Discrete confocal quadrics and checkerboard incircular nets

Jan Techter
TU Berlin
February 28, 2020

Discrete confocal quadrics

[1] A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics.
I. An integrable systems approach,
Journal of Integrable Systems (2016)

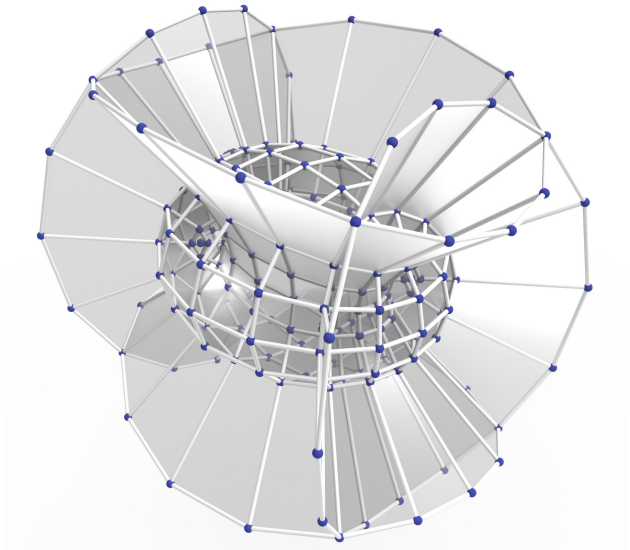
[2] A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics.
II. A geometric approach to general parametrization,
International Mathematics Research Notices (2018)

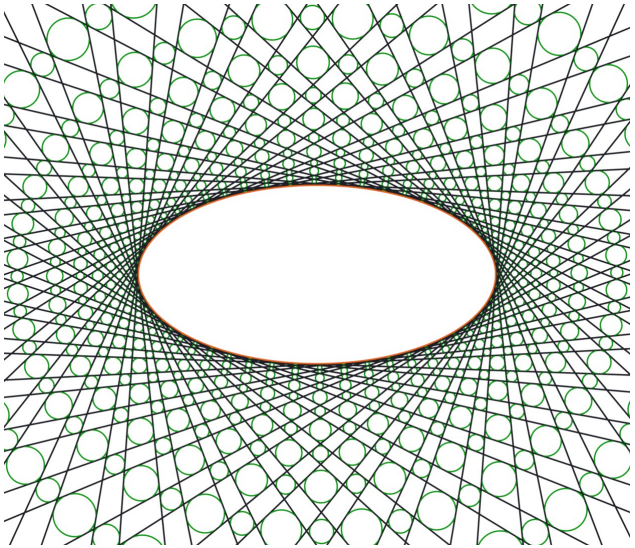
Checkerboard incircular nets

[3] A.I. Bobenko, W.K. Schief, J. Techter.
Checkerboard incircular nets.
Laguerre geometry and parametrisation,
Geometriae Dedicata (2019)

What and why

- **confocal quadrics** constitute an important example of orthogonal coordinate systems with isothermic coordinate surfaces
- no previous structure preserving discretization of confocal quadrics
- **new discrete orthogonality** constraint allowed for such a discretization
- identified incircular nets as a special case
- classification and parametrization of **checkerboard incircular nets**





Nets

Let $U \subset \mathbb{R}^M$ be open and connected. Then a smooth regular map

$$\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N, \qquad \boldsymbol{s} = (s_1, \ldots, s_M) \mapsto \boldsymbol{x}(s_1, \ldots, s_M)$$

is called an M-**dimensional (smooth regular) net**.

- $M = 1$: parametrized curves
- $M = 2$: parametrized surfaces
- $M = N$: coordinate systems

Orthogonal nets

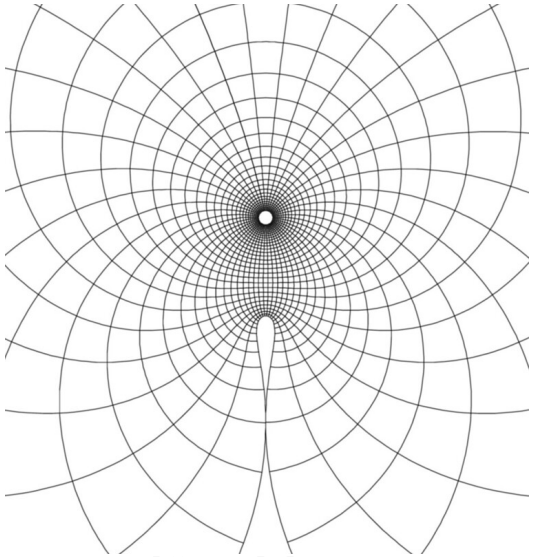
A net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ is called **orthogonal** if

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0, \qquad i, j = 1, \ldots, M, \; i \neq j.$$

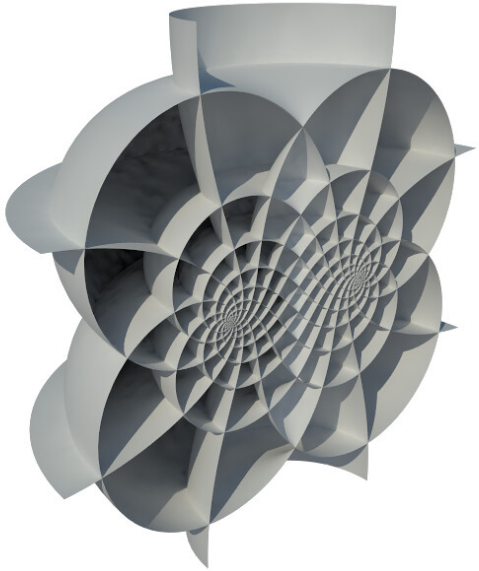
- Möbius invariant
- diagonal first fundamental form

$$\mathrm{I} = H_1^2 \mathrm{d}s_1^2 + \ldots + H_M^2 \mathrm{d}s_M^2,$$

where $H_i^2 = \langle \partial_i \boldsymbol{x}, \partial_i \boldsymbol{x} \rangle$ are called **Lamé coefficients**.



$N = M = 2$



$N = M = 3$

[Pictures by Daniel Piker (CC BY 2.0)]

Theorem (Dupin)

For $M \geqslant 3$ every orthogonal net is conjugate.

- Classical formulation for $M = N = 3$:
  The coordinate surfaces of a triply orthogonal system intersect in common curvature lines.

conjugate nets

A net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ is called **conjugate** if

$$\partial_i \partial_j \boldsymbol{x} \wedge \partial_i \boldsymbol{x} \wedge \partial_j \boldsymbol{x} = 0, \qquad i, j = 1, \ldots, \; i \neq j.$$

- projectively invariant
- condition on every two-dimensional subnet

Discrete nets

1. A map

$$\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N, \qquad \boldsymbol{n} = (n_1, \ldots, n_M) \mapsto \boldsymbol{x}(\boldsymbol{n})$$

   is called an M-**dimensional discrete net**.

2. Denote the **difference operators** by

$$\Delta_i \boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i) - \boldsymbol{x}(\boldsymbol{n})$$

   for every $\boldsymbol{n} \in \mathbb{Z}^M$ and $i = 1, \ldots, M$.

Circular nets as discrete orthogonal nets.

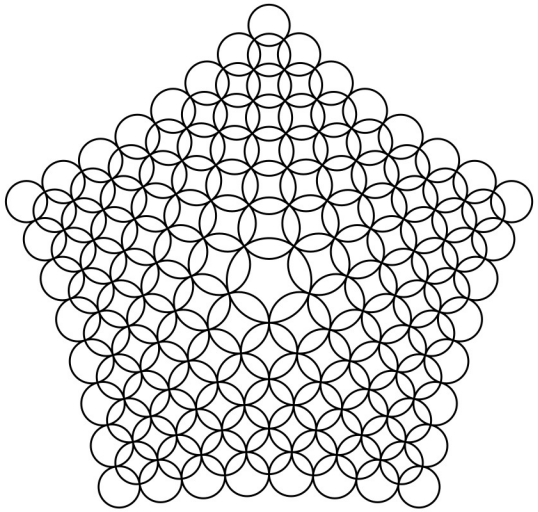
Circular nets

A discrete net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$ is called a **circular net** if all its elementary quadrilaterals are circular, i.e., each four points

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j)), \quad i,j = 1,\dots,M,\ i \neq j,$$

lie on a circle.

- Möbius invariant



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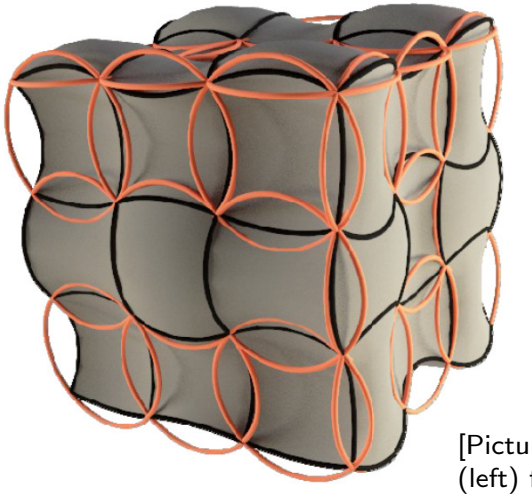


$N = M = 3$

[Pictures:
(left) from DDG book
(right) by M.R. Jimenez, L.M. María]

Circular nets as discrete orthogonal nets.

Circular nets

A discrete net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$ is called a **circular net** if all its elementary quadrilaterals are circular, i.e., each four points

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j)), \quad i,j=1,\dots,M,\ i\neq j,$$

lie on a circle.

- Möbius invariant
- definition already includes discrete conjugacy (even for $M=2$)

Discrete conjugate nets

A discrete net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$ is called **conjugate**, or a **Q-net**, if

$$\Delta_i\Delta_j\boldsymbol{x} \wedge \Delta_i\boldsymbol{x} \wedge \Delta_j\boldsymbol{x} = 0, \qquad i,j=1,\dots,M,\ i\neq j,$$

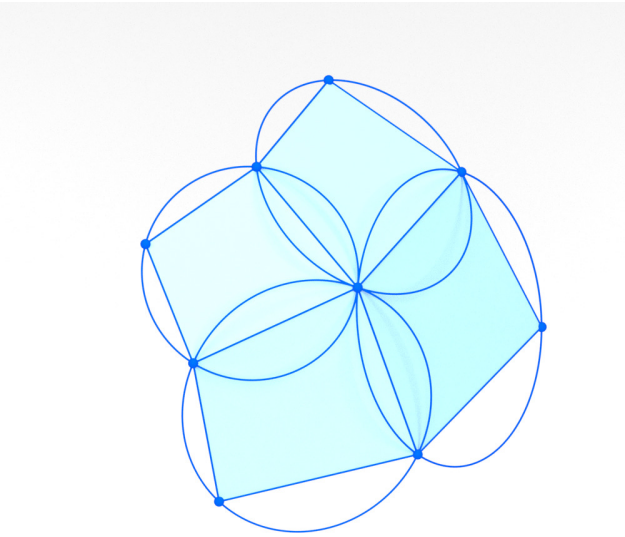
or equivalently, if all its elementary quadrilaterals are coplanar.

Curvature line parametrization

Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ a parametrized surface. Then

$$\boldsymbol{x} \text{ curvature line parametrization } \Leftrightarrow \boldsymbol{x} \text{ conjugate and orthogonal.}$$

- Let $\boldsymbol{m} : U \to \mathbb{S}^3$ be the stereographic projection of $\boldsymbol{x}$ to the sphere
- $\boldsymbol{x} : U \to \mathbb{R}^3$ curvature line param. $\Leftrightarrow \boldsymbol{m} : U \to \mathbb{S}^3$ conjugate
- Möbius invariant
- discretized by circular nets (discrete conjugate nets in $\mathbb{S}^3$)



Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrized surface.

- the tangent plane of $\boldsymbol{x}$ at $(s_1, s_2) \in U$ is given by

$$\left\{ \boldsymbol{X} \in \mathbb{R}^3 \mid \boldsymbol{\nu}(s_1, s_2) \cdot \boldsymbol{X} = d \right\}$$

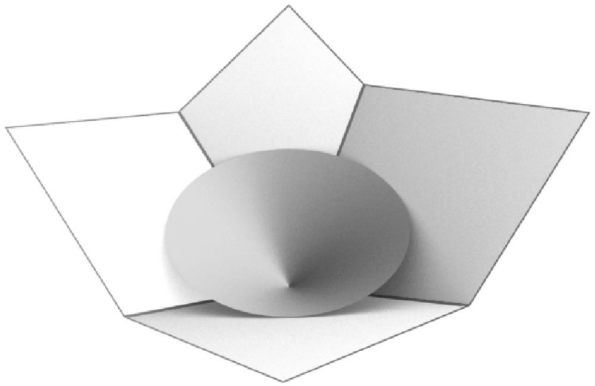
  where

$$\boldsymbol{\nu} = \frac{\partial_1 \boldsymbol{x} \times \partial_2 \boldsymbol{x}}{\|\partial_1 \boldsymbol{x} \times \partial_2 \boldsymbol{x}\|}, \qquad \boldsymbol{\nu} \cdot \boldsymbol{x} - d = 0.$$

- identify oriented tangent planes with points on the Blaschke cylinder

$$\boldsymbol{v} = (\boldsymbol{\nu}, -d) \in \mathcal{Z} = \mathbb{S}^2 \times \mathbb{R}$$

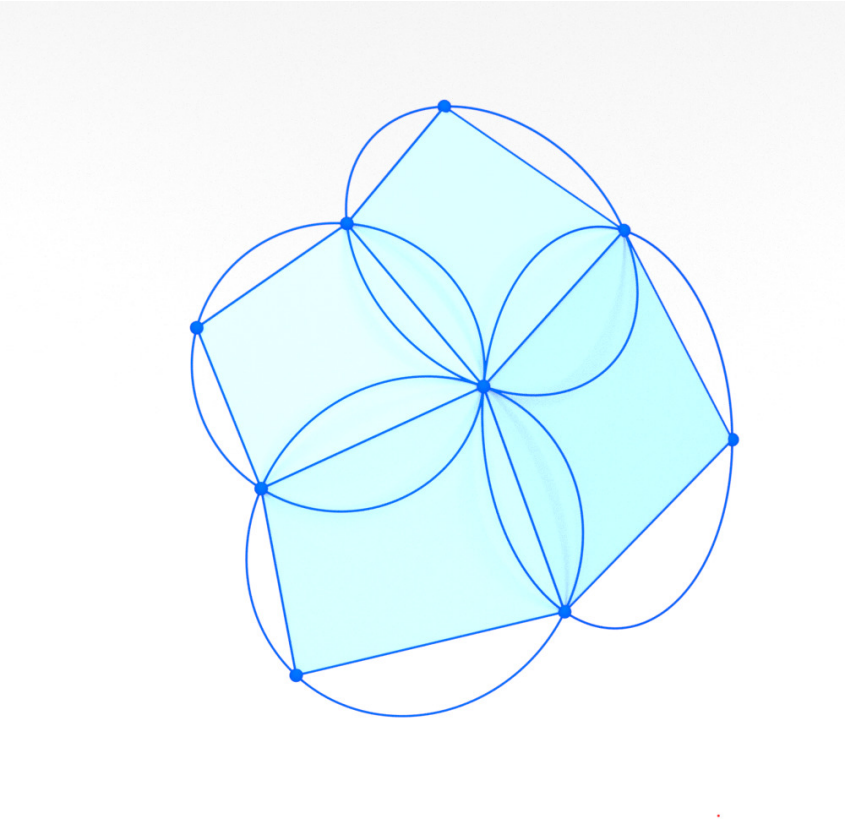
- $\boldsymbol{x} : U \to \mathbb{R}^3$ curvature line parametrization $\Leftrightarrow$ $\boldsymbol{v} : U \to \mathcal{Z}$ conjugate
- Laguerre invariant
- discretized by conical nets (discrete conjugate nets in $\mathcal{Z}$)



[Picture from DDG book]

Associated circular and conical nets [PW]

For every circular net there is a three-parameter family of associated conical nets (and vice versa).



