C_2 respectively, describe the combinatorics of the new complex.

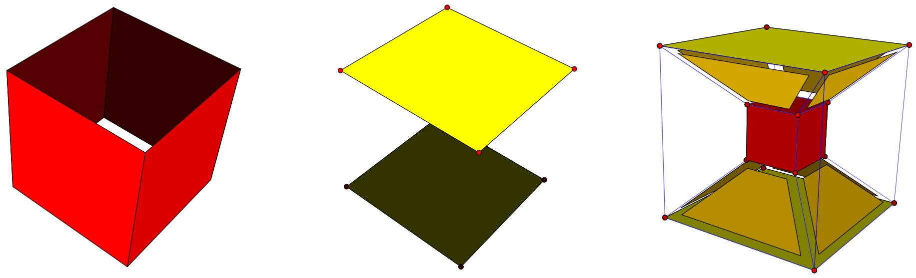
If C is a PL sphere then the cubical fissure $\text{fis}_C(C_1, C_2)$ of C between C_1 and C_2 is still a PL sphere.

Example. Let C be the boundary complex of the 3-cube. The complexes are given by facets meaning that the complex is made up of these facets and all their faces. Take the two subcomplexes with the following facets:

$$S_1 = \{(-00), (0{-}0), (+00), (0{+}0)\}, \quad S_2 := \{(00-), (00+)\}.$$

The union of S_1 and S_2 is C. The facets of the cubical fissure of C between S_1 and S_2 is:

$$\begin{aligned}\text{fis}_C(S_1, S_2) = \{&(-00+), (0{-}0{+}), (+00+), (0{+}0{+}),\\ &(-0{-}0), (0{-}{-}0), (+0{-}0), (0{+}{-}0),\\ &(-0{+}0), (0{-}{+}0), (+0{+}0), (0{+}{+}0),\\ &(00{-}{-}), (00{+}{-})\}.\end{aligned}$$



(a) Subcomplex S_1 (b) Subcomplex S_2 (c) Cubical fissure

Figure 3.9: The cubical fissure of two subcomplexes of the 3-cube is a subcomplex of the Schlegel diagram of the 4-cube.

As the complexes S_1 and S_2 are subcomplexes of the 3-cube, the resulting fissure is part of the 4-cube, shown in Figure 3.9.

3.5 Neighborly cubical polytopes

Joswig and Ziegler [11] first achieved to construct neighborly cubical polytopes. They constructed high dimensional cubes as deformed products (cf. Amenta and Ziegler [2]) of intervals. These have the property that an orthogonal projection to some of the coordinates preserves the skeleton needed.

The facets of the neighborly cubical polytopes obtained from the projection of deformed cubes are given by the Cubical Gale Eveness Condition of Joswig and Ziegler [11, Thm. 18].

Theorem 3.17 (Cubical Gale Evenness Condition). *The facets of the neighborly cubical polytope* $\mathrm{ncp}_d(n)$ *are given by vectors* $\alpha \in \{0, \pm 1\}^n$ *with* $d-1$ *zeros. They are classified by the number* $\mathbf{p}$ *of leading* ± 1*'s:*

- $\mathbf{p = 0}$. $\alpha_1 = 0$, *and* $|\alpha|$ *satisfies the simplicial Gale Evenness Condition: between any two values* $\alpha_i, \alpha_j \in \{\pm 1\}$ *there is an even number of zeros.*
- $\mathbf{0 < p < n - d + 1}$. $\alpha = (\underbrace{-1, +1, \ldots, (-1)^{p-1}, \sigma}_{p}, 0, \alpha^{(n-p-1)})$, *with* $\sigma \in \{-1, +1\}$, $\alpha^{(n-p-1)} \in \{0, \pm 1\}^{n-p-1}$ *and:*
  1. $|\alpha^{(n-p-1)}|$ *satisfies the simplicial Gale Evenness Condition, and*

2. *if* $\sigma = (-1)^{p+1}$, *then* $\alpha^{(n-p-1)}$ *starts with an even number of zeros; if* $\sigma = (-1)^p$, *then* $\alpha^{(n-p-1)}$ *starts with an odd number of zeros.*

- $\mathbf{p = n - d + 1}$. $\alpha = (\underbrace{-1, +1, \ldots, (-1)^{p-1}, \sigma}_{p=n-d+1}, 0, \ldots, 0)$ *with* $\sigma \in \{-1, +1\}$.

3.6 Neighborly cubical spheres

Babson, Billera and Chan [3] constructed cubical analogs of neighborly simplicial spheres in 1997. They gave an inductive construction to prove the existence of neighborly cubical PL spheres.

To construct a neighborly cubical d-sphere they start off with the boundary of the $(d+1)$-cube. Then they perform a sequence of cubical fissures along the mirror complex of the boundary of a cyclic polytope. For each ordering of the vertices of the cyclic polytope the cubical fissure and thus the neighborly cubical sphere obtained may differ. As it turns out by Proposition 3.28, in odd dimension the ordering does not matter.

The goal of this section is to give an explicit formula for the combinatorics of the neighborly cubical spheres in a more general setting of neighborly increasing sequences of simplicial balls.

3.6.1 Neighborly increasing sequences

The important structure for the inductive construction is the pulling triangulation and the neighborliness of the cyclic polytopes. A first generalization can be achieved by taking any neighborly simplicial polytope instead of cyclic polytopes to construct neighborly cubical spheres. Further we define neighborly increasing sequences of simplicial balls which abstract the combinatorial structure of pulling triangulations according to an ordering of the vertices of neighborly simplicial polytopes. The combinatorics of this more general construction is described in Theorem 3.25.

We start off with a lemma connecting the neighborliness of the cone over a simplicial ball with the neighborliness of the simplicial ball.

Lemma 3.18. *Let* B_{i-1} *be a simplicial* $(d-1)$*-ball on* $i-1$ *vertices and* $T_i = B_{i-1} * v_i$ *a simplicial* d*-ball for* $d \geq 2$*. Then* ∂T_i *is* k*-neighborly if and only if* ∂B_{i-1} *is* $(k-1)$*-neighborly and* B_{i-1} *is* k*-neighborly.*

Proof. The boundary ∂T_i is equal to $(\partial B_{i-1} * v_i) \cup B_{i-1}$ since $T_i = B_{i-1} * v_i$. If ∂T_i is k-neighborly all k-subsets of $[i]$ are faces of ∂T_i. Because ∂T_i is the union of $\partial B_{i-1} * v_i$ and B_{i-1}, all $(k-1)$-subsets of $[i-1]$ must be contained in ∂B_{i-1} and B_{i-1} must contain all k-subsets of $[i-1]$. This means that ∂B_{i-1} is $(k-1)$-neighborly and B_{i-1} is k-neighborly.

Conversely, if ∂B_{i-1} is $(k-1)$-neighborly and B_{i-1} is k-neighborly then $\partial T_i = (\partial B_{i-1} * v_i) \cup B_{i-1}$ obviously contains all k-subsets of $[i]$. □

This motivates the following definition

Definition 3.19. Let $V = \{v_0, \ldots, v_n\}$ be a set of vertices, B_i be a simplicial $(d-1)$-ball on the vertex set $V_i = \{v_0, \ldots, v_i\}$ for $i = d, \ldots, n-1$ with $d \geq 2$. Then $(T_i)_{i=d+1}^{n}$ is an *increasing sequence* of simplicial d-balls if

$$T_i = B_{i-1} * v_i \quad \text{with} \quad B_{i-1} \subseteq \partial T_{i-1} \quad \text{for } i = d+1, \ldots, n.$$

It is a *neighborly increasing sequence* if ∂T_n is a neighborly $(d-1)$-sphere.

With Lemma 3.18 and induction we obtain a proposition describing neighborly increasing sequences.

Proposition 3.20. *Let $(T_i)_{i=d+1}^{n}$ be an increasing sequence of d-balls for $d \geq 2$ with B_i defined as in the above definition. Then the following are equivalent:*

(i) $(T_i)_{i=d+1}^{n}$ is a neighborly increasing sequence.

(ii) ∂T_i is a neighborly $(d-1)$-sphere for all $i = d+1, \ldots, n$.

(iii) For all $i = d+1, \ldots, n$

- *B_{i-1} is $\lfloor \frac{d}{2} \rfloor$-neighborly and*
- *∂B_{i-1} is $(\lfloor \frac{d}{2} \rfloor - 1)$-neighborly.*

Since the convex hull of a subset of the vertices of a neighborly simplicial polytope is a neighborly simplicial polytope (cf. Proposition 2.6) the above definition is a correct generalization of the sequences of pulling triangulations of cyclic polytopes used by Babson, Billera, and Chan. As a consequence we obtain

Corollary 3.21. *Take an abitrary ordering of the vertices of a neighborly simplicial polytope. Then there exists a realization such that the induced pulling triangulations form a neighborly increasing sequence of simplicial balls.*

In fact, all neighborly increasing sequences of 2-dimensional simplicial balls are nothing but the pulling triangulations of the cyclic 2-polytopes, i.e. n-gons. In dimension three all neighborly increasing sequences are indeed still sequences of pulling triangulations of 3-polytopes.

Proposition 3.22. *Let P be a simplicial 3-polytope, $B \subsetneq \partial P$ a simplicial 2-ball containing all vertices of P. Then the pulling triangulation of $B * v$ is the pulling triangulation of a 3-polytope.*

Proof. By Steinitz's Theorem (cf. Ziegler [19, Chapter 4]) every planar 3-connected graph on more than three vertices corresponds to the graph of a 3-polytope. This allows us to have a look at the local change of the graph and not to consider the embedding of the polytope in $\mathbb{R}^3$. Since B contains all vertices of P, the complement B^C of B in the boundary of P is a triangulated k-gon without interior vertices (see Figure 3.10). We proceed by

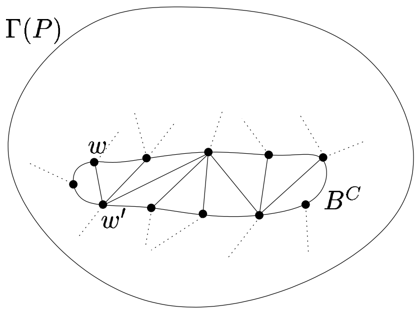


Figure 3.10: Subgraph of the graph of P induced by B^C

induction on the number of vertices k on the boundary of B^C. We show that the graph obtained by replacing the triangulation of B^C with a new vertex connected to all the boundary vertices of B^C is still planar and 3-connected (see Figure 3.11). For $k = 3$, B^C is a triangle. It is replaced by a K_4 and

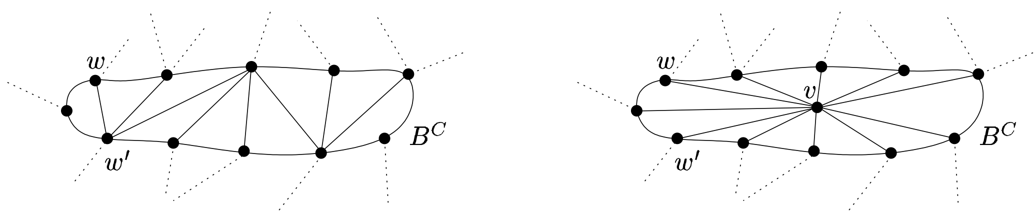


Figure 3.11: Local change of the graph of the boundary of P

remains planar and 3-connected.

Now let B^C be a triangulated k-gon for some $k > 3$. Then there exists a diagonal (w, w') through B^C splitting it into a triangle and a $(k-1)$-gon (easily proved by induction). By induction, the graph G' obtained by replacing the triangulation of the $(k-1)$-gon with the new vertex v connected to all vertices is a planar 3-connected graph. But since there exist three disjoint paths from w to w' in $G' \backslash (w, w')$ (see Figure 3.12), the graph obtained by replacing the edge $(w.w')$ with the edge (u, v) is 3-connected and planar.

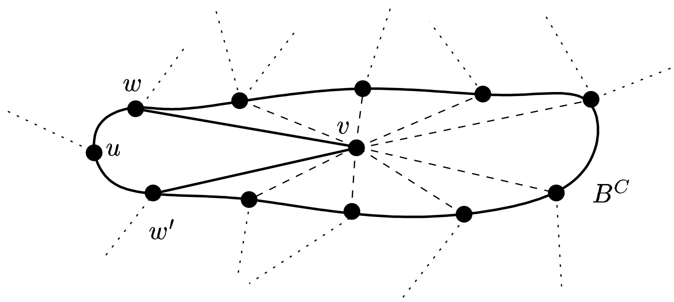


Figure 3.12: Three disjoint paths connect the vertices w and w'.

Thus by Steinitz's Theorem, there exists a 3-polytope with this graph. The pulling triangulation of this polytope with respect to v is combinatorially equivalent to $B * v$. □

Every neighborly increasing sequence of 3-balls $(T_i)_{i=4}^n$ begins with the 3-simplex T_4. Further Proposition 3.22 assures that if T_i is the pulling triangulation of a polytope then also is T_{i+1}. Thus we get the following characterization.

Corollary 3.23. *All neighborly increasing sequences of simplicial* 3*-balls are sequences of* 3*-balls each combinatorially isomorphic to the pulling triangulation of a simplicial* 3*-polytope.*

This does not mean that each sequence arises from a vertex ordering of *one* neighborly simplicial polytope.

In the following we show that neighborly increasing sequences of simplicial balls are a *real* generalization of sequences of pulling triangulations of neighborly simplicial polytopes. Therefore we construct a neighborly increasing sequence not arising from a neighborly simplicial polytope. The smallest example of a non-polytopal neighborly simplicial 3-sphere on 10 vertices is the Altshuler sphere N^{10}_{425} constructed in his enumeration of neighborly simplicial 3-manifolds on 10 vertices (cf. Altshuler [1]). It was shown to be non-polytopal by Bokowski and Garms [5], where it is called M^{10}_{425}.

Proposition 3.24. *There exist neighborly increasing sequences of simplicial 4-balls such that not all boundaries are polytopal PL 3-spheres.*

Proof. The proof is to construct a neighborly increasing sequence of the Altshuler sphere N^{10}_{425}. The combinatorics of the sequence $(T_i)_{i=5}^{10}$ of simplicial 4-balls with $\partial T_{10} = N^{10}_{425}$ is given in Appendix A.2. We used the application `topaz` of the `polymake` system to verify that T_i is a 4-ball for $i = 5, \dots, 10$ and that ∂T_{10} is the neighborly simplicial PL 3-sphere N^{10}_{425}. □

We also give a little insight into the problems which make it difficult to construct a neighborly increasing sequence of an arbitrary neighborly simplicial sphere in Appendix A.1.

3.6.2 Generalization of the BBC-construction

The neighborly increasing sequences of simplicial balls presented above lead to a generalization of the construction of Babson, Billera and Chan [3, Theorem 3.1]. Further we deduce a direct formula describing the combinatorics of neighborly cubical spheres.