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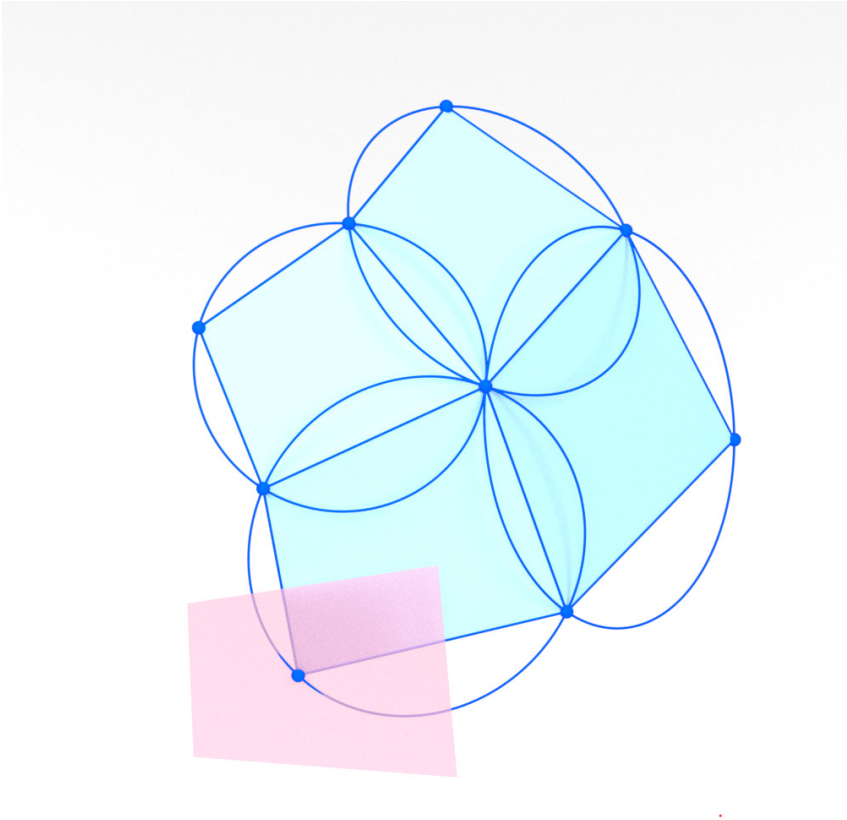

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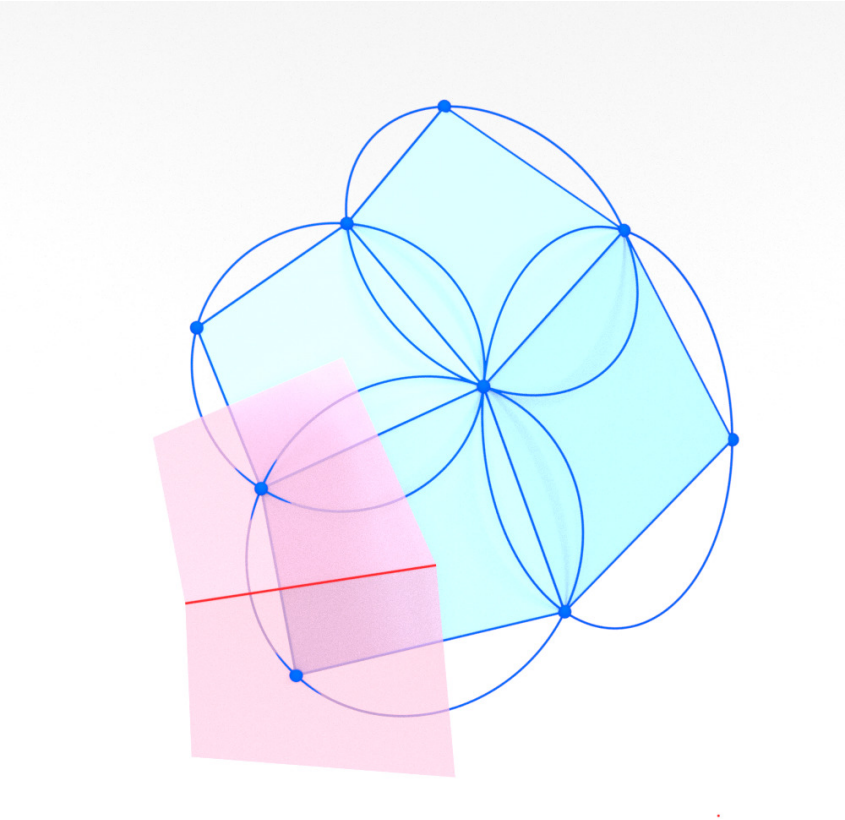

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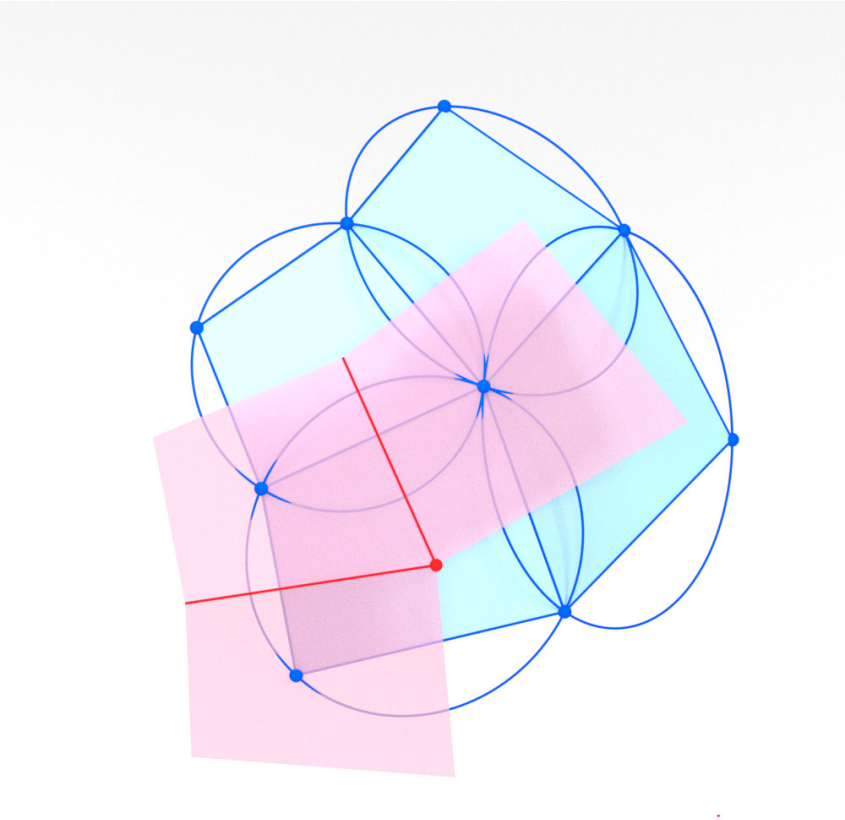

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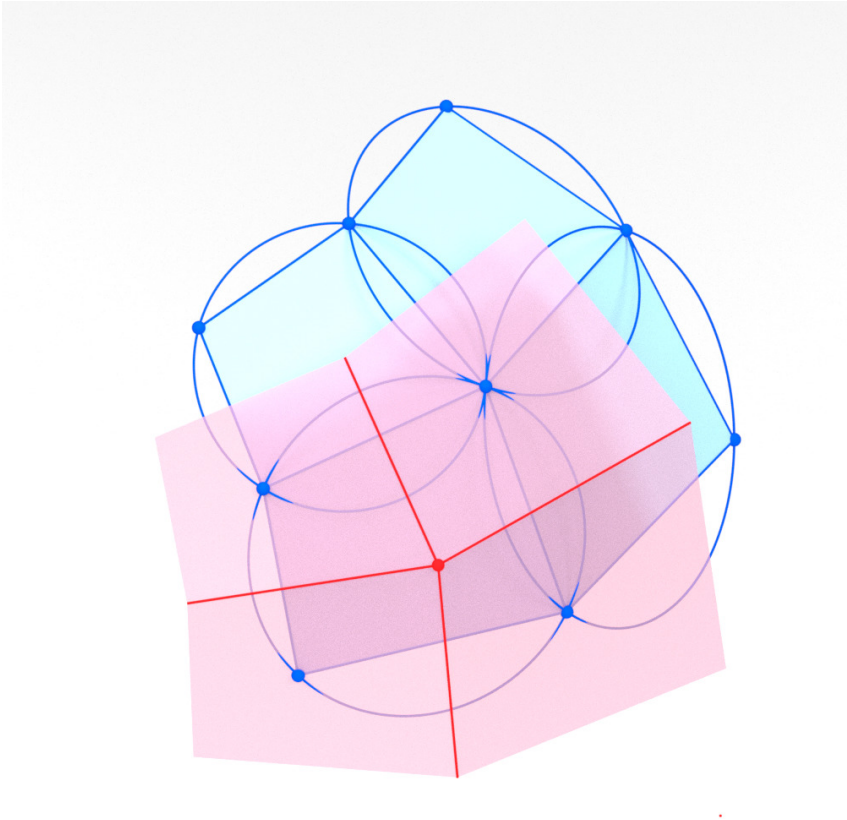

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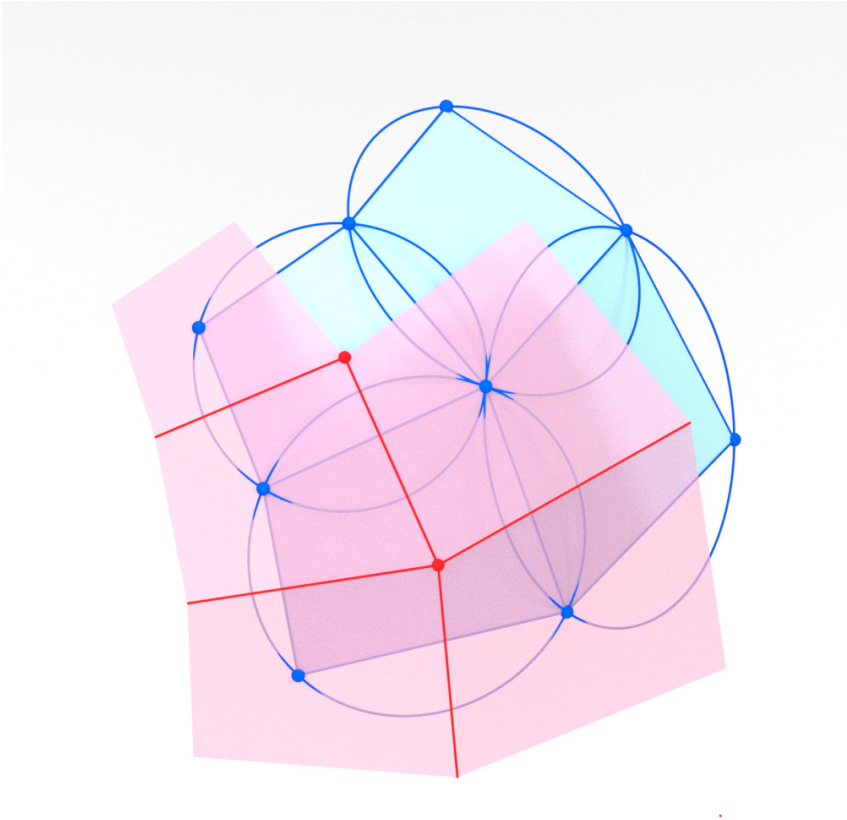

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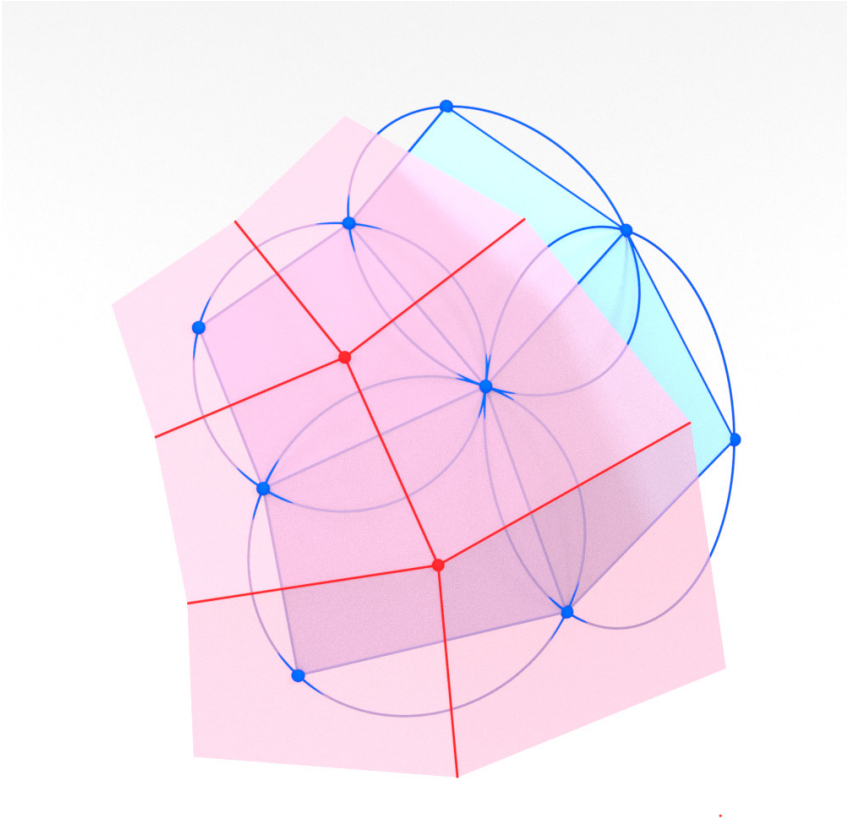

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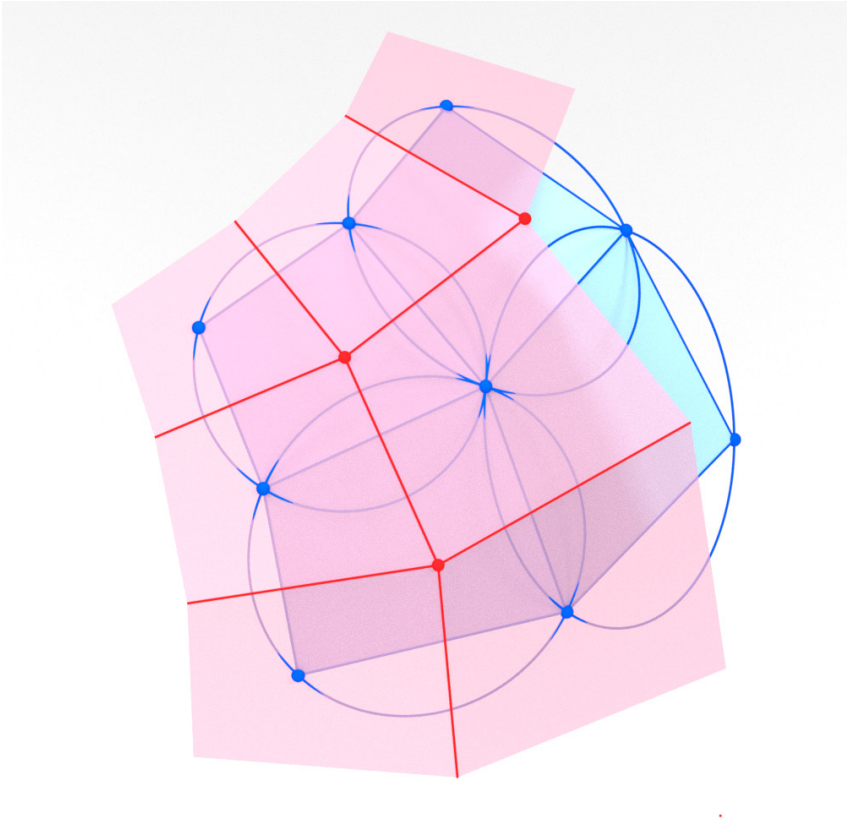

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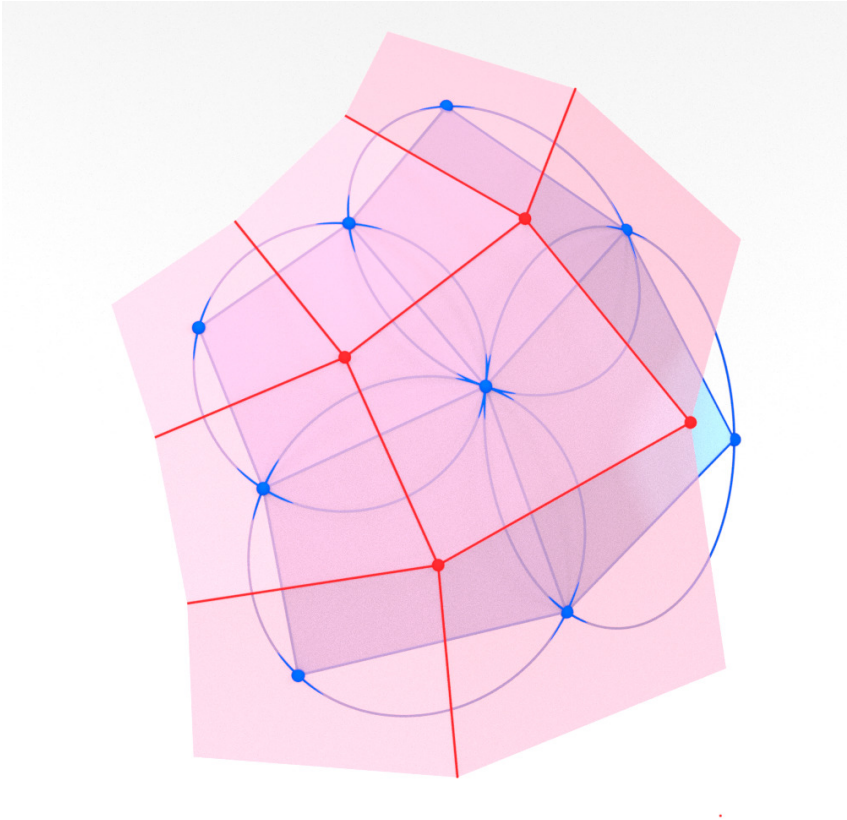