Theorem 3.25. *Take a neighborly increasing sequence of simplicial $(d-1)$-balls $(T_i)_{i=d}^{n-1}$ with $n > d > 2$. Then there exists a neighborly cubical d-sphere $\mathrm{ncs}_d((T_i)_{i=d}^{n-1})$ with the $\lfloor (d-1)/2 \rfloor$-skeleton of the n-cube. Its facets are the vectors $\alpha \in \{0, \pm 1\}^n$ with d zeros satisfying the following properties:*

- $\mathbf{p = 0}$. $\alpha_n = 0$ *and* $|\alpha^{(n-1)}| := (|\alpha_1|, \dots, |\alpha_{n-1}|) \in \partial T_{n-1}$
- $\mathbf{0 < p < n - d}$. $\alpha = (\alpha^{(n-p-1)}, 0, \underbrace{\alpha_{n-p+1} = \sigma, -1, \dots, -1}_{p})$, *where* $\sigma \in \{-1, +1\}$, $\alpha^{(n-p-1)} \in \{0, \pm 1\}^{n-p-1}$ *with*
  1. $|\alpha^{(n-p-1)}| \in \partial T_{n-p-1}$ *and*
  2. *if* $\sigma = +1$, *then* $|(\alpha^{(n-p-1)}, 0)| \in T_{n-p}$*;* *if* $\sigma = -1$, *then* $|(\alpha^{(n-p-1)}, 0)| \notin T_{n-p}$.
- $\mathbf{p = n - d}$. $\alpha = (0, \dots, 0, \underbrace{\sigma, -1, \dots, -1}_{p=n-d})$ *with* $\sigma \in \{-1, +1\}$.

Proof. For $i = d, \dots, n-2$ let B_i be the facets of the boundary of T_i such that $T_{i+1} = B_i * v_{i+1}$ as in Definition 3.19 of the increasing sequence.

To build the neighborly cubical d-sphere on 2^n vertices we inductively define S_k for $k = d+1, \dots, n$, where S_n is the neighborly cubical sphere

$\mathrm{ncs}_d((T_i)_{i=d}^{n-1})$. To start with, S_{d+1} is set to the boundary of the $(d+1)$-cube. Then for $k = d+1, \ldots, n-1$ we recursively define :

$$\begin{aligned} S_{k+1} &:= \mathrm{fis}_{S_k}(\mathrm{M}(T_k), S_k \setminus \mathrm{M}(T_k)) \\ &= (\mathrm{M}(T_k) \times \{+1\}) \cup (\partial\,\mathrm{M}(T_k) \times \{0\}) \cup ((S_k \setminus \mathrm{M}(T_k)) \times \{-1\}) \end{aligned} \quad (3.1)$$

Since we start with the boundary of the $(d+1)$-cube and proceed by fissuring, all S_k are PL d-spheres.

We verify three properties of this definition to prove the theorem:

1. $\mathrm{M}(T_k) \subseteq S_k$ such that the fissuring is possible and S_{k+1} is defined for each $k = d+1, \ldots, n-1$.

2. S_k is neighborly for $k = d+1, \ldots, n$.

3. The combinatorics are correct.

By definition $T_k = B_{k-1} * v_k$ with $B_{k-1} \subset \partial T_{k-1}$ for $k = d+1, \ldots, n-1$. For the corresponding mirror complexes in their vector representation this means

$$\begin{aligned} \mathrm{M}(T_k) &= \mathrm{M}(B_{k-1} * v_k) \\ &= \mathrm{M}(B_{k-1}) \times \{0\} \\ &\subseteq \mathrm{M}(\partial T_{k-1}) \times \{0\} \\ &= \partial\,\mathrm{M}(T_{k-1}) \times \{0\} \\ &\subseteq S_k. \end{aligned}$$

To see that S_k is neighborly it suffices to show that the $\lfloor(d-1)/2\rfloor$-skeleton of the k-cube is included in the $\lfloor(d-1)/2\rfloor$-skeleton of S_k. Since S_k is a subcomplex of the k-cube the following inclusion holds for their $\lfloor(d-1)/2\rfloor$-skeleta

$$(S_k)_{\lfloor(d-1)/2\rfloor} \subseteq (\mathrm{C}_k)_{\lfloor(d-1)/2\rfloor}.$$

We have already shown, that $\mathrm{M}(T_k) \subseteq S_k$ for all $k = d+1, \ldots, n-1$. As $\partial\,\mathrm{M}(T_k) \subseteq \mathrm{M}(T_k)$ and $S_k \subseteq \mathrm{C}_k$ we get the following for the $\lfloor(d-1)/2\rfloor$-skeleta:

$$(\partial\,\mathrm{M}(T_k))_{\lfloor(d-1)/2\rfloor} \subseteq (\mathrm{M}(T_k))_{\lfloor(d-1)/2\rfloor} \subseteq (S_k)_{\lfloor(d-1)/2\rfloor} \subseteq (\mathrm{C}_k)_{\lfloor(d-1)/2\rfloor}.$$

But ∂T_k is a neighborly $(d-2)$-sphere it has the $(\lfloor(d-1)/2\rfloor - 1)$-skeleton of the $(k-1)$-simplex. Thus $\mathrm{M}(\partial T_k) = \partial\,\mathrm{M}(T_k)$ has the $\lfloor(d-1)/2\rfloor$-skeleton of the k-cube and equality holds in the above inclusion.

By definition S_n contains $\mathrm{M}(\partial T_{n-1}) \times \{0\}$. Since the boundary of T_{n-1} is simplicially neighborly, the cone over ∂T_{n-1} is also simplicially $\lfloor(d-1)/2\rfloor$-neighborly, i.e. has the $\lfloor(d-3)/2\rfloor$-skeleton of the $(n-1)$-simplex. Thus the mirror complex of this cone has the $\lfloor (d-1)/2 \rfloor$-skeleton of the n-cube. In the vector representation the mirror complex of the cone is given by

$$\mathrm{M}((\partial T_{n-1}) \times \{0\}) = \mathrm{M}(\partial T_{n-1}) \times \{0\}.$$

This is a subcomplex of S_n which proves that S_n is cubically neighborly.

Finally, we need to prove that the vector representation for the facets is the one induced by the inductive definition of S_k. We proceed by induction on k.

For $k = d+1$, S_{d+1} is the boundary of the $(d+1)$-cube. The facets of S_{d+1} are the vectors $\alpha \in \{0, \pm 1\}^{d+1}$ with exactly one ± 1. The facets of $T_d = \Delta_{d-1}$ are the vectors in $\{0,1\}^d$ with one 1 as well. Thus all facets of S_{d+1} are either type 0 or type 1:

facet				type
±1	0	$\cdots$	0	0
0	±1	$\ddots$	$\vdots$	0
$\vdots$	$\ddots$	$\ddots$	0	0
0	$\cdots$	0	±1	1

The induction step is split into two claims showing the vector description and the inductive definition of S_k yield the same combinatorics.

Claim. *The facets of the vector representation given by the theorem are contained in the facets of the inductive definition via fissuring of Equation (3.1).*

We analyze the different types of facets of the vector representation.

- **type 0:** The facets of type 0 are the vectors $\alpha = (\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in \partial T_k$. This is equivalent to $\alpha^{(k)} \in \mathrm{M}(\partial T_k)$ and thus $(\alpha^{(k)}, 0) \in \mathrm{M}(\partial T_k) \times \{0\}$.

- **type 1:** The facets of type 1 are the vectors $\alpha = (\alpha^{(k-1)}, 0, \sigma)$ with $|\alpha^{(k-1)}| \in \partial T_{k-1}$ and

$$\begin{aligned} \sigma = +1 \quad &\text{if} \quad (\alpha^{(k-1)}, 0) \in T_k \quad \text{or} \\ \sigma = -1 \quad &\text{if} \quad (\alpha^{(k-1)}, 0) \notin T_k. \end{aligned}$$

  By induction $(\alpha^{(k-1)}, 0)$ is a type 0 facet of S_k. Since $T_k = B_{k-1} * v_k$ with $B_{k-1} \subseteq \partial T_{k-1}$ we obtain:

  - ▶ if $(|\alpha^{(k-1)}|, 0) \in T_k$ then $(\alpha^{(k-1)}, 0) \in \mathrm{M}(T_k)$ and thus $(\alpha^{(k-1)}, 0, +1) \in S_{k+1}$;

  - ▶ if $(|\alpha^{(k-1)}|, 0) \notin T_k$ then $(\alpha^{(k-1)}, 0) \in S_k \setminus \mathrm{M}(T_k)$ and thus $(\alpha^{(k-1)}, 0, -1) \in S_{k+1}$.

- **type $\mathbf{l = 2, \ldots, k - d}$:** The facets of type l are the vectors

$$(\alpha^{(k-l)}, 0, \sigma, -1, \ldots, -1) \in \{0, \pm 1\}^{k+1}$$

  with $\sigma \in \{-1, +1\}$ and $\alpha^{(k-l)} \in \{0, \pm 1\}^{k-l}$ are the facets of type l. Taking only the first k entries of $\alpha = (\alpha^{(k)}, -1)$, it follows by induction that $\alpha^{(k)}$ is a type $l-1$ facet of S_k. Since all facets of $\mathrm{M}(T_k)$ contain the vertex k, $\alpha^{(k)}$ is not contained in $\mathrm{M}(T_k)$. Further it is not contained in $\mathrm{M}(\partial T_k)$ because it has d zero entries. Thus $\alpha^{(k)}$ is a facet of $S_k \setminus \mathrm{M}(T_k)$ and $(\alpha^{(k)}, -1) \in S_{k+1}$.

Claim. *The facets of the inductive definition via fissuring of Equation (3.1) are contained in the facets of the vector representation given by the theorem.*

There are three different kinds of facets of S_{k+1} according to the inductive definition in Equation (3.1). We need to show that all kinds of facets of S_{k+1} come up in one types of the combinatorics given by the theorem.

- The facets of $\mathrm{M}(T_k) \times \{+1\}$ are the vectors $(\alpha^{(k)}, +1) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in T_k$ and $\alpha^{(k)}_k = 0$ since by definition of the increasing sequence all facets of T_k contain the vertex k. These are some of the facets of type 1 of S_{k+1}.

- The facets of $\mathrm{M}(\partial T_k) \times \{0\}$ are the vectors $(\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$ with $|\alpha^{(k)}| \in \partial T_k$. These are the facets of type 0 of S_{k+1}.

- Let $\alpha^{(k)} \in S_k \setminus \mathrm{M}(T_k)$. By induction, we get the vector representation of the facets of S_k.

  If $\alpha^{(k)}$ is a facet of type l with $0 < l \leq k - d$ then $\alpha^{(k)}_k \neq 0$ and thus is not in $\mathrm{M}(T_k)$. By appending a -1 we get a facet of type $l + 1$ of S_{k+1}.

  The facets $(\alpha^{(k-1)}, 0) \in S_k$ with $|\alpha^{(k-1)}| \in \partial T_{k-1}$ of type 0. Those facets may be partitioned into two parts

  - ▶ $(|\alpha^{(k-1)}|, 0) \in T_k$ or
  - ▶ $(|\alpha^{(k-1)}|, 0) \notin T_k$.

  If $(|\alpha^{(k-1)}|, 0) \in T_k$ then $(\alpha^{(k-1)}, 0) \notin S_k \setminus \mathrm{M}(T_k)$. If $(\alpha^{(k-1)}, 0) \notin \mathrm{M}(T_k)$ then $(\alpha^{(k-1)}, 0) \in S_k \setminus \mathrm{M}(T_k)$ and $(\alpha^{(k-1)}, 0, -1)$ is a facet of type 1 of S_{k+1}.

We conclude by setting $\mathrm{ncs}_d((T_i)_{i=d}^n) = S_n$. This is the neighborly cubical sphere with the combinatorics given by the theorem.

□

This theorem generalizes the construction of Babson, Billera, and Chan to special sequences of neighborly simplicial balls. For example consider the neighborly simplicial 4-polytope on 8 vertices constructed by Grünbaum [9, Theorem 7.2.3] which is not combinatorially equivalent to the cyclic 4-polytope on 8 vertices. By Corollary 3.21 any ordering of the vertices yields a neighborly increasing sequence. We constructed the neighborly cubical sphere with our `polymake` client and verified that it is not combinatorially equivalent to the neighborly cubical polytope $\mathrm{ncp}_6(9)$ constructed by Joswig and Ziegler. The vector representation of the facets of the sphere is shown on the cover.

Example. Take a look at the neighborly cubical sphere constructed from the 5-gon. The resulting sphere is a neighborly cubical 3-sphere with the graph of the 6-cube. We use the notation of the proof of Theorem 3.25 and follow the inductive construction.

Let T_i denote the pulling triangulation of the i-gon for $i = 4, 5$. The facets of the triangulations and their boundaries in the notation of Equation 2.1 are (cf. Figure 3.13)

	4-gon	5-gon
T_i	0 0 1 0 1 0 0 0	0 0 1 1 0 1 0 0 1 0 1 1 0 0 0
∂T_i	**0 0 1 1** **1 0 0 1** **1 1 0 0** 0 1 1 0	0 0 1 1 1 1 0 0 1 1 1 1 0 0 1 1 1 1 0 0 0 1 1 1 0

The facets of $B_4 \subseteq \partial T_4$ such that $T_5 = B_4 * v_4$ are marked in bold.

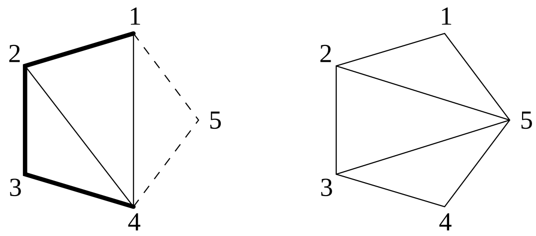
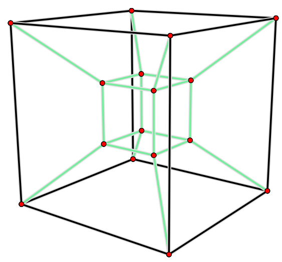


Figure 3.13: The pulling triangulation of a 4-gon and a 5-gon. The facets of B_4 of the 4-gon such that $B_4 * v_5$ is the pulling triangulation of the 5-gon are bold.

Let S_4 be the boundary of the 4-cube. In the vector representation it corresponds to

±1	0	0	0
0	±1	0	0
0	0	±1	0
0	0	0	±1

Schlegel diagram of S_4



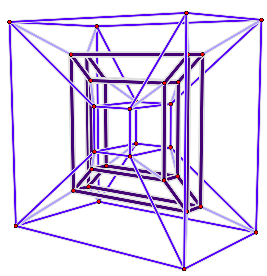
Define S_5 and S_6 similar to Equation (3.1) by

$$\begin{aligned} S_k := &\operatorname{fis}_{S_{k-1}}(\mathrm{M}(T_{k-1}), S_{k-1} \setminus \mathrm{M}(T_{k-1})) \\ = &(\mathrm{M}(T_{k-1}) \times \{+1\}) \cup \\ &(\partial\,\mathrm{M}(T_k) \times \{0\}) \cup \\ &((S_{k-1} \setminus \mathrm{M}(T_{k-1})) \times \{-1\}) \end{aligned}$$

To construct S_5 we need to calculate the mirror complex of T_4 and append a +1 (printed in bold), append a 0 to the boundary of T_4 and append a −1 to all facets of S_4 that are not contained in $\mathrm{M}(T_4)$. This yields

±1	**0**	**0**	**0**	**+1**
0	±1	0	0	−1
0	**0**	**±1**	**0**	**+1**
0	0	0	±1	−1
0	0	±1	±1	0
±1	0	0	±1	0
±1	±1	0	0	0
0	±1	±1	0	0