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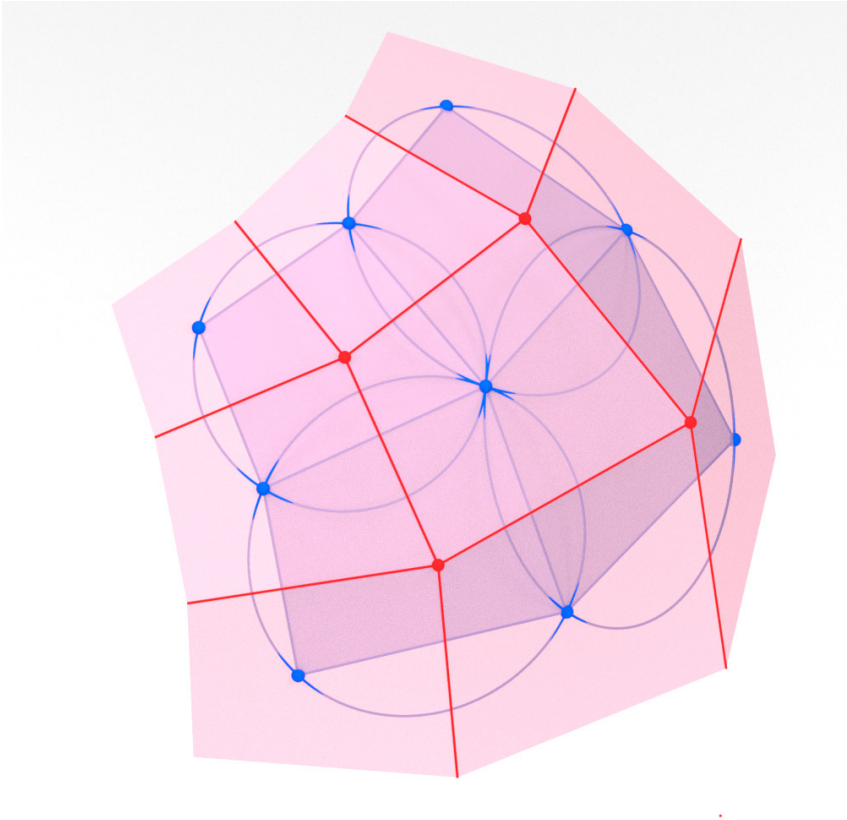

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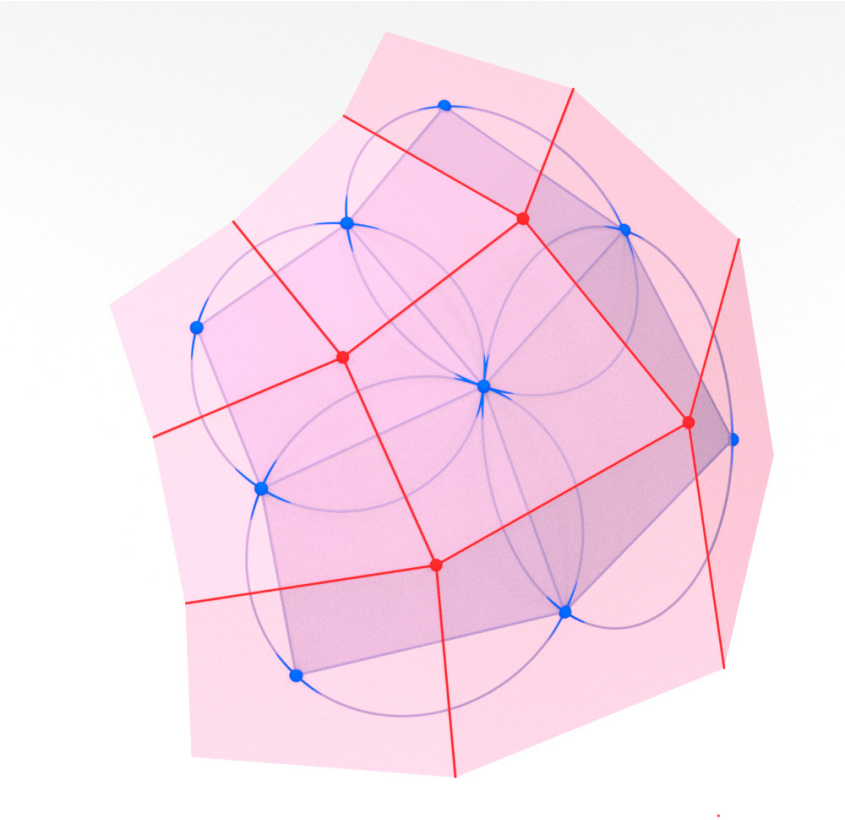

- pairs of circular and conical nets have **orthogonal dual edges**

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circular nets
point representation
Möbius invariant

conical nets
tangential representation
Laguerre invariant

orthogonal pairs
combined representation
similarity invariant

- pairs of circular and conical nets have **orthogonal dual edges**

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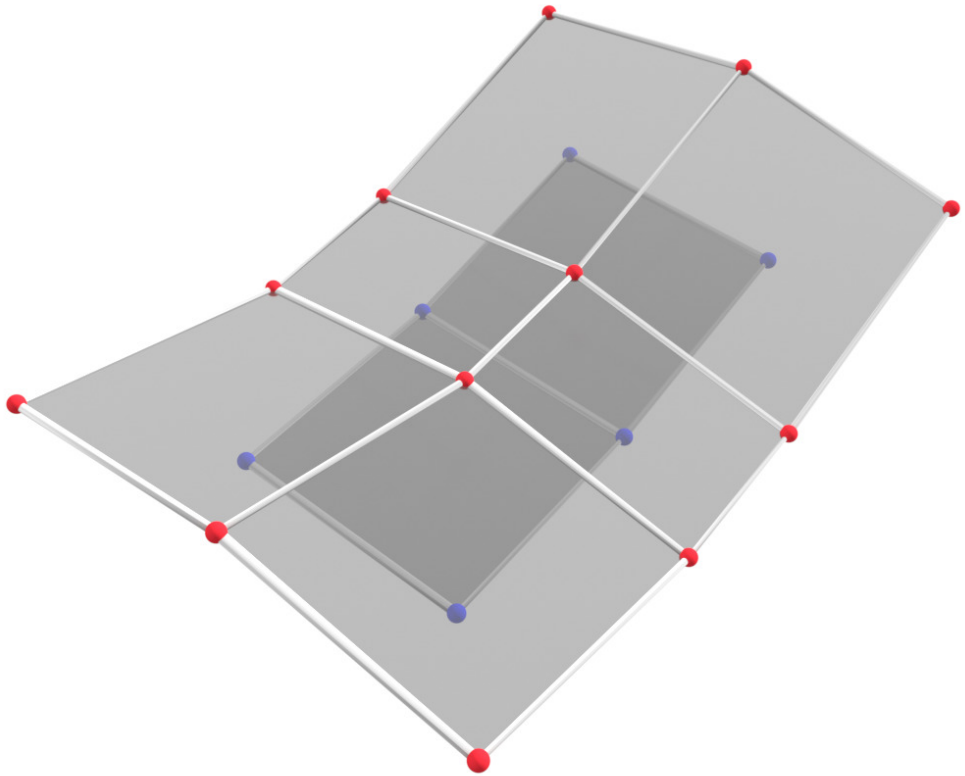
General pairs of dual orthogonal nets [PW, PJWP]



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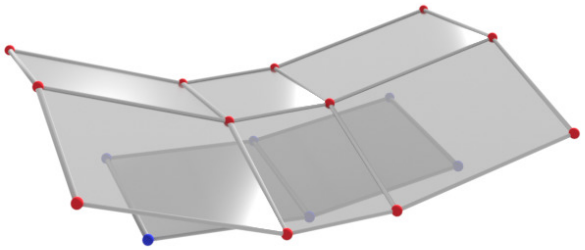

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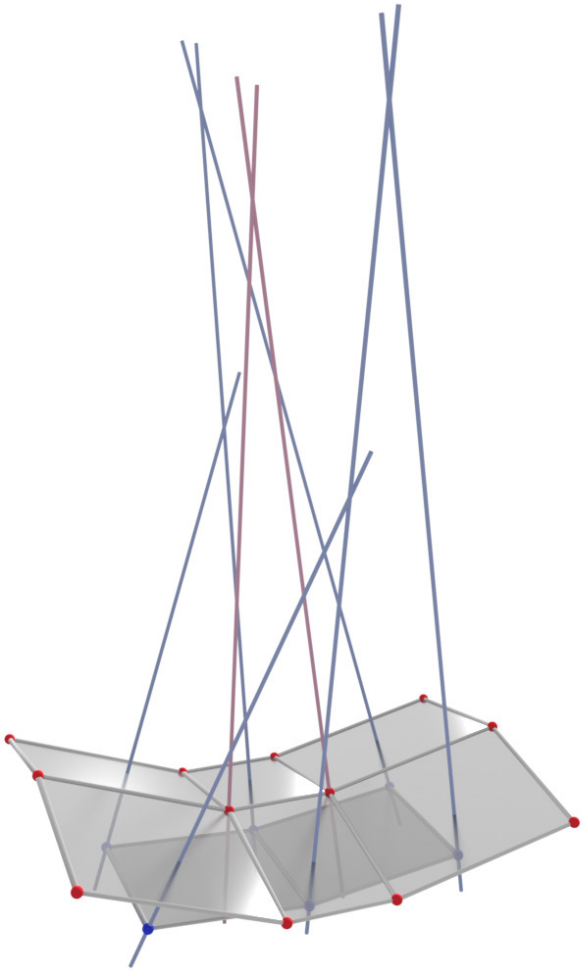
General pairs of dual orthogonal nets [PW, PJWP]

- one normal line per vertex-face pair



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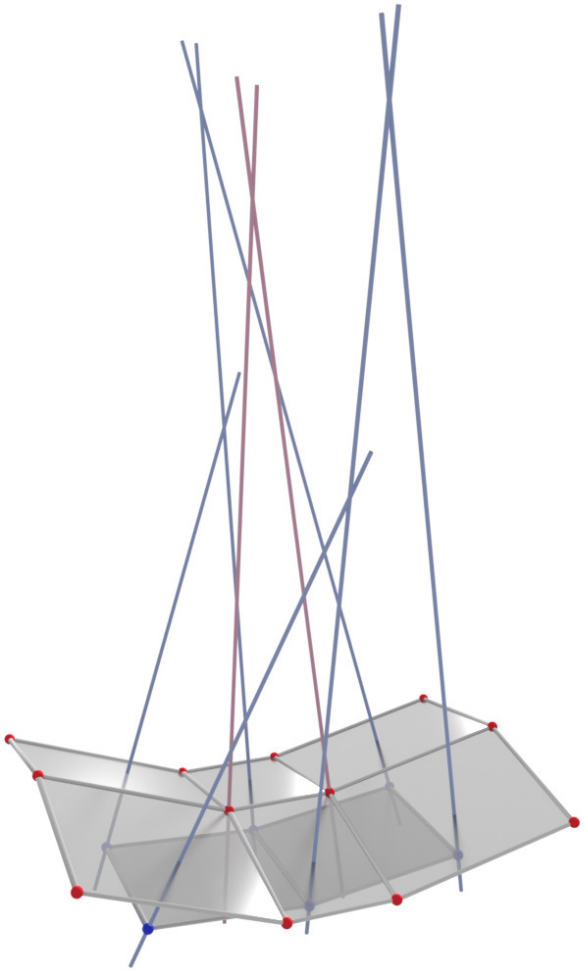
General pairs of dual orthogonal nets [PW, PJWP]

- one normal line per vertex-face pair
- adjacent normals intersect



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General pairs of dual orthogonal nets [PW, PJWP]

- one normal line per vertex-face pair
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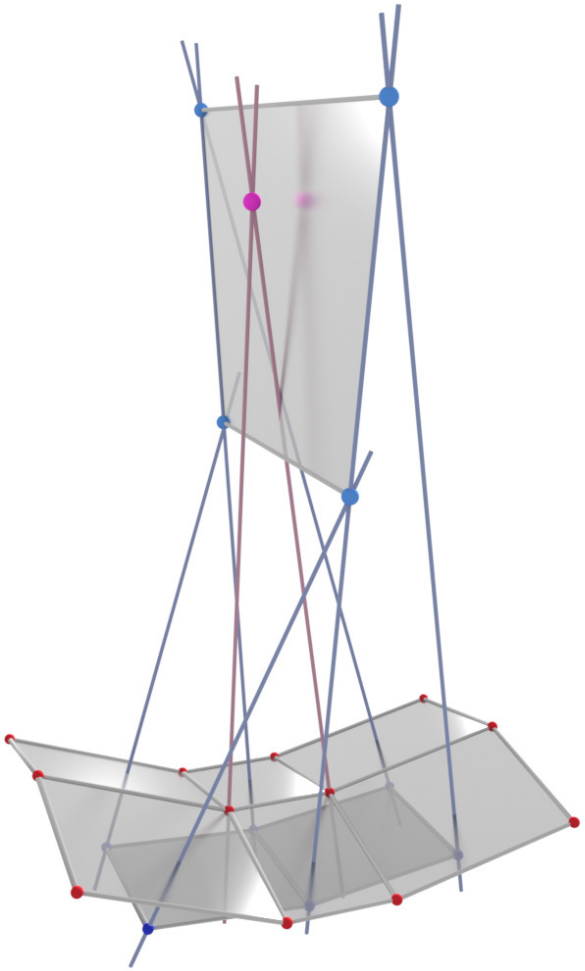
- discrete focal surfaces



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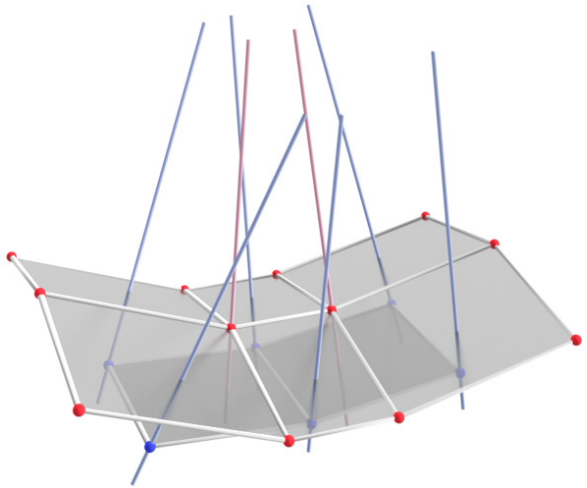
General pairs of dual orthogonal nets [PW, PJWP]

- one normal line per vertex-face pair
- adjacent normals intersect
- discrete focal surfaces
- discrete parallel surfaces

  (extension to triply orthogonal system with $H_3^2 = H_3^2(n_3)$)



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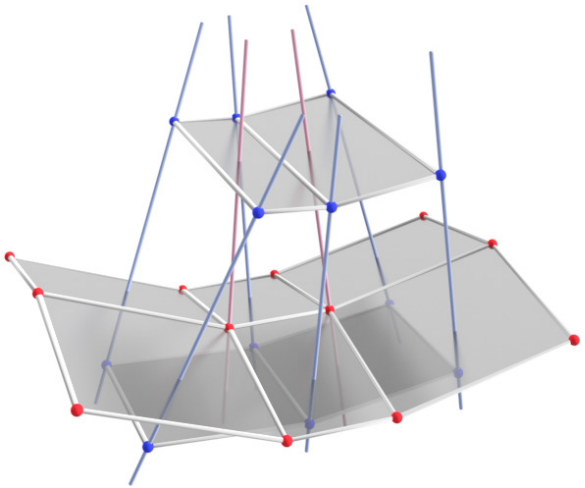

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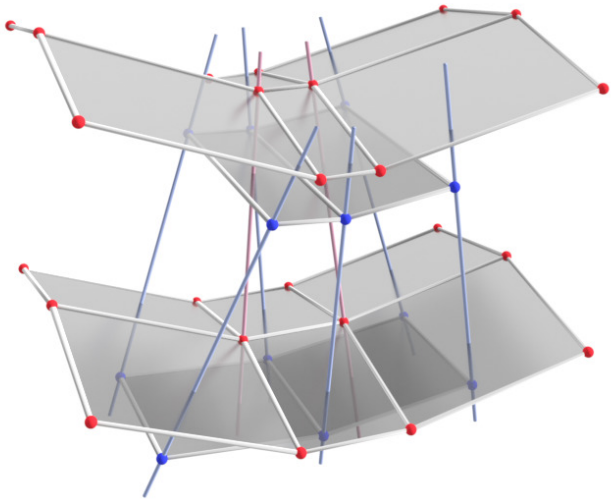
General pairs of dual orthogonal nets [PW, PJWP]

- one normal line per vertex-face pair
- adjacent normals intersect
- discrete focal surfaces
- discrete parallel surfaces

  (extension to triply orthogonal system with $H_3^2 = H_3^2(n_3)$)
- characterization of discrete channel surfaces and Dupin cyclides



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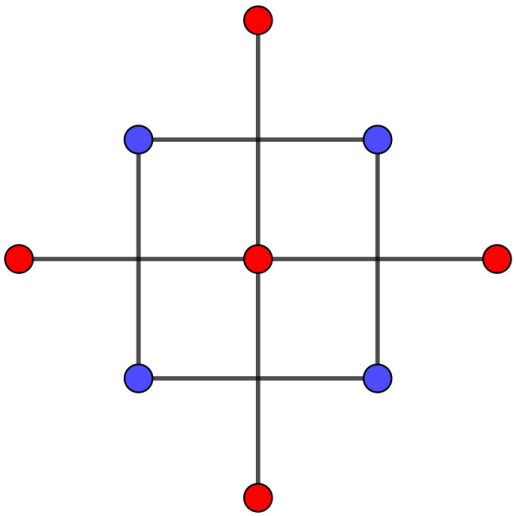
Pairs of dual discrete nets

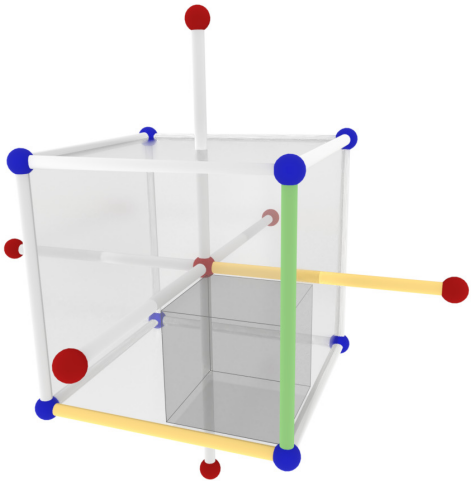
A map

$$\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \tfrac{1}{2}\right)^M \to \mathbb{R}^N$$

is called a **pair of dual discrete nets**.



$\mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2$



$\mathbb{Z}^3 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^3$

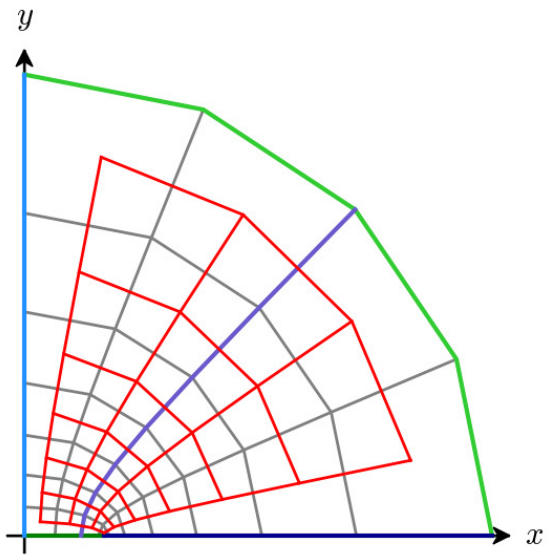
Orthogonal pairs of dual discrete nets [1]

A pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathbb{R}^N$ is called **orthogonal** if every pair of dual edges is orthogonal in $\mathbb{R}^N$, i.e.,

$$\langle \Delta_i \boldsymbol{x}(\boldsymbol{n}), \Delta_j \boldsymbol{x}(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}) \rangle = 0, \qquad i, j = 1, \ldots, M$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^M$ with $\sigma_i = 1$ and $\sigma_j = -1$.

- invariant under translation of each of its two discrete subnets
- invariant under similarity transformations
  (Möbius invariant version possible)



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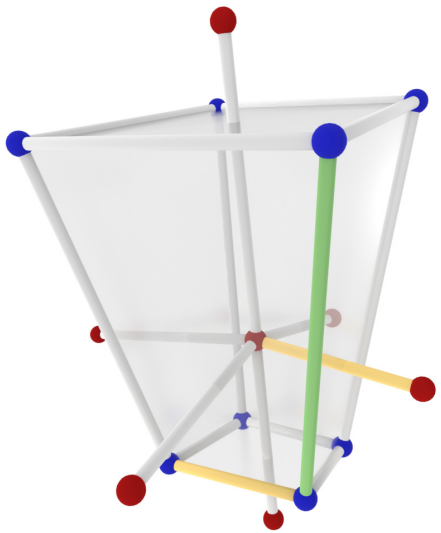


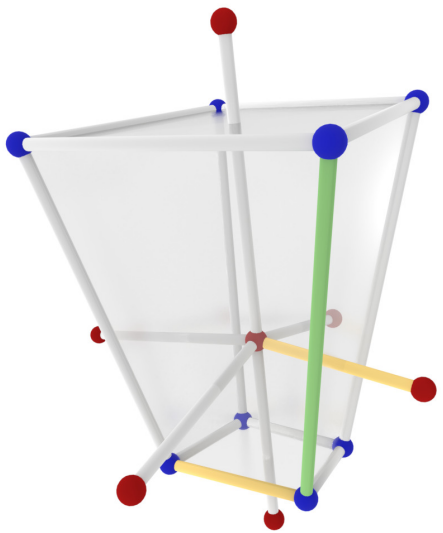
$N = M = 3$

Theorem (discrete Dupin, [2])

Let $M \geqslant 3$ *and* $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathbb{R}^N$ *be an orthogonal pair of dual discrete nets. Then its two discrete subnets*

$$\boldsymbol{x}\big|_{\mathbb{Z}^M} \text{ and } \boldsymbol{x}\big|_{\left(\mathbb{Z}+\frac{1}{2}\right)^M}$$

are discrete conjugate nets, i.e., have planar faces.

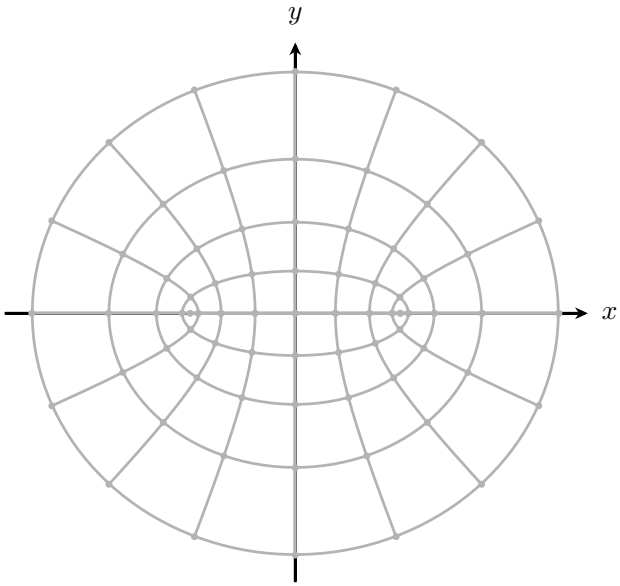


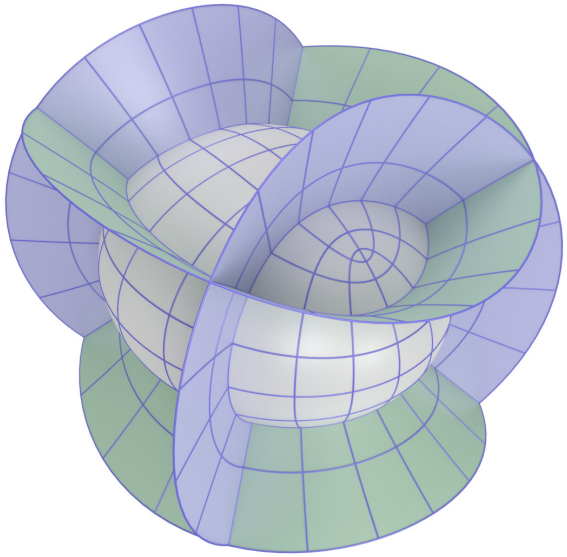
$N = M = 3$

Confocal quadrics

Given $a_1 > a_2 > \ldots > a_N$.
The one-parameter family of confocal quadrics is given by:

$$Q(u) = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \sum_{k=1}^{N} \frac{x_k^2}{a_k + u} = 1 \right\}, \quad u \in \mathbb{R}.$$

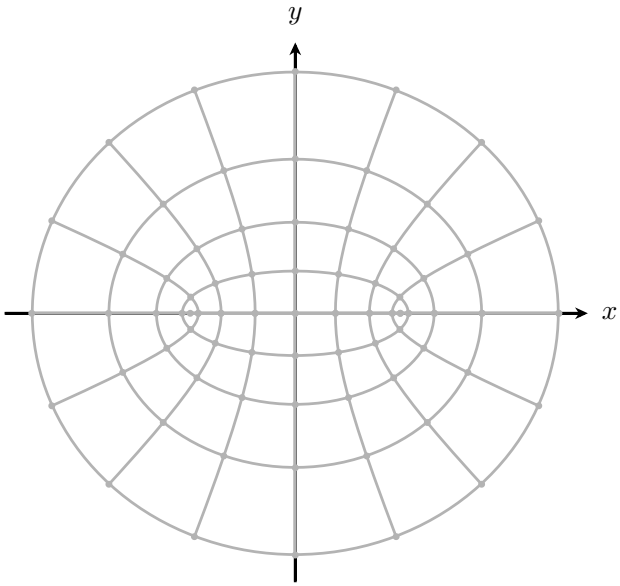


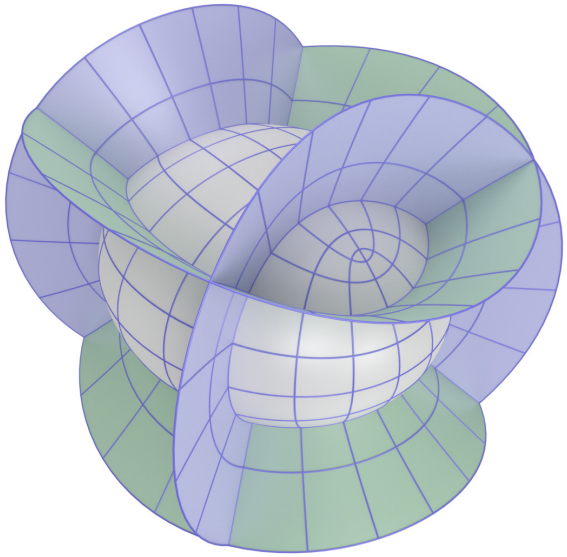


- given $(x_1, \ldots, x_N) \in \mathbb{R}^N$ with $x_1 \cdots x_N \neq 0$ the N roots $-a_1 < u_1 < -a_2 < u_2 < \cdots < -a_N < u_N$ of equation

$$\sum_{k=1}^{N} \frac{x_k^2}{a_k + u} = 1$$

  correspond to N confocal quadrics $Q(u_i)$ of different (affine) type
- two quadrics $Q(u_i)$ and $Q(u_j)$ of different type intersect orthogonally
- two quadrics $Q(u_i)$ and $Q(\tilde{u}_i)$ of the same type do not intersect





Confocal coordinates

We call a coordinate system $\boldsymbol{x} : \mathbb{R}^N \supset U \to \mathbb{R}^N$ a **confocal coordinate system** if its coordinate hypersurfaces consist of confocal quadrics.

- $\boldsymbol{x}$ is a confocal coordinate system if and only if

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{s})^2}{a_k + u_i(s_i)} = 1, \quad i = 1, \ldots, N$$

  with some $a_1 > \ldots > a_N$,
  and some functions $u_1(s_1), \ldots, u_N(s_N)$ in the intervals $(-a_1, -a_2), \ldots, (-a_N, \infty)$.
- a confocal coordinate system is uniquely determined by
  - the constants a_k (confocal family) and
  - the functions u_i (reparametrization along coordinate lines)

Main Theorem 1 [2]

If a coordinate system $\boldsymbol{x} : \mathbb{R}^N \supset U \to \mathbb{R}^N$ satisfies the two conditions:

i) $\boldsymbol{x}$ **factorizes**, in the sense that

$$x_k(\boldsymbol{s}) = f_1^k(s_1) f_2^k(s_2) \cdots f_N^k(s_N), \quad k = 1, \ldots, N$$

with $f_i^k(s_i) \neq 0$ and $\left(f_i^k\right)'(s_i) \neq 0$, and

ii) $\boldsymbol{x}$ is **orthogonal**, that is,

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0 \quad for \quad i \neq j,$$

then all coordinate hypersurfaces are confocal quadrics,
i.e., $\boldsymbol{x}$ is a confocal coordinate system.

Discrete confocal coordinates [2]

A discrete coordinate system $\boldsymbol{x} : \left(\frac{1}{2}\mathbb{Z}\right)^N \supset U \to \mathbb{R}^N$ is called a **discrete confocal coordinate system** if it satisfies two conditions:

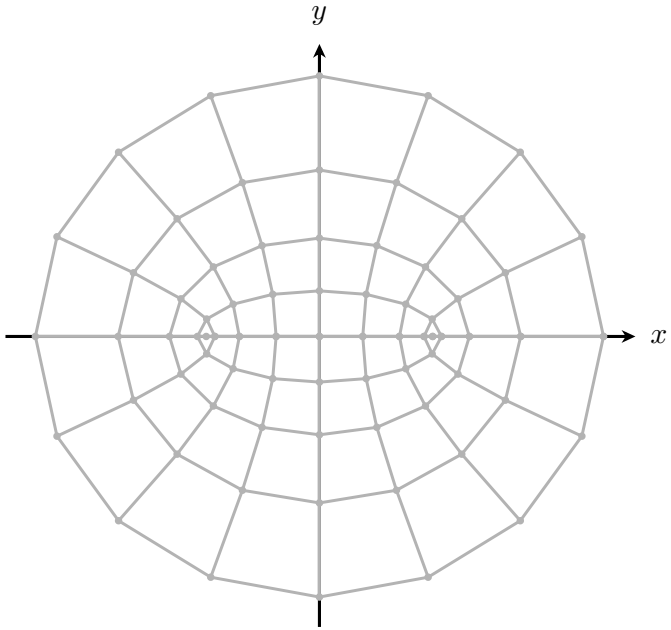
i) $\boldsymbol{x}$ **factorizes**, in the sense that for any $\boldsymbol{n} \in \mathcal{U}$

$$x_k(\boldsymbol{n}) = f_1^k(n_1) f_2^k(n_2) \cdots f_N^k(n_N), \quad k = 1, \ldots, N$$

with $f_i^k(n_i) \neq 0$ and $\Delta f_i^k(n_i) \neq 0$, and

ii) $\boldsymbol{x}$ is **orthogonal** (all pairs of dual subnets).