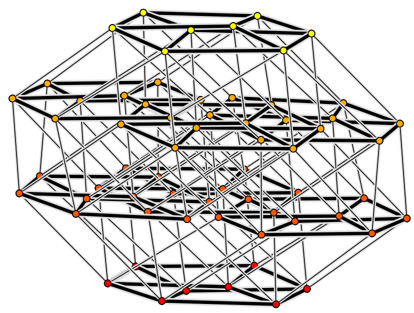
Schlegel diagram of S_5



The inner cube in the Schlegel diagram is encircled by a torus which is the mirror complex of the boundary of the quadrilateral ∂T_4. Repeating this procedure we obtain the vector representation of a neighborly cubical 3-sphere with the graph of the 6-cube

$$\begin{array}{ccccc|c}
\pm1 & 0 & 0 & 0 & +1 & -1 \\
0 & \pm1 & 0 & 0 & -1 & -1 \\
0 & 0 & \pm1 & 0 & +1 & -1 \\
0 & 0 & 0 & \pm1 & -1 & -1 \\
\mathbf{0} & \mathbf{0} & \mathbf{\pm1} & \mathbf{\pm1} & \mathbf{0} & \mathbf{+1} \\
\mathbf{\pm1} & \mathbf{0} & \mathbf{0} & \mathbf{\pm1} & \mathbf{0} & \mathbf{+1} \\
\mathbf{\pm1} & \mathbf{\pm1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{+1} \\
0 & \pm1 & \pm1 & 0 & 0 & -1 \\
\hline
0 & 0 & \pm1 & \pm1 & \pm1 & 0 \\
\pm1 & 0 & 0 & \pm1 & \pm1 & 0 \\
\pm1 & \pm1 & 0 & 0 & \pm1 & 0 \\
\pm1 & \pm1 & \pm1 & 0 & 0 & 0 \\
0 & \pm1 & \pm1 & \pm1 & 0 & 0
\end{array}$$

Graph of S_6



S_6 has the 1-skeleton of the 6-cube. Thus S_6 is a neighborly cubical 3-sphere with the graph of the 6-cube.

3.6.3 Neighborly cubical spheres from cyclic polytopes

This general construction yields a closed formula for the combinatorics of the neighborly cubical sphere constructed from pulling triangulations of cyclic polytopes.

Corollary 3.26. *For $i = d, \ldots, n-1$ let T_i be the pulling triangulation with respect to the last vertex v_i of the standard $(d-1)$-dimensional cyclic polytope $\mathrm{cyc}_{d-1}(i)$ with vertices $v_k = (k, k^2, \ldots, k^{d-1})$ for $k = 1, \ldots, i$.*

Then the facets of the neighborly cubical d-sphere $\mathrm{ncs}_d((T_i)_{i=d}^{n-1})$ with the $\lfloor (d-1)/2 \rfloor$-skeleton of the n-cube for $n > d > 2$ are the vectors $\alpha = (\alpha_i)_{i=1}^{n}$ in $\{0, \pm1\}^n$ with d zeros of the following forms:

- $\mathbf{p = 0}$. *$\alpha_n = 0$, and $|\alpha|$ satisfies the simplicial Gale Evenness Condition.*
- $\mathbf{0 < p < n - d}$. *$\alpha = (\alpha^{(n-p-1)}, 0, \underbrace{\alpha_{n-p+1} = \sigma, -1, \ldots, -1}_{p})$, where $\sigma \in \{-1, +1\}$, $\alpha^{(n-p-1)} \in \{0, \pm1\}^{n-p-1}$ such that*

    1. *$|\alpha^{(n-p-1)}|$ satisfies the simplicial Gale Evenness Condition and*

2. *if* $\sigma = +1$, *then* $\alpha^{(n-p-1)}$ *ends with an even number of zeros;*
   if $\sigma = -1$, *then* $\alpha^{(n-p-1)}$ *ends with an odd number of zeros.*

- $\mathbf{p = n - d}$. $\alpha = (0, \dots, 0, \underbrace{\sigma, -1, \dots, -1}_{p=n-d})$ *with* $\sigma \in \{-1, +1\}$.

As a consequence we obtain an isomorphism of the neighborly cubical spheres constructed from special vertex orderings of the cyclic polytopes and the neighborly cubical polytopes of Joswig and Ziegler.

Corollary 3.27. *Let* $d \geq 4$ *and* T_i *be the pulling triangulation of the cyclic polytope* $\mathrm{cyc}_{d-2}(i)$ *with respect to the last vertex as in Corollary 3.26. Then the neighborly cubical sphere* $\mathrm{ncs}_{d-1}((T_i)_{i=d-1}^{n-1})$ *and the boundary of the neighborly cubical polytopes* $\mathrm{ncp}_d(n)$ *are combinatorially isomorphic. The isomorphism is given by inverting the order and then flipping the even bits:*

$$\begin{aligned} \Phi : \{0, \pm 1\}^n &\to \{0, \pm 1\}^n, \\ (\alpha_1, \alpha_2, \dots, \alpha_n) &\mapsto (\alpha_n, -\alpha_{n-1}, \dots, (-1)^n \alpha_2, (-1)^{n+1} \alpha_1) \end{aligned}$$

By Corollary 2.11 one obtains a result on the uniqueness of the construction for odd sphere dimension, since the pulling triangulations with respect to any vertex of even dimensional cyclic polytopes are combinatorially equivalent.

Corollary 3.28. *Let* $\pi \in \mathbb{S}_n$ *be a permutation and let* $(v_1, \dots, v_{n-1})$ *and* $(v_{\pi(1)}, \dots, v_{\pi(n-1)})$ *be two orderings of the vertices of the cyclic d-polytope of* ***even*** *dimension on* $n > d \geq 2$ *vertices. For* $i = d+1, \dots, n$ *let* T_i *and* $\widetilde{T}_i$ *be the pulling triangulations of* $\mathrm{conv}(v_1, \dots, v_i)$ *and* $\mathrm{conv}(v_{\pi(1)}, \dots, v_{\pi(i)})$ *with respect to* v_i *and* $v_{\pi(i)}$*, respectively. Then the neighborly cubical spheres* $\mathrm{ncs}_{d+1}((T_i)_{i=d+1}^n)$ *and* $\mathrm{ncs}_{d+1}((\widetilde{T}_i)_{i=d+1}^n)$ *are isomorphic.*

In other words, the neighborly cubical spheres in odd dimension constructed by Babson, Billera, and Chan using pulling triangulations of cyclic polytopes are exactly the boundaries of the neighborly cubical polytopes constructed by Joswig and Ziegler.

For even sphere dimension, i.e. odd dimensional cyclic polytopes, this is not true. The smallest possible example is the $(2k-1)$-dimensional cyclic polytope on $2k+1$ vertices where the automorphism group action has two orbits and there exist two different pulling triangulations. But as the following proposition shows, the corresponding spheres are combinatorially isomorphic, since the mirror complexes are complements in the boundary of the cube.

Proposition 3.29. *The neighborly cubical d-spheres constructed from the pulling triangulations of the cyclic $(d-1)$-polytopes on $d+1$ vertices are combinatorially isomorphic for all $d \geq 3$.*

Proof. For odd dimension, the dimension of the corresponding cyclic polytope is even. Thus it is a special case of Corollary 3.28.

For even dimension $d = 2k$, the dimension of the corresponding cyclic polytope is odd. By Theorem 2.10 the action of the automorphism group on the vertices has two orbits: the vertices with odd and even indices in the standard increasing order. The cyclic $(d-1)$-polytope on $d+1$ is the free sum of Δ_{k+1} with vertices $v_1, v_3, \ldots, v_{2k+1}$ and Δ_k with vertices $v_2, v_4, \ldots, v_{2k}$. Its facets are the joins of facets of Δ_k and facets of Δ_{k+1}.

The simplices of the pulling triangulation with respect to the odd vertex v_{2k+1} are joins of Δ_{k+1} and a facet of Δ_k. In the vector notation this is all the vectors in $\{0,1\}^{d+1}$ with one 1 at an even position. So the facets of the neighborly cubical sphere are the union of the mirror complex of the boundary of $\mathrm{cyc}_{d-1}(d+1)$ with a zero appended and:

$$2k+1 \left\{ \begin{array}{cccccccc} \pm1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & \pm1 & 0 & 0 & \vdots & & 0 & +1 \\ 0 & 0 & \pm1 & 0 & \vdots & & 0 & -1 \\ 0 & 0 & 0 & \pm1 & 0 & \cdots & 0 & +1 \\ 0 & \ldots & \ldots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 & +1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \pm1 & -1 \end{array} \right.$$

The simplices of the pulling triangulation with respect to an even vertex are the vectors in $\{0,1\}^{d+1}$ with one 1 at an odd position. Thus the facets of the neighborly cubical sphere for this triangulation are exactly the facets of the above neighborly cubical sphere with the last ±1 flipped.