- stepsize-$\frac{1}{2}$-lattice contains 2^{N-1} orthogonal pairs



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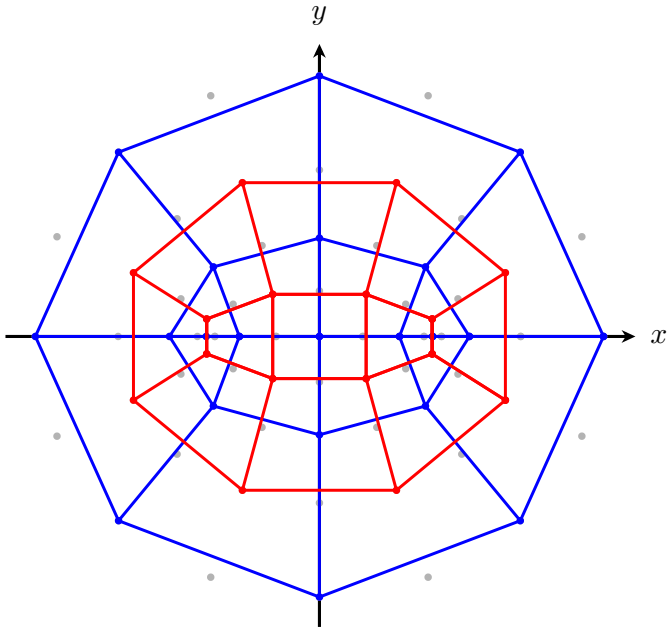

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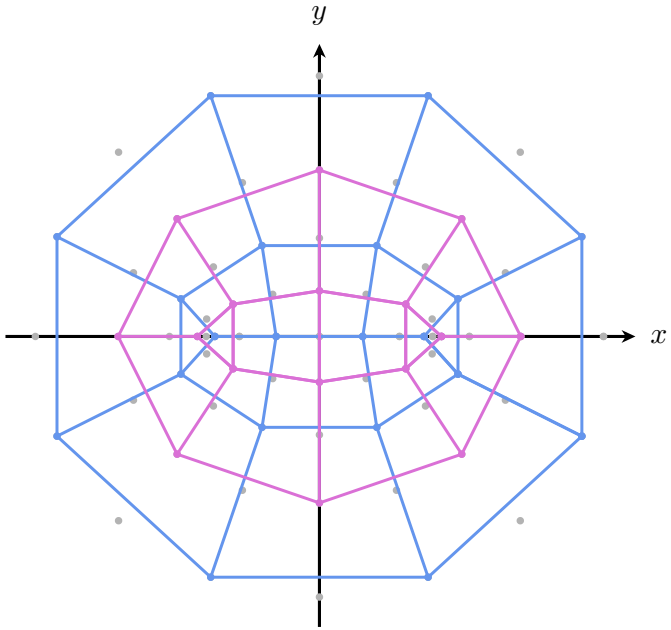


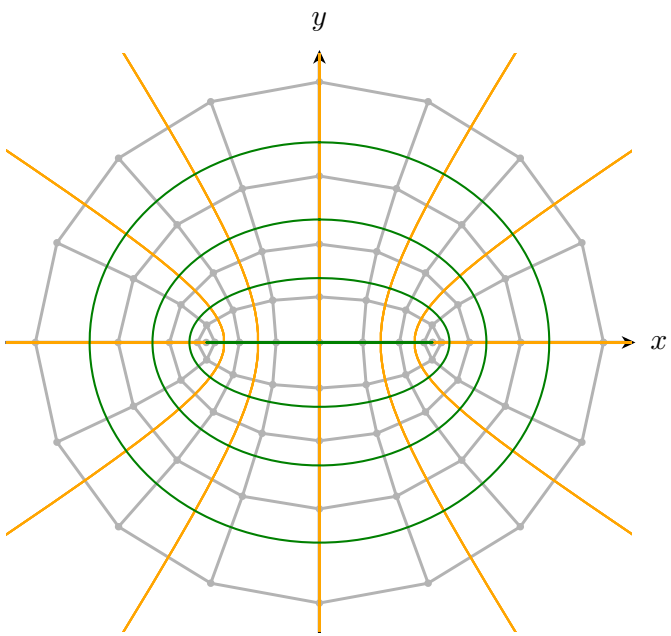
Main Theorem 2 [2]

For a discrete confocal coordinate system, there exist $a_1, \dots, a_N \in \mathbb{R}$, and sequences $u_i : \left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right) \to \mathbb{R}$, $i = 1, \dots, N$, such that

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n}) x_k(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})}{a_k + u_i(n_i + \frac{1}{4}\sigma_i)} = 1, \qquad i = 1, \dots, N.$$

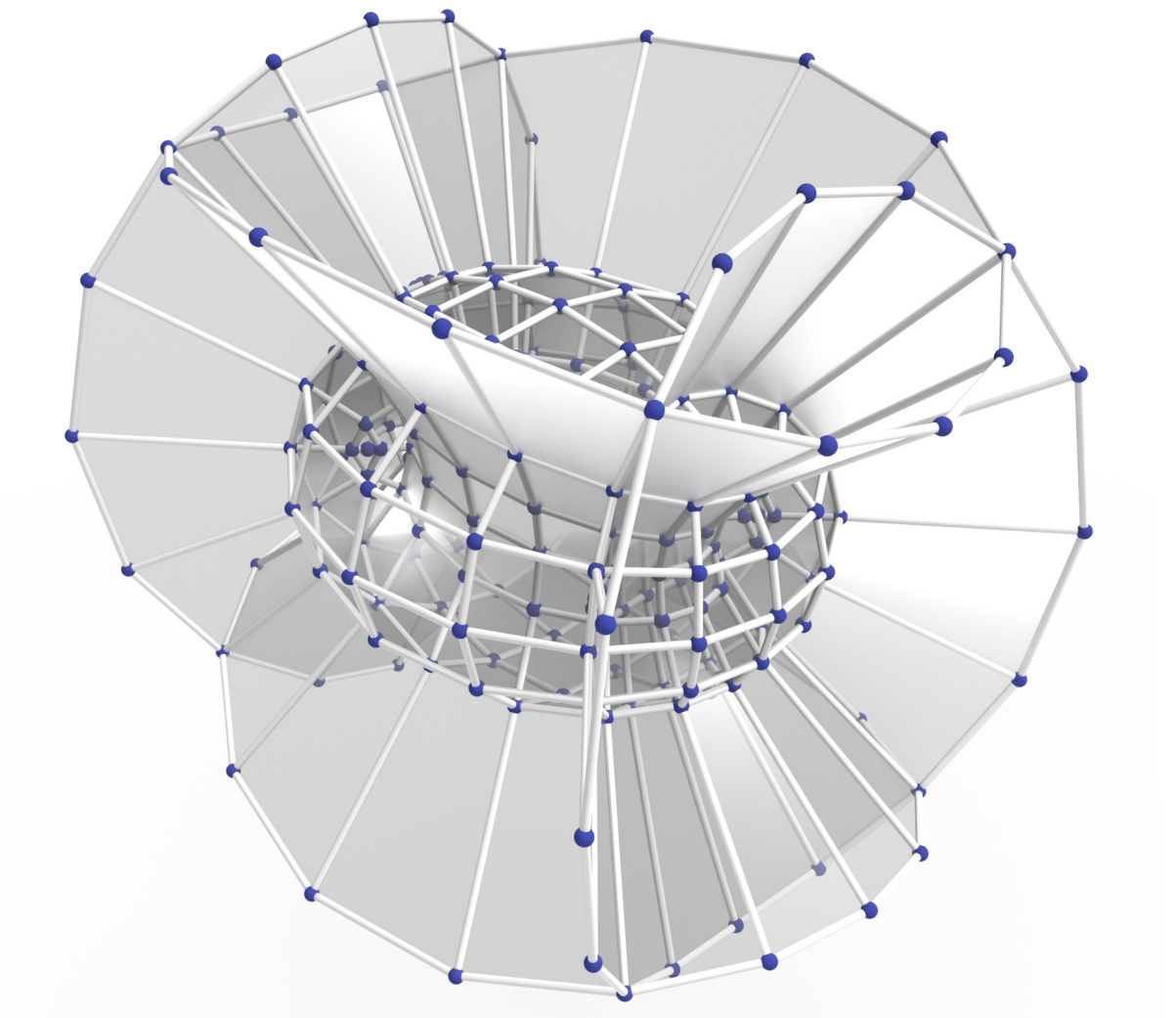
for any $\boldsymbol{n} \in \mathcal{U}$ and $\boldsymbol{\sigma} \in \{\pm 1\}^N$.

- proof by "discretization of smooth proof"
- geometric construction by polarity (generalizable to dual pencils)
- reparametrization captured in the functions u_i



$\left[\sum_{k=1}^{N} \frac{x_k(\boldsymbol{s})^2}{a_k + u_i(s_i)} = 1\right]$

Example ($N = 3$): Explicit solution by solving functional equations for f_i^k in terms of elliptic functions [2].



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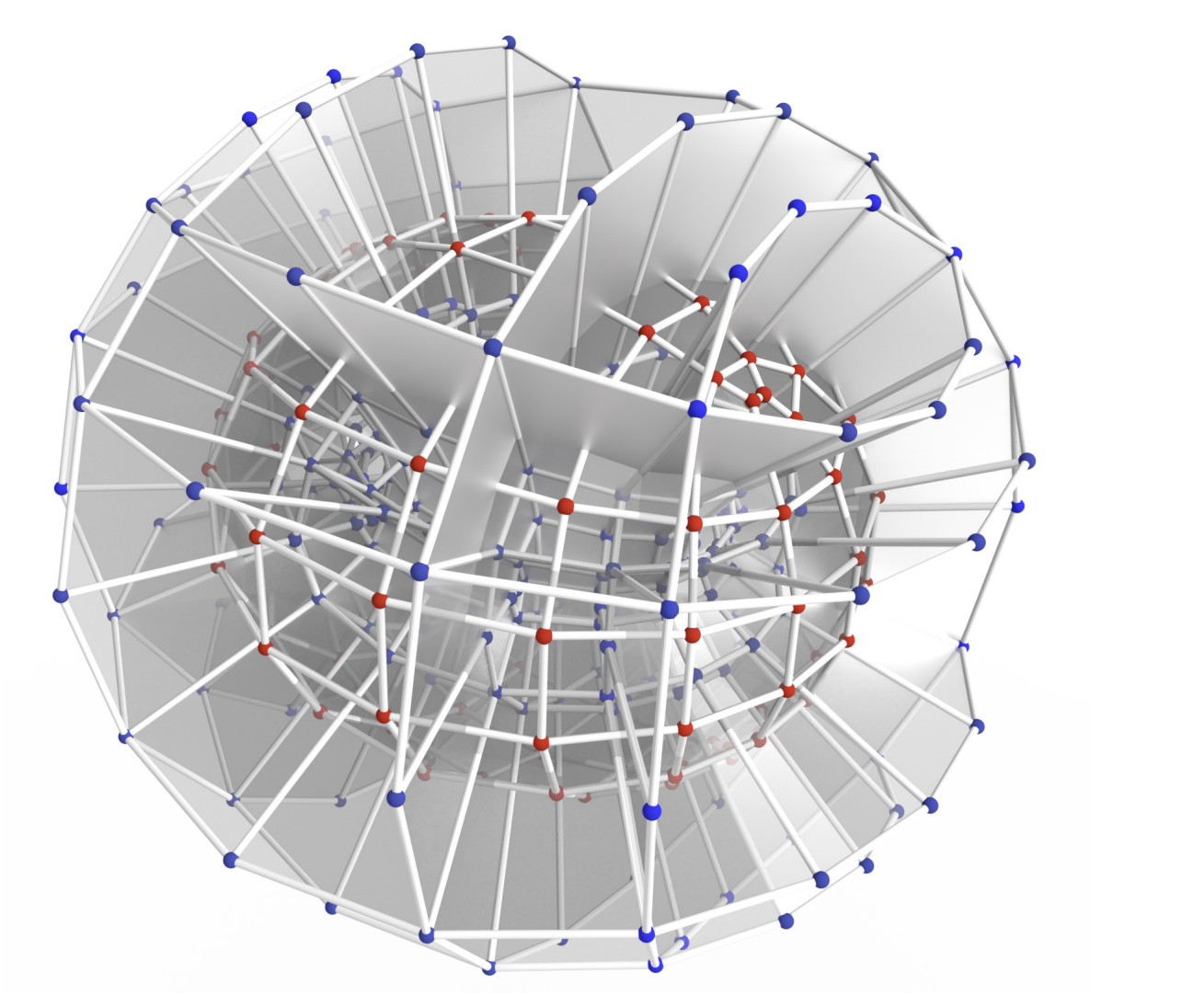
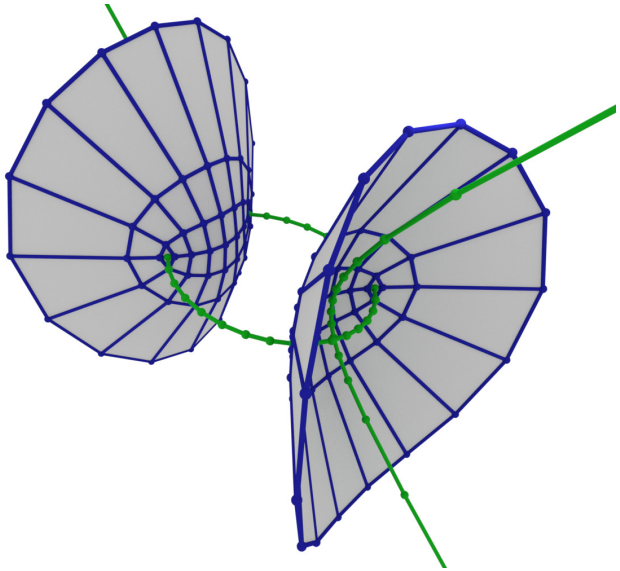
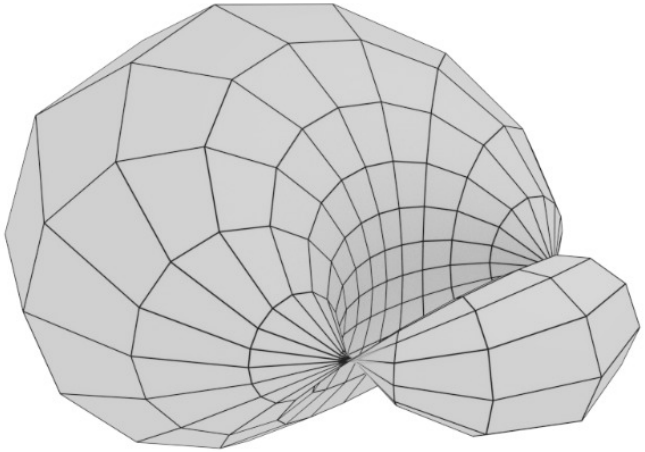


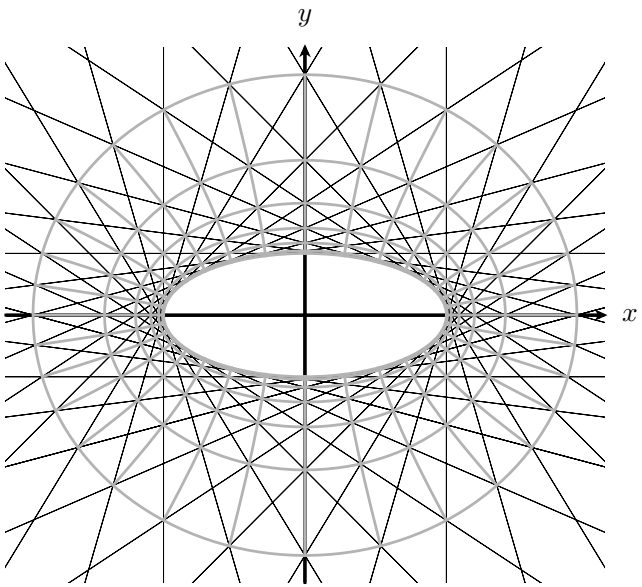
Properties of discrete confocal coordinates

- factorizable
- discrete orthogonal
- geometric construction
  (generalizable to dual pencils and thus to hyperbolic / elliptic geometry)
- 2D-subnets discrete isothermic
- satisfy generalized discrete Euler-Poisson-Darboux equation (3D-consistent)
- discrete umbilical points (focal conics)
  and corresponding discrete Dupin cyclides





Example with straight diagonal lines.



- explicit parametrization [2]

[AB] A.V. Akopyan, A.I. Bobenko. Incircular nets and confocal conics, *Trans. AMS* **370:4** (2018) 2825–2854.

[Bö] W. Böhm. Verwandte Sätze über Kreisvierseitnetze, *Arch. Math. (Basel)* **21** (1970) 326–330.

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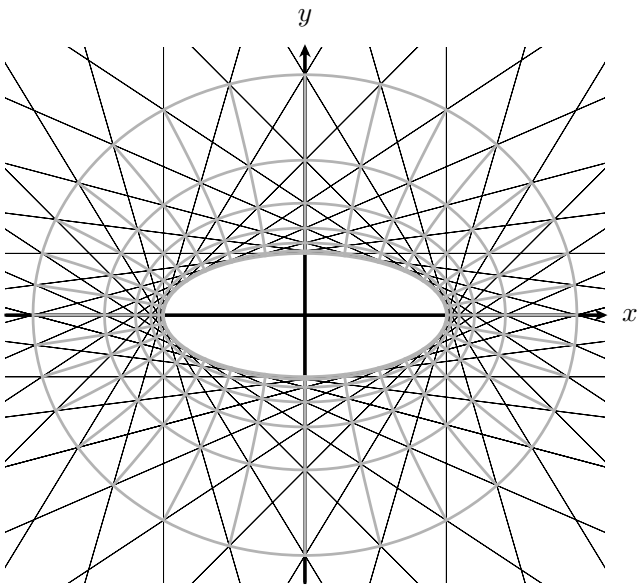

- explicit parametrization [2]
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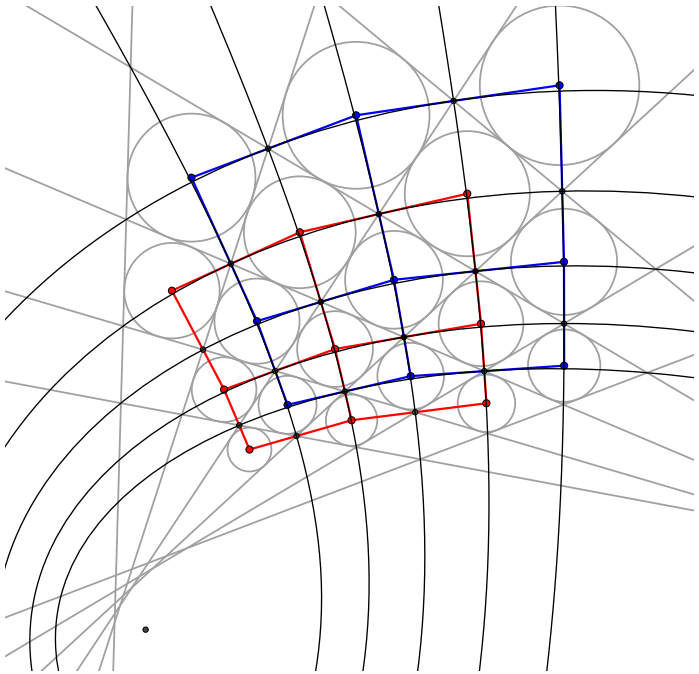

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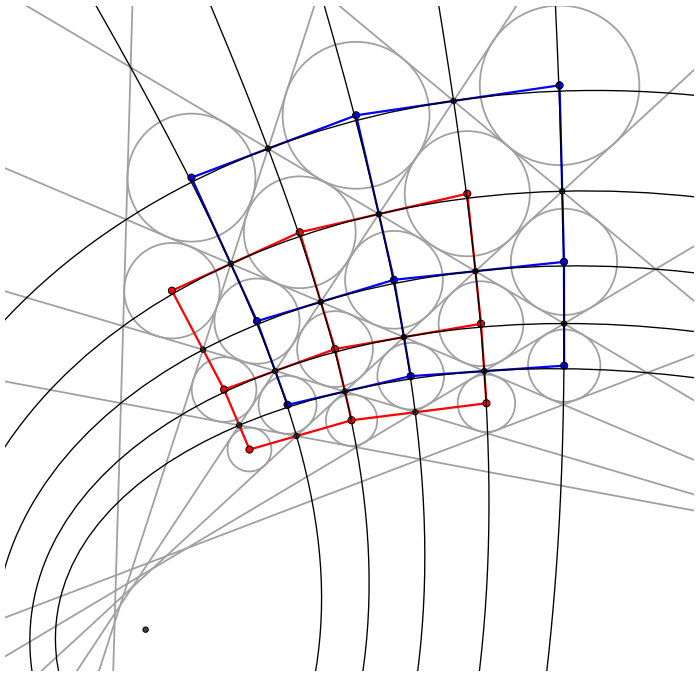

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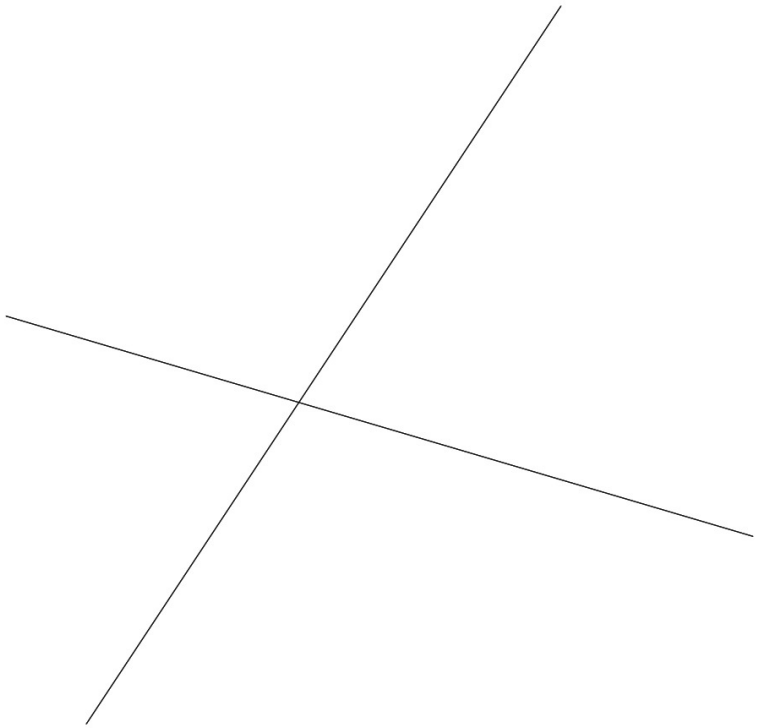
Elementary construction of **incircular nets** [Bö, AB]



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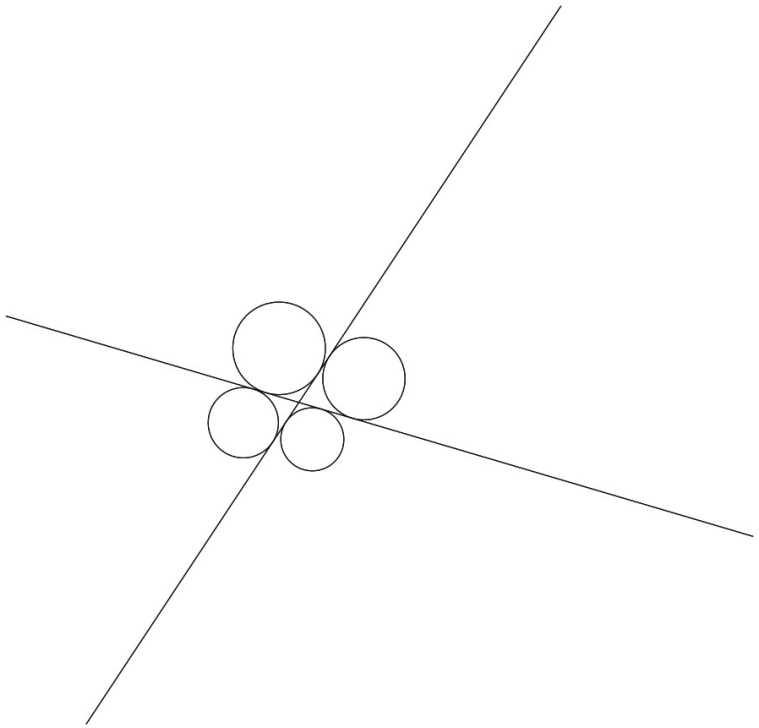

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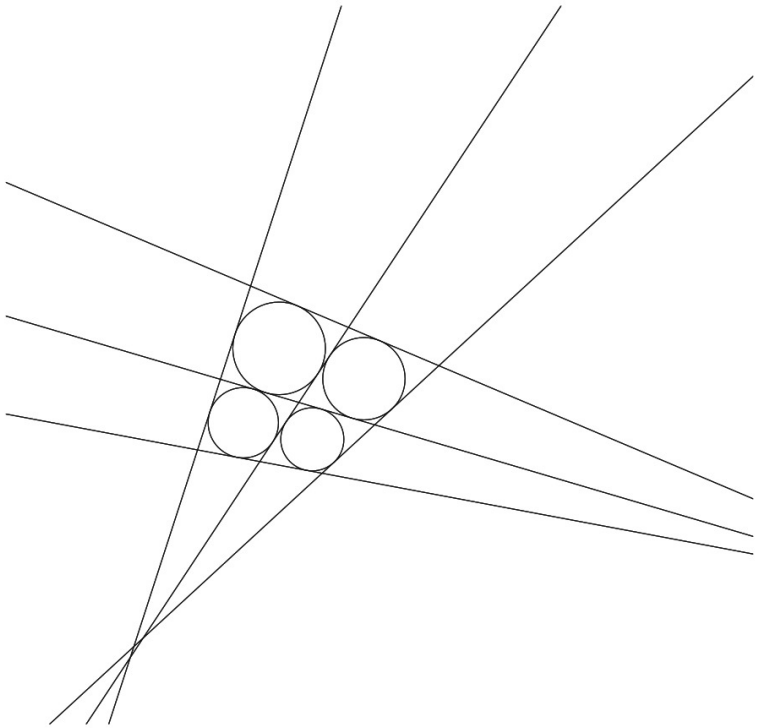

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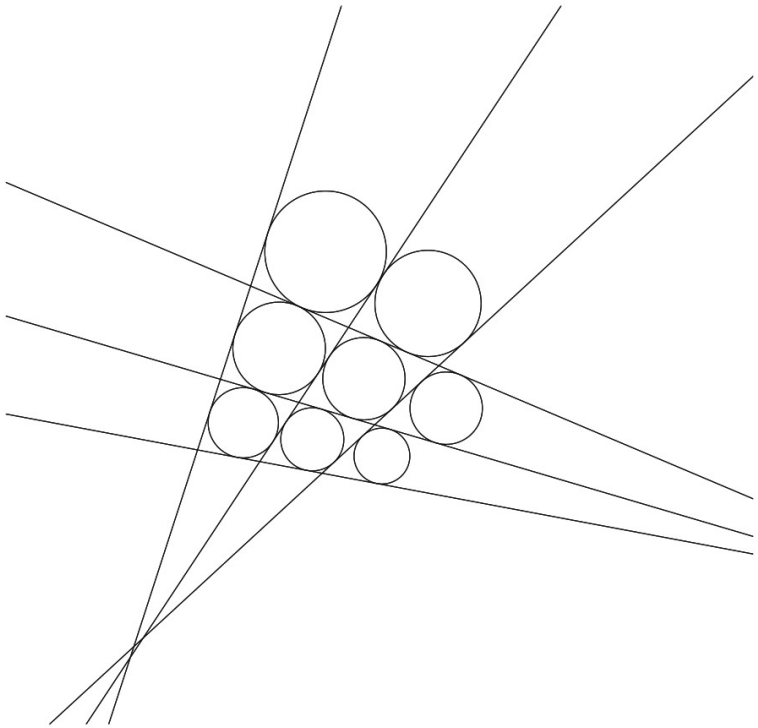

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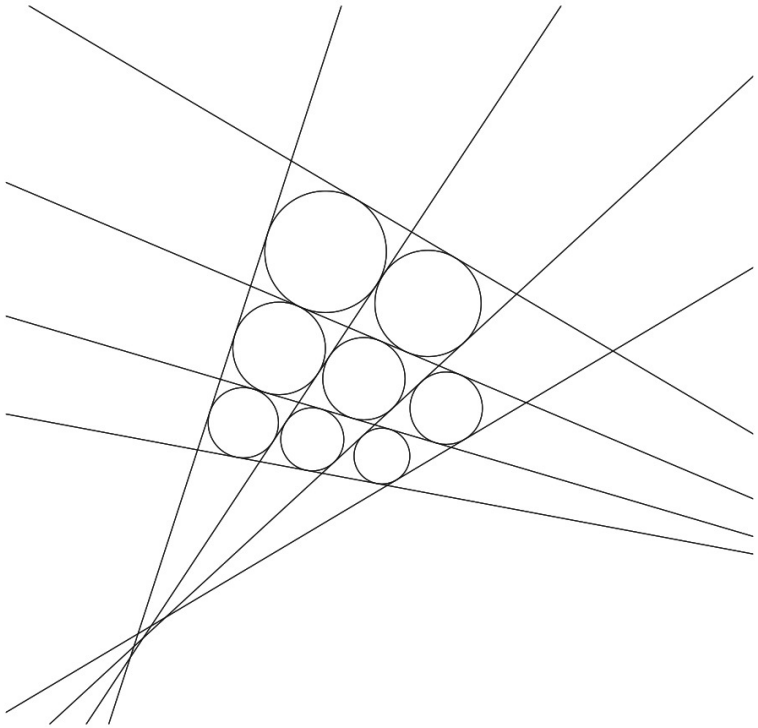

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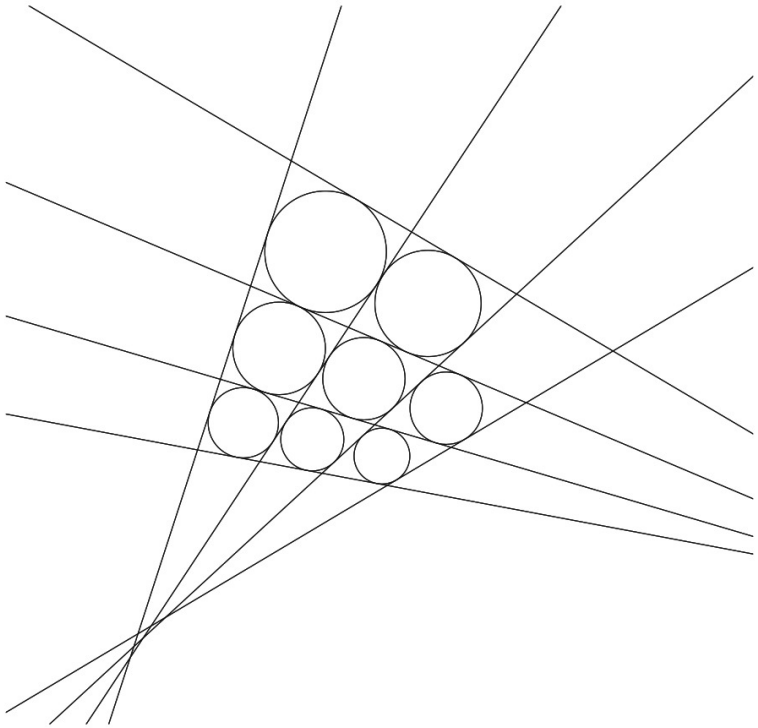

- incidence theorem (existence of the 9-th circle)

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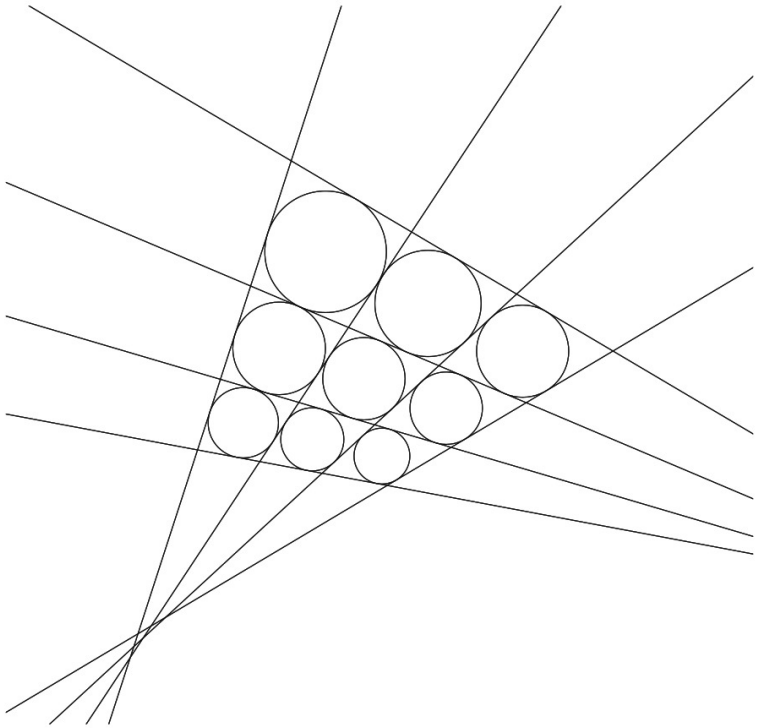