Consequently, the neighborly cubical spheres constructed from the two different pulling triangulations of the cyclic $(d-1)$-polytope on $d+1$ vertices are isomorphic. □

In the next example we have a look at the different orderings of the vertices of the cyclic 3-polytope on six vertices. These are the smallest parameters for which the neighborly cubical sphere could be different.

Example. Since the neighborly cubical spheres constructed from cyclic 3-polytopes on five vertices are combinatorially isomorphic, only the index of the last vertex is important. By Corollary 2.12 the cyclic polytope $\mathrm{cyc}_3(6)$ has

						4	5	6
0	0	0	**1**	**1**	**1**	*	*	*
0	1	0	0	**1**	**1**		*	*
0	1	1	0	0	**1**			*
0	1	1	**1**	0	0	*		
0	0	1	**1**	**1**	0	*	*	
1	0	0	**1**	**1**	0	*	*	
1	1	0	0	**1**	0		*	
1	1	1	0	0	0			

Table 3.1: Facets of the cyclic polytope $\mathrm{cyc}_3(6)$. The stars indicate the facets not seen from the corresponding vertex.

$\lceil 6/3 \rceil = 3$ pulling triangulations. Each pulling triangulation corresponds to one vertex ordering and one orbit

(a) $\{1,2,3,4,5,6\}$: orbit $\{1,6\}$,

(b) $\{1,2,3,4,6,5\}$: orbit $\{2,5\}$ and

(c) $\{1,2,3,5,6,4\}$: orbit $\{3,4\}$.

To get the pulling triangulation with respect to a vertex we need the facets of $\mathrm{cyc}_3(6)$ which do not contain this vertex. The facets of the cyclic polytope are given by Gale's Evenness Condition. The facets not containing a given vertex are listed in Table 3.1. The simplices of the pulling triangulation are:

(a) $\{1,2,3,6\}$,$\{1,3,4,6\}$,$\{1,4,5,6\}$

(b) $\{1,2,3,5\}$,$\{1,3,4,5\}$,$\{1,2,5,6\}$,$\{2,3,5,6\}$,$\{3,4,5,6\}$

(c) $\{1,2,3,4\}$,$\{1,4,5,6\}$,$\{1,2,4,6\}$,$\{2,3,4,6\}$

The combinatorics of the neighborly cubical spheres corresponding to the three different triangulations are listed in Table 3.2. These are not combinatorially equivalent verified with `polymake` [8].

3.7 Surfaces of high genus

In this section we present an easy way to realize polyhedral surfaces of 'unusually large genus' in $\mathbb{R}^3$. By 'unusually large' we mean that the number of vertices is less than the genus of the surface.

1	2	3	4	5	6		1	2	3	4	6	5		1	2	3	5	6	4	
0	0	0	0	±	+	-	0	0	0	0	±	+	-	0	0	0	0	±	+	-
0	0	0	±	0	-	-	0	0	0	±	0	-	-	0	0	0	±	0	-	-
0	0	±	0	0	+	-	0	0	±	0	0	+	-	0	0	±	0	0	+	-
0	±	0	0	0	-	-	0	±	0	0	0	-	-	0	±	0	0	0	-	-
±	0	0	0	0	+	-	±	0	0	0	0	+	-	±	0	0	0	0	+	-
0	0	0	±	±	0	+	0	0	0	±	±	0	+	0	0	0	±	±	0	+
0	±	0	0	±	0	+	0	±	0	0	±	0	+	0	±	0	0	±	0	-
0	±	±	0	0	0	+	0	±	±	0	0	0	-	0	±	±	0	0	0	+
0	0	±	±	0	0	-	0	0	±	±	0	0	+	0	0	±	±	0	0	+
±	0	0	±	0	0	-	±	0	0	±	0	0	+	±	0	0	±	0	0	+
±	±	0	0	0	0	-	±	±	0	0	0	0	+	±	±	0	0	0	0	-
0	0	0	±	±	±	0	0	0	0	±	±	±	0	0	0	0	±	±	±	0
0	±	0	0	±	±	0	0	±	0	0	±	±	0	0	±	0	0	±	±	0
0	±	±	0	0	±	0	0	±	±	0	0	±	0	0	±	±	0	0	±	0
0	±	±	±	0	0	0	0	±	±	±	0	0	0	0	±	±	±	0	0	0
0	0	±	±	±	0	0	0	0	±	±	±	0	0	0	0	±	±	±	0	0
±	0	0	±	±	0	0	±	0	0	±	±	0	0	±	0	0	±	±	0	0
±	±	0	0	±	0	0	±	±	0	0	±	0	0	±	±	0	0	±	0	0
±	±	±	0	0	0	0	±	±	±	0	0	0	0	±	±	±	0	0	0	0

Table 3.2: The facets of three neighborly cubical 4-spheres with the 1-skeleton of the 7-cube.

The surfaces considered here are cubical polyhedral surfaces where each vertex has degree q. They were first desribed by Coxeter [6] in 1937 in his paper on regular skew polyhedra in dimensions three and four. In 1956 Ringel [17] rediscovered these surfaces analyzing problems concerning the graph of the n-dimensional cube. He pointed out that the surfaces are of lowest genus amongst all surfaces on which the graph of the n-cube may be drawn without self intersection. Further he gave an explicit combinatorial description of the surface as a 2-dimensional subcomplex of the n-cube. In 1982 McMullen, Schulz and Wills [14, 15] introduced a much more general class of polyhedral surfaces called equivelar surfaces $\mathcal{M}_{p,q}$.

Definition 3.30. An *equivelar surface* $\mathcal{M}_{p,q}$ is a polyhedral surface such that all 2-faces are p-gons and all vertices have degree q.

In particular, they inductively constructed an embedding of the surfaces $\mathcal{M}_{4,q}$ in $\mathbb{R}^3$. The surfaces described by Coxeter and Ringel are a subfamily of the equivelar surfaces $\mathcal{M}_{4,q}$.

Equivelar surfaces and mirror complexes

We combine the results of Coxeter and Ringel and connect them to mirror complexes of q-gons used by Babson, Billera, and Chan.

Let Q be the boundary of a q-gon for $q > 2$ and $\mathrm{M}(Q)$ its mirror complex. With vertices of Q labelled in cyclic order we obtain the same vector representation as Ringel [17, p. 17] of $\mathrm{M}(Q)$:

$$\begin{array}{cccccccc} 0 & 0 & \pm1 & \pm1 & \cdots & \pm1 & \pm1 & \pm1 \\ \pm1 & 0 & 0 & \pm1 & \cdots & \pm1 & \pm1 & \pm1 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ \pm1 & \pm1 & \pm1 & \pm1 & \cdots & \pm1 & 0 & 0 \\ 0 & \pm1 & \pm1 & \pm1 & \cdots & \pm1 & \pm1 & 0 \end{array}$$

By Proposition 3.14 the vertex figure of every vertex of $\mathrm{M}(Q)$ is isomorphic to Q. Thus the mirror complex is a PL 2-manifold which is embedded in the 2-skeleton of the q-cube.

The obvious way to realize a 2-dimensional subcomplex of the q-cube in $\mathbb{R}^5$ is in the Schlegel diagram of the 6-dimensional neighborly cubical polytope $\mathrm{ncp}_6(q)$ of Joswig and Ziegler: Since the neighborly cubical polytope $\mathrm{ncp}_6(q)$ has the 2-skeleton of the q-cube, $\mathrm{M}(Q)$ is contained in the boundary of $\mathrm{ncp}_6(q)$. The Schlegel diagram of $\mathrm{ncp}_6(q)$ is embedded in $\mathbb{R}^5$, thus $\mathrm{M}(Q)$ may be realized in $\mathbb{R}^5$.

The mirror complex of the pulling triangulation of the q-gon is a subcomplex of the facets of type $p = 0$ of $\mathrm{ncs}_3((T_i)_{i=3}^{q-1})$. Thus the mirror complex of the q-gon itself is a subcomplex of $\mathrm{ncs}_3((T_i)_{i=3}^{q-1})$. By Corollary 3.27 $\mathrm{ncs}_3((T_i)_{i=3}^{q-1})$ is isomorphic to the boundary of the neighborly cubical polytope $\mathrm{ncp}_4(q)$. Hence the mirror complex of the q-gon can be realized as a subcomplex of the Schlegel diagram of $\mathrm{ncp}_4(q)$ in $\mathbb{R}^3$.

The genus of this surface may easily be calculated from its f-vector:

$$f(\mathrm{M}(Q)) = (2^q, 2^{q-1}q, 2^{q-2}q).$$

The Euler-characteristic of the mirror complex of the q-gon is:

$$\chi(\mathrm{M}(Q)) = 2^q - 2^{q-1}q + 2^{q-2}q = 2^{q-2}(4-q).$$

This leads to the genus: $g(q) = 1 + 2^{q-3}(q-4) \in \mathcal{O}(f_0 \cdot \log f_0)$.

Thus we get a surface with higher genus than vertices for the case $q = 12$. The mirror complex of the 12-gon has $2^{12} = 4096$ vertices and genus $g(12) = 4097$.

q	3	4	5	6	7	8	9	10	11	12	..
$f_0(\mathrm{M}(q-gon))$	8	16	32	64	128	256	512	1024	2048	4096	..
genus $\mathrm{M}(q-gon)$	0	1	5	17	49	129	321	769	1793	4097	..

Examples

Let us have a look at the realizations of the mirror complexes of the q-gon for $q = 4, 5$ in the Schlegel diagram of $\mathrm{ncp}_4(q)$.

The mirror complex of the quadrilateral is a surface of genus 1 in the Schlegel diagram of the neighborly cubical polytope $\mathrm{ncp}_4(4)$: a torus embedded as a subcomplex of the Schlegel diagram of the 4-cube (cf. Figure 3.14).

The mirror complex of the pentagon is a surface of genus 5 embedded in the Schlegel diagram of $\mathrm{ncp}_4(5)$ (cf. Figure 3.15).

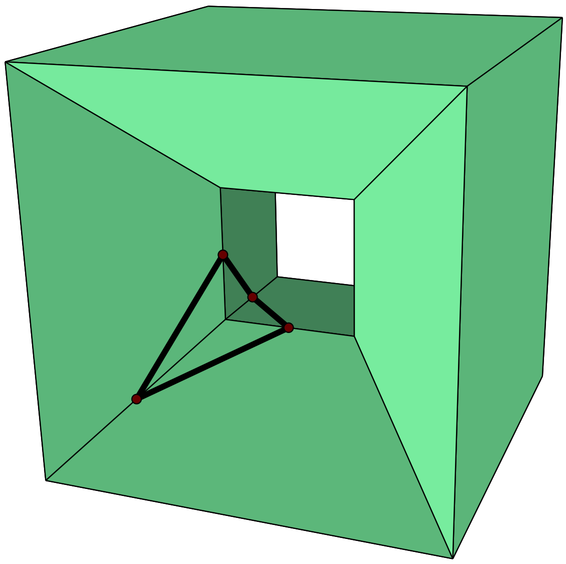


Figure 3.14: The mirror complex of the 4-gon in the Schlegel diagram of the 4-cube (the neighborly cubical polytope $\mathrm{ncp}_4(4)$)

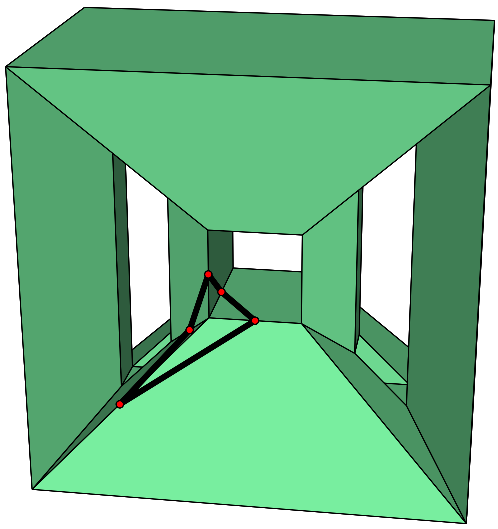


Figure 3.15: The mirror complex of the 5-gon in the Schlegel diagram of the neighborly cubical polytope $\mathrm{ncp}_4(5)$

Appendix A

Neighborly increasing sequences of spheres

We show that the concept of neighborly increasing sequences of simplicial balls is more general than sequences of pulling triangulations of neighborly simplicial polytopes. In dimension two and three all neighborly increasing sequences arise from pulling triangulations of neighborly simplicial polytopes. In dimension four there is a neighborly increasing sequence of 4-balls $(T_i)_{i=5}^{10}$ such that the boundary of T_{10} is the neighborly Altshuler 3-sphere N_{425}^{10} on 10 vertices. Since N_{425}^{10} is not polytopal this implies that neighborly increasing sequences are more general than sequences of pulling triangulations.

In the following we describe the problems of the construction for an arbitrary neighborly simplicial PL sphere. Finally we give the exact combinatorics of the neighborly increasing sequence $(T_i)_{i=5}^{10}$ constructed for N_{425}^{10}.

A.1 Problems of the construction

Given a neighborly simplicial PL $(d-1)$-sphere S_n on n vertices in a given order $(v_1, \ldots, v_n)$ we want to construct a neighborly increasing sequence of d-balls $(T_i)_{i=d+1}^{n}$ with $\partial T_n = S_n$. If we *had* an algorithm to construct T_{n-1} from $S_n = \partial T_n$ we *could* construct the whole sequence by induction. Therefore it suffices to illustrate problems for one step of the construction.

Let S_n be a neighborly simplicial PL $(d-1)$-sphere. Then the deletion $B_{n-1} := \mathrm{del}(S_n, v_n)$ is a $(d-1)$-ball and $T_n := B_{n-1} * v_n$ is the pulling triangulation of a d-ball with boundary S_n (cf. Figure A.1 for $d = 2$).

To build a neighborly increasing sequence we have to construct a triangulated d-ball T_{n-1} which contains B_{n-1} in its boundary. Therefore we have to find a simplicial $(d-1)$-ball B'_{n-1} on the vertices $(v_1, \ldots, v_{n-1})$

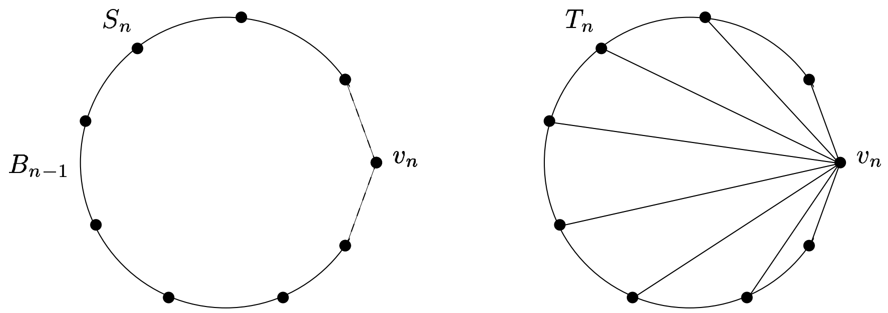


Figure A.1: The pulling triangulation T_n with respect to v_n is a 2-ball whose boundary S_n is a 1-sphere. $B_{n-1} = \mathrm{del}(S_n, v_n,)$ is a 1-ball in $\partial T_n = S_n$.

such that the intersection of B_{n-1} and B'_{n-1} is exactly the common boundary $\partial B_{n-1} = \partial B'_{n-1}$. Then by identifying the vertices we glue B_{n-1} and B'_{n-1} along their common boundary and obtain a PL $(d-1)$-sphere S_{n-1} on $n-1$ vertices $(v_1, \ldots, v_{n-1})$. If the $(d-1)$-balls B_{n-1} and B'_{n-1} intersect in some interior face then the identification does not yield a $(d-1)$-sphere (cf. Figure A.2). Similar to T_n, we obtain T_{n-1} as $T_{n-1} := B_{n-2} * v_{n-1}$ with $B_{n-2} := \mathrm{del}(S_{n-1}, v_{n-1})$.