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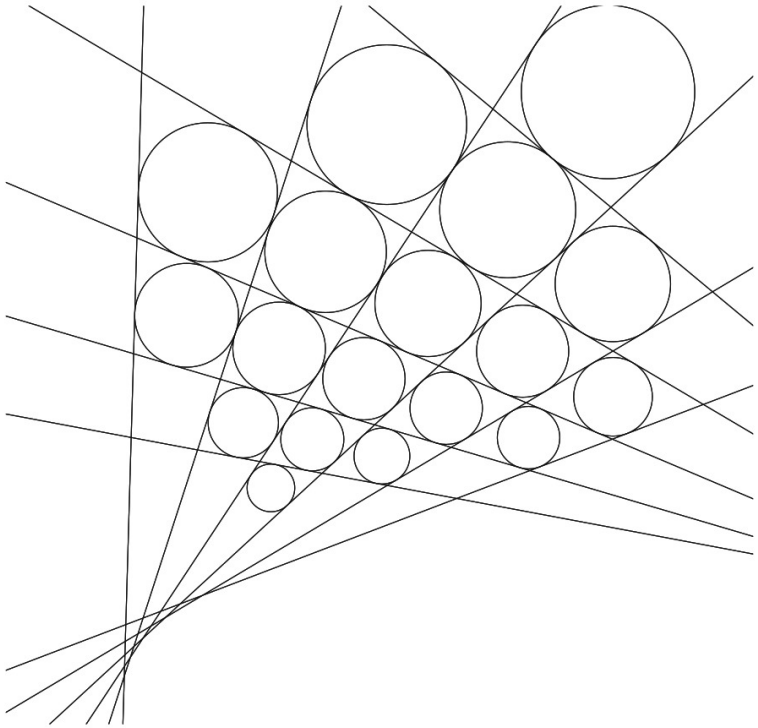

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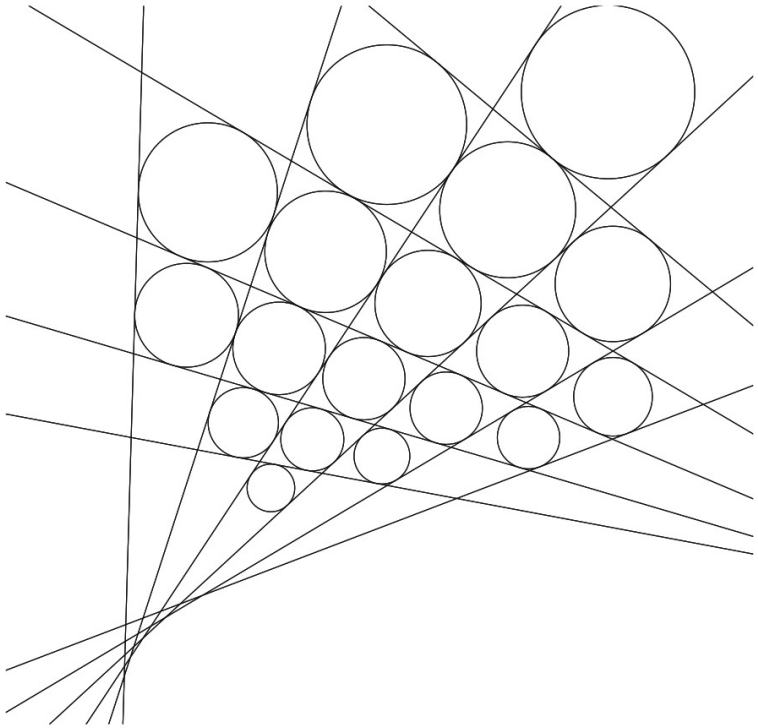
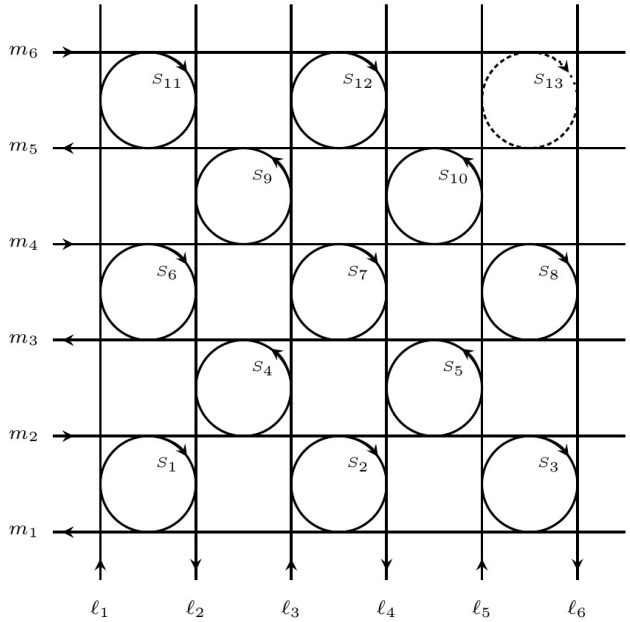
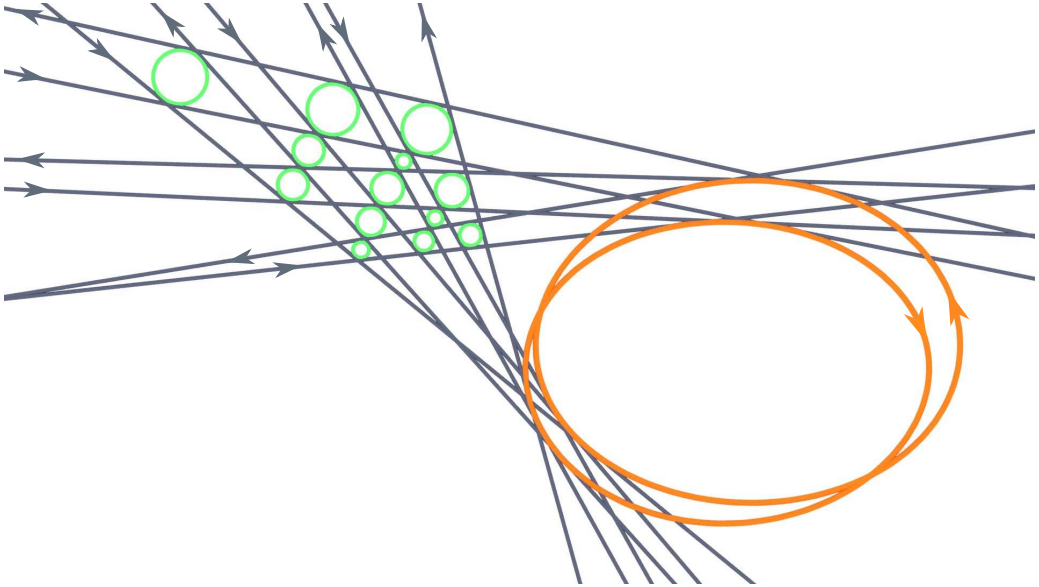


- incidence theorem (existence of the 9-th circle)
- all lines touch a common conic
- proofs easier in Laguerre geometric generalization

Laguerre geometric generalization: **checkerboard incircular nets** [AB]

- two one-parameter families of oriented lines

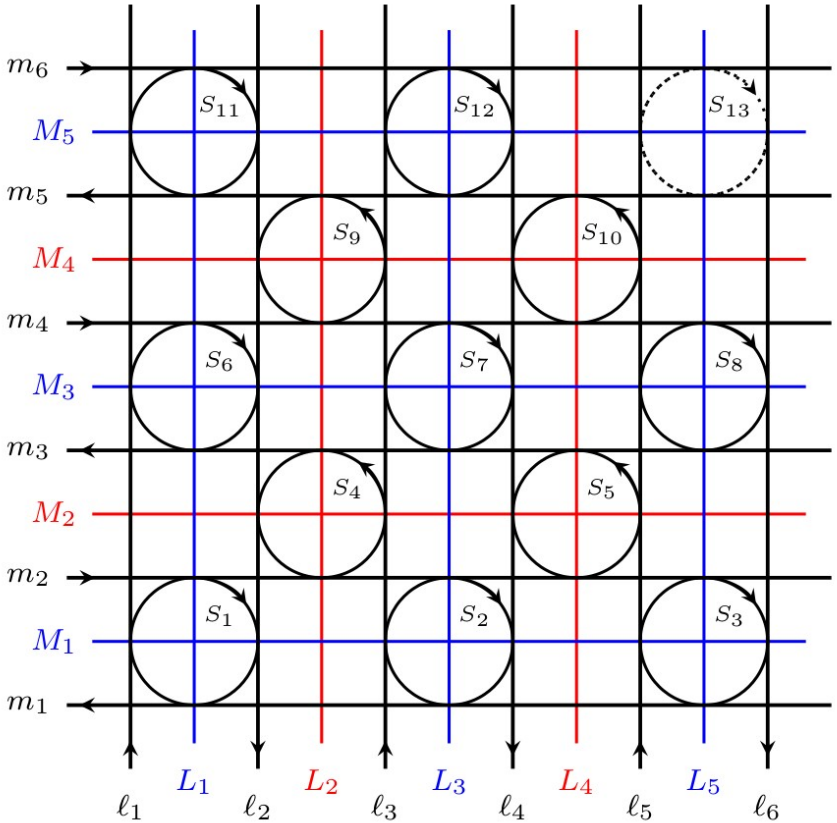


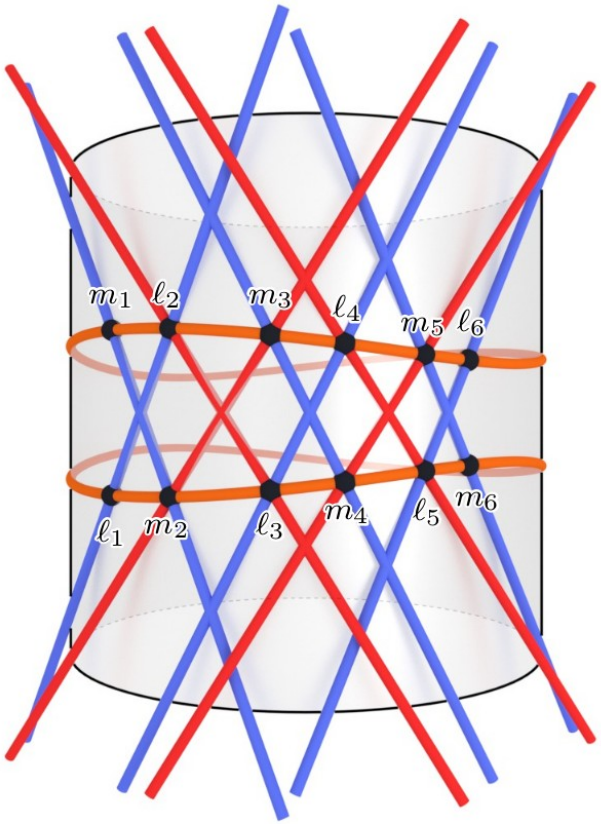


Main Theorem 3 [AB, 3]

1. incidence theorem (existence of the 13-th circle)
2. all lines of a checkerboard incircular net touch a common hypercycle

Proof





oriented lines $\longleftrightarrow$ points on the Blaschke cylinder $\mathcal{Z}$

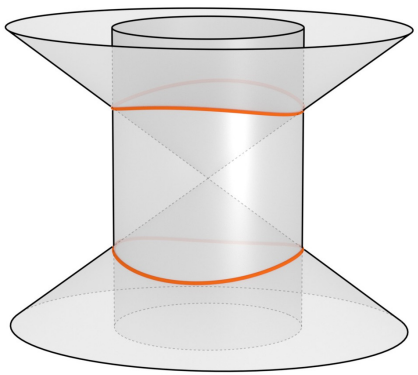
oriented circels $\longleftrightarrow$ planes

Main Theorem 4 [BI, 3]

1. classification of hypercycles
2. parametrization of checkerboard incircular nets

Example: base curve of an ellipse

- the hypercycle corresponding to an ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$



is given by the intersection of the Blaschke cylinder $\mathcal{Z}$

$$v^2 + w^2 = 1$$

with the cone

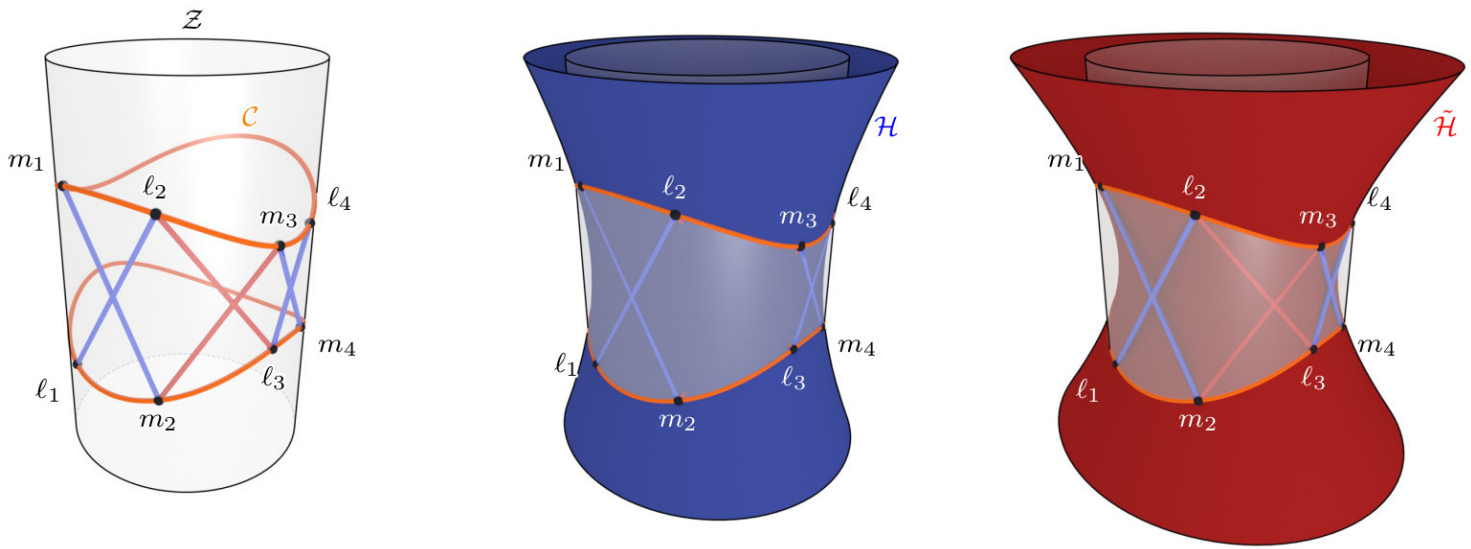
$$\alpha^2 v^2 + \beta^2 w^2 = d^2$$

- or equivalently by the base curve of the pencil

$$(\alpha^2 + \lambda)v^2 + (\beta^2 + \lambda)w^2 = d^2 + \lambda$$

[BI] W. Blaschke. Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene, Separatdruck aus "Monatshefte f. Mathematik u. Physik", XXI, Hamburg (1910).