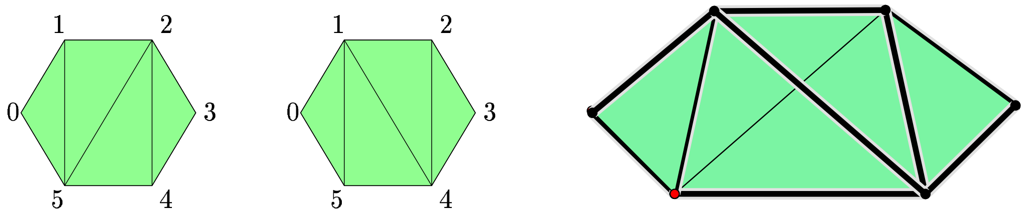


Figure A.2: By identifying the vertices of the two 2-balls (left) we obtain a 2-sphere with 'ears' (right) which is not a 2-sphere.

The crux of the construction in Appendix A.2 was to find the simplicial ball B'_{k-1} without introducing new vertices such that $B_{k-1} \cap B'_{k-1} = \partial B_{k-1}$ for $k = 10, \ldots, 5$. Since N^{10}_{425} is a PL 3-sphere the boundary ∂B_{k-1} of the 3-ball B_{k-1} is a PL 2-sphere which can be visualized with `polymake` and JavaView [16]. Thus we could easily try different 3-balls B'_{k-1} with $\partial B'_{k-1} = \partial B_{k-1}$ and succeeded in constructing a neighborly increasing sequence.

A.2 A neighborly increasing sequence of N^{10}_{425}

As we saw in the example on page 52 the neighborly increasing sequence depends on the ordering of the vertices. Thus this is not *the*, but *a* neighborly increasing sequence of N^{10}_{425}.

The whole construction used the application `topaz` of `polymake` [8] to check whether the definition of the neighborly increasing sequence is satisfied, i.e. that T_i is a 4-ball for $i = 5, \ldots, 10$ and the $\partial T_{10} = N^{10}_{425}$ is neighborly. The facets of N^{10}_{425} are taken from Bokowski and Garms [5] (notation: $M^{10}_{425} = N^{10}_{425}$). `topaz` confirmed that N^{10}_{425} is a neighborly simplicial PL 3-sphere on 10 vertices.

In the second coloumn of the following table the vertex sets of the facets of the neighborly increasing sequence of 4-balls $(T_i)^{10}_{i=5}$ with $\partial T_{10} = N^{10}_{425}$ is given. The third coloumn contains the facets of the boundary ∂T_i. The facets of ∂T_i are partitioned into B_i (**bold**) and B'_i such that $T_{i+1} = B_i * v_{i+1}$.

	Facets of T_i	Facets of ∂T_i
T_{10}	{0 1 4 5 9}, {0 1 2 3 9}, {2 3 6 7 9}, {0 1 6 7 9}, {2 3 4 5 9}, {4 5 6 7 9}, {0 1 4 8 9}, {0 1 2 6 9}, {0 1 3 5 9}, {0 2 3 6 9}, {2 3 4 8 9}, {2 3 5 7 9}, {2 4 5 8 9}, {0 4 5 6 9}, {1 3 4 5 9}, {0 4 6 7 9}, {2 6 7 8 9}, {3 5 6 7 9}, {2 5 7 8 9}, {1 3 4 8 9}, {0 3 5 6 9}	{0 1 4 5}, {0 1 2 3}, {2 3 6 7}, {0 1 6 7}, {2 3 4 5}, {4 5 6 7}, {0 1 4 8}, {0 1 2 6}, {0 1 3 5}, {0 2 3 6}, {2 3 4 8}, {2 3 5 7}, {2 4 5 8}, {0 4 5 6}, {1 3 4 5}, {0 4 6 7}, {2 6 7 8}, {3 5 6 7}, {2 5 7 8}, {1 3 4 8}, {0 3 5 6}, {4 5 8 9}, {2 3 8 9}, {6 7 8 9}, {0 1 8 9}, {0 1 7 9}, {1 2 3 9}, {4 5 7 9}, {1 6 7 9}, {2 6 8 9}, {0 4 8 9}, {1 3 8 9}, {5 7 8 9}, {0 4 7 9}, {1 2 6 9}
T_9	{0 1 4 5 8}, {0 1 2 3 8}, {2 3 6 7 8}, {0 1 6 7 8}, {2 3 4 5 8}, {4 5 6 7 8}, {0 1 2 6 8}, {0 1 3 5 8}, {0 2 3 6 8}, {2 3 5 7 8}, {0 4 5 6 8}, {1 3 4 5 8}, {0 4 6 7 8}, {3 5 6 7 8}, {0 3 5 6 8}	**{0 1 4 5}, {0 1 2 3}, {2 3 6 7}, {0 1 6 7}, {2 3 4 5}, {4 5 6 7}, {0 1 4 8}, {0 1 2 6}, {0 1 3 5}, {0 2 3 6}, {2 3 4 8}, {2 3 5 7}, {2 4 5 8}, {0 4 5 6}, {1 3 4 5}, {0 4 6 7}, {2 6 7 8}, {3 5 6 7}, {2 5 7 8}, {1 3 4 8}, {0 3 5 6},** {0 1 7 8}, {1 2 3 8}, {4 5 7 8}, {1 6 7 8}, {0 4 7 8}, {1 2 6 8}

	Facets of T_i	Facets of ∂T_i
T_8	{1 2 3 4 7}, {0 1 4 5 7}, {0 1 2 3 7}, {2 3 4 5 7}, {0 1 2 6 7}, {0 1 3 5 7}, {0 2 3 6 7}, {0 4 5 6 7}, {1 3 4 5 7}, {0 3 5 6 7}	**{0 1 4 5}, {0 1 2 3}, {2 3 6 7}, {0 1 6 7}, {2 3 4 5}, {4 5 6 7}, {0 1 2 6}, {0 1 3 5}, {0 2 3 6}, {2 3 5 7}, {0 4 5 6}, {1 3 4 5}, {0 4 6 7}, {3 5 6 7}, {0 3 5 6},** {1 2 3 4}, {1 2 6 7}, {2 4 5 7}, {0 1 4 7}, {1 2 4 7}
T_7	{1 2 3 4 6}, {0 1 4 5 6}, {0 1 2 3 6}, {2 3 4 5 6}, {0 1 3 5 6}, {1 3 4 5 6}	**{1 2 3 4}, {0 1 4 5}, {0 1 2 3}, {2 3 4 5}, {0 1 2 6}, {0 1 3 5}, {2 3 5 6}, {0 4 5 6}, {1 3 4 5}, {0 3 5 6},** {0 2 3 6}, {2 4 5 6}, {0 1 4 6}, {1 2 4 6}
T_6	{1 2 3 4 5}, {0 1 2 3 5}, {0 1 2 4 5}	**{1 2 3 4}, {0 1 4 5}, {0 1 2 3}, {2 3 4 5}, {0 1 3 5}, {1 3 4 5},** {0 1 2 4}, {0 2 4 5}, {0 2 3 5}
T_5	{0 1 2 3 4}	**{1 2 3 4}, {0 1 2 3}, {0 1 2 4},** {0 1 3 4}, {0 2 3 4}

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