Example (cont'd): parametrization / construction



- parametrize the base curve by elliptic functions

$$\boldsymbol{v}_\pm(\psi) = \begin{pmatrix} v(\psi) \\ w(\psi) \\ d_\pm(\psi) \end{pmatrix} = \begin{pmatrix} \mathrm{cn}(\psi,k) \\ \mathrm{sn}(\psi,k) \\ \pm\alpha\,\mathrm{dn}(\psi,k) \end{pmatrix}, \quad k = \sqrt{1-\frac{\beta^2}{\alpha^2}},$$

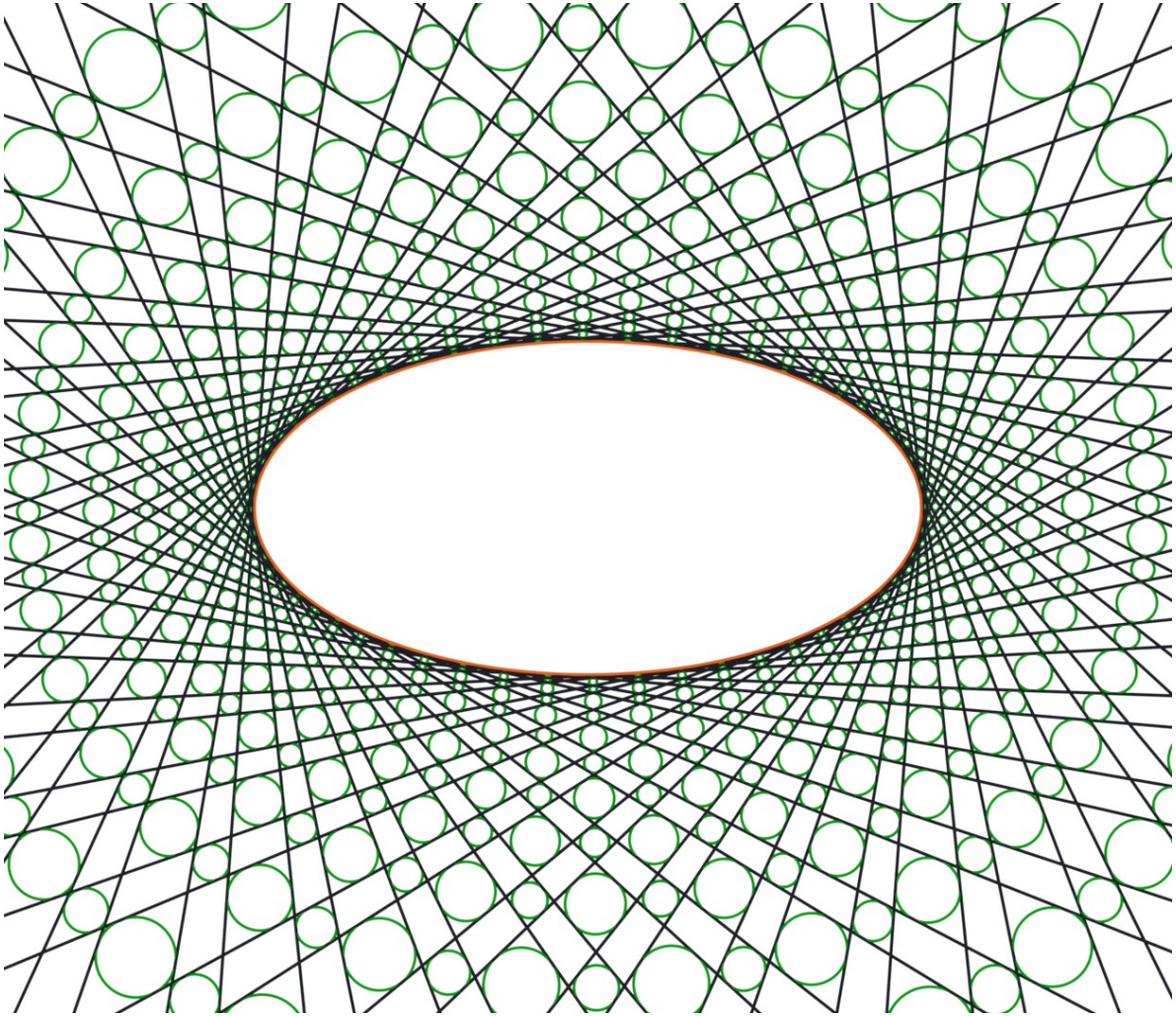
- choose two hyperboloids in the pencil by

$$\lambda = \alpha^2\,\mathrm{cs}^2\left(\frac{s}{2},k\right), \qquad \tilde{\lambda} = \alpha^2\,\mathrm{cs}^2\left(\frac{\tilde{s}}{2},k\right)$$

- alternate the corresponding shifts s and $\tilde{s}$ to switch component along the generators (Poncelet map)

$$\boldsymbol{v}_-(\psi) \;\to\; \boldsymbol{v}_+(\psi+s) \;\to\; \boldsymbol{v}_-(\psi+s+\tilde{s}) \;\to\; \cdots$$

- constrain $s + \tilde{s}$ to obtain periodic patterns



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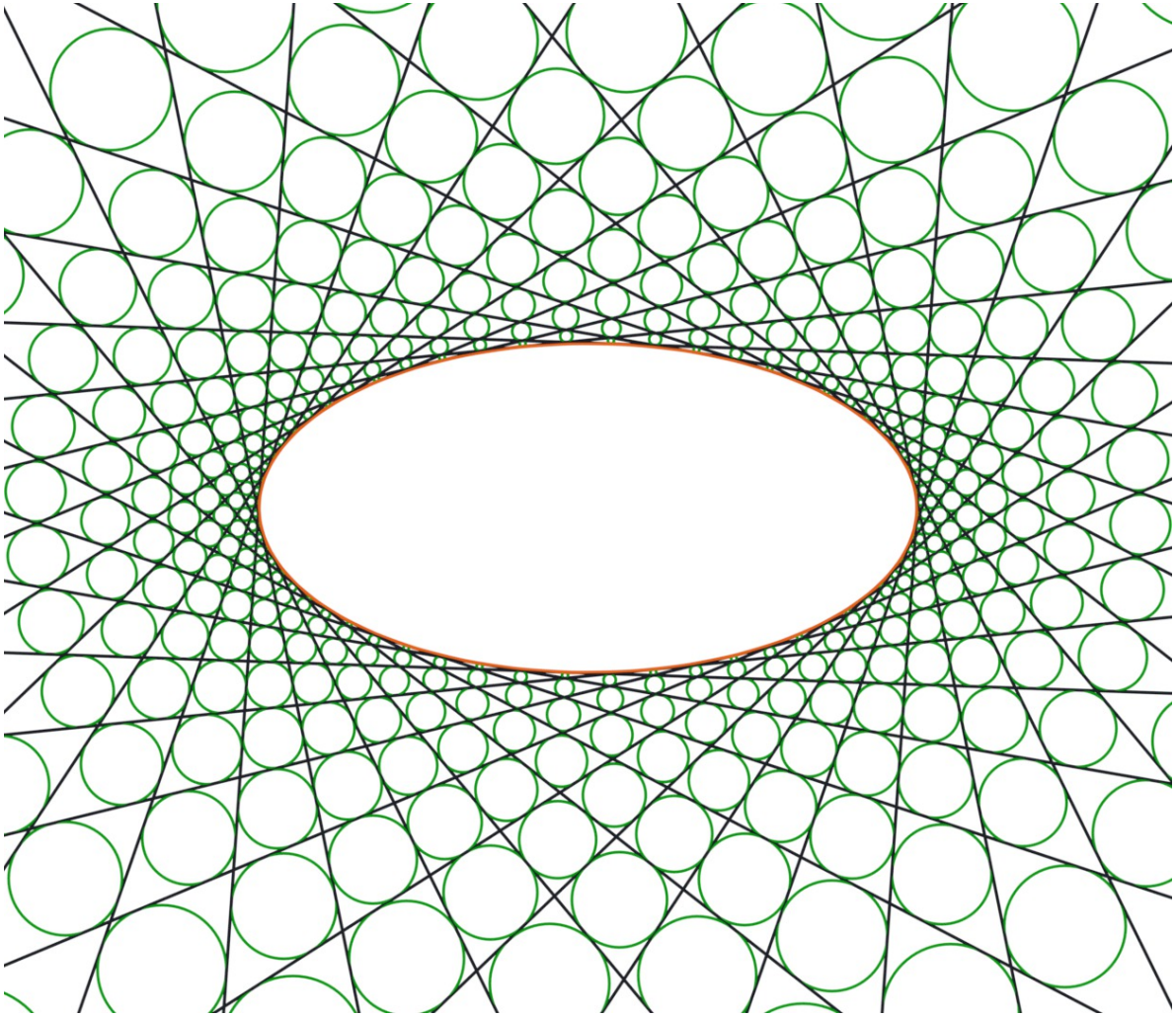
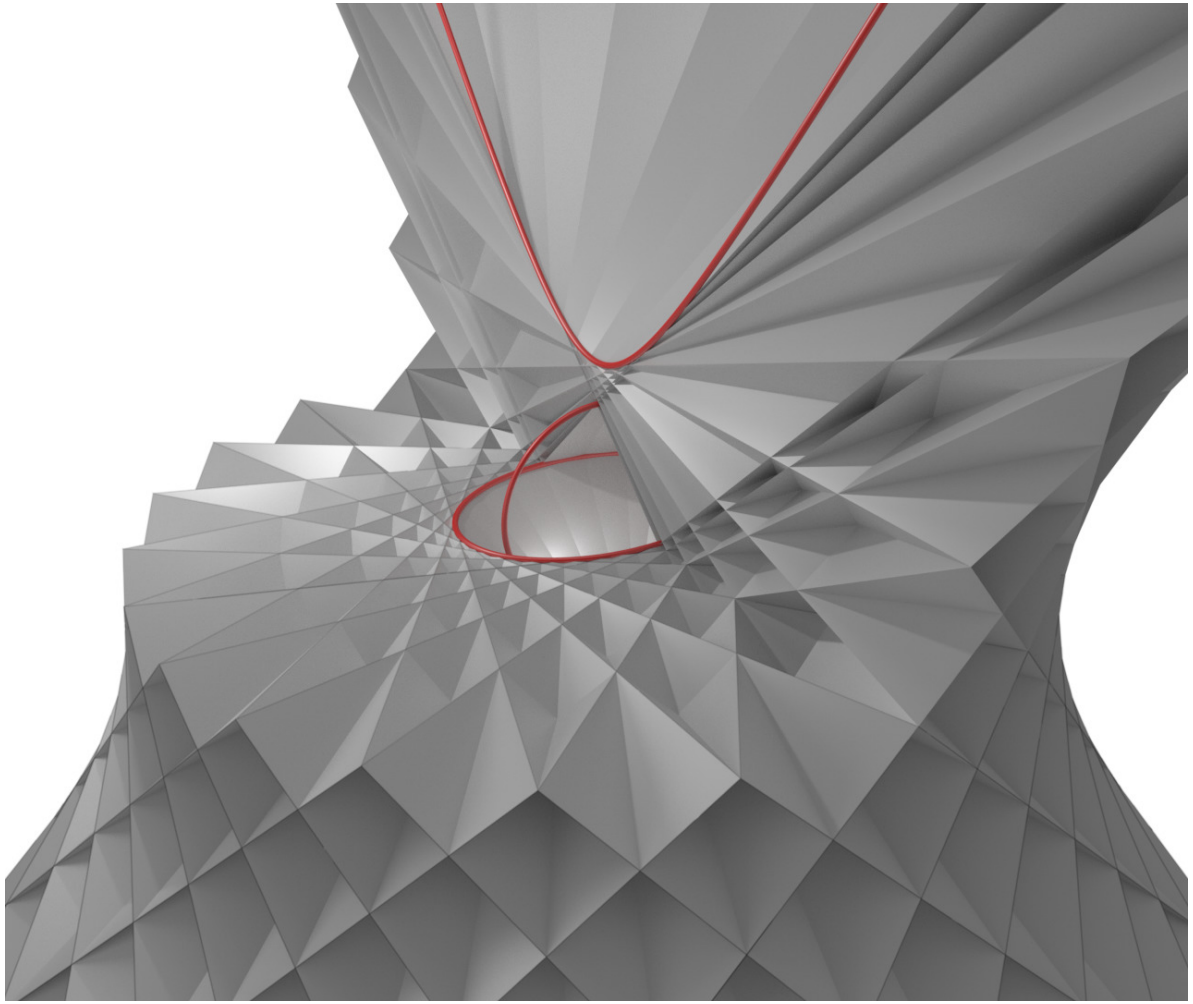


Summary

- **Main Theorem 1**: characterization of confocal coordinates by factorizability and orthogonality
- **Definition:** new discrete orthogonality
- **Definition:** discrete confocal coordinates (including arbitrary parametrizations)
- **Main Theorem 2:** discrete version of Main Theorem 1
- geometric construction of discrete confocal quadrics
- relation to checkerboard incircular nets
- **Main Theorem 3:** incidence theorem / touching hypercycle
- **Main Theorem 4:** classification of hypercycles / parametrization of checkerboard incircular nets

Thank you!



How to obtain confocal coordinates (specific parametrization)?

$$\left[\sum_{k=1}^{N} \frac{x_k(\boldsymbol{s})^2}{a_k + u_i(s_i)} = 1\right]_{i=1,\dots,N} \quad \Leftrightarrow \quad \left[x_k(\boldsymbol{s})^2 = \frac{\prod_{i=1}^{N}(u_i(s_i) + a_k)}{\prod_{i\neq k}(a_k - a_i)}\right]_{k=1,\dots,N}$$

- $\boldsymbol{x}$ may equivalently be written as

$$x_k(\boldsymbol{s}) = \frac{f_1^k(s_1)\cdots f_N^k(s_N)}{\prod_{i=1}^{k-1}\sqrt{a_i - a_k}\prod_{i=k+1}^{N}\sqrt{a_k - a_i}}$$

$$\left(f_i^k(s_i)\right)^2 = \begin{cases} u_i(s_i) + a_k, & k \leqslant i, \\ -(u_i(s_i) + a_k), & k > i. \end{cases}$$

- Consistency equations for f_i^k are given by

$$\begin{cases} \left(f_i^1(s_i)\right)^2 - \left(f_i^k(s_i)\right)^2 = a_1 - a_k, & k \leqslant i, \\ \left(f_i^1(s_i)\right)^2 + \left(f_i^k(s_i)\right)^2 = a_1 - a_k, & k > i. \end{cases}$$

How to obtain discrete confocal coordinates?

$$\left[\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n})x_k(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})}{a_k+u_i(n_i+\frac{1}{4}\sigma_i)} = 1\right]_{i=1\ldots,N} \quad \Leftrightarrow \quad \left[x_k(\boldsymbol{n})x_k(\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{\sigma}) = \frac{\prod_{j=1}^{N}(u_i(n_i+\frac{1}{4}\sigma_i)+a_k)}{\prod_{j\neq k}(a_k-a_j)}\right]_{k=1,\ldots,N}$$

- $\boldsymbol{x}$ may equivalently be written as

$$x_k(\boldsymbol{n}) = \frac{f_1^k(n_1)\cdots f_N^k(n_N)}{\prod_{i=1}^{k-1}\sqrt{a_i-a_k}\prod_{i=k+1}^{N}\sqrt{a_k-a_i}}$$

$$f_i^k(n_i)f_i^k(n_i+\tfrac{1}{2}) = \begin{cases} u_i(n_i+\frac{1}{4})+a_k, & k\leqslant i, \\ -(u_i(n_i+\frac{1}{4})+a_k), & k>i. \end{cases}$$

- Consistency equations for f_i^k are given by

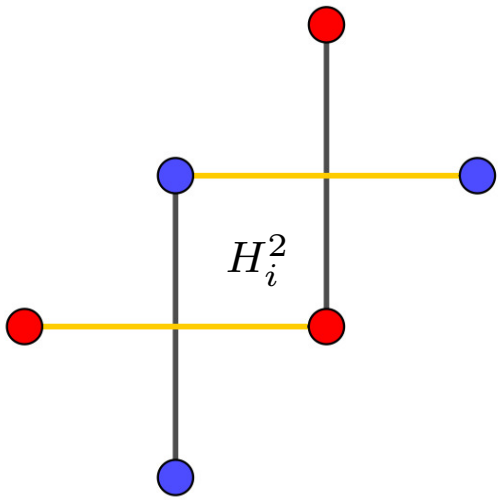
$$\begin{cases} f_i^1(n_i)f_i^1(n_i+\frac{1}{2}) - f_i^k(n_i)f_i^k(n_i+\frac{1}{2}) = a_1-a_k, & k\leqslant i, \\ f_i^1(n_i)f_i^1(n_i+\frac{1}{2}) + f_i^k(n_i)f_i^k(n_i+\frac{1}{2}) = a_1-a_k, & k>i. \end{cases}$$

discrete Lamé coefficients

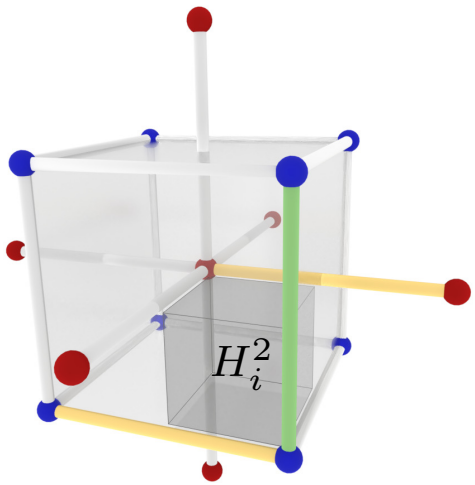
For a pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathbb{R}^N$ the **discrete Lamé coefficients** $H_i^2 : \left(\mathbb{Z} + \frac{1}{4}\right)^M \to \mathbb{R}$ are defined by

$$H_i^2(\boldsymbol{n} + \tfrac{1}{4}\boldsymbol{\sigma}) = \begin{cases} \langle \Delta_i \boldsymbol{x}(\boldsymbol{n}), \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) \rangle, & \sigma_i = 1 \\ \langle \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n}), \Delta_i \boldsymbol{x}(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) \rangle, & \sigma_i = -1 \end{cases}$$

for all $\boldsymbol{n} \in \mathbb{Z}^M$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^M$,
where $\bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}(\boldsymbol{n}) - \boldsymbol{x}(\boldsymbol{n} - \boldsymbol{e}_i)$.



$M = 2$



$M = 3$

Isothermic nets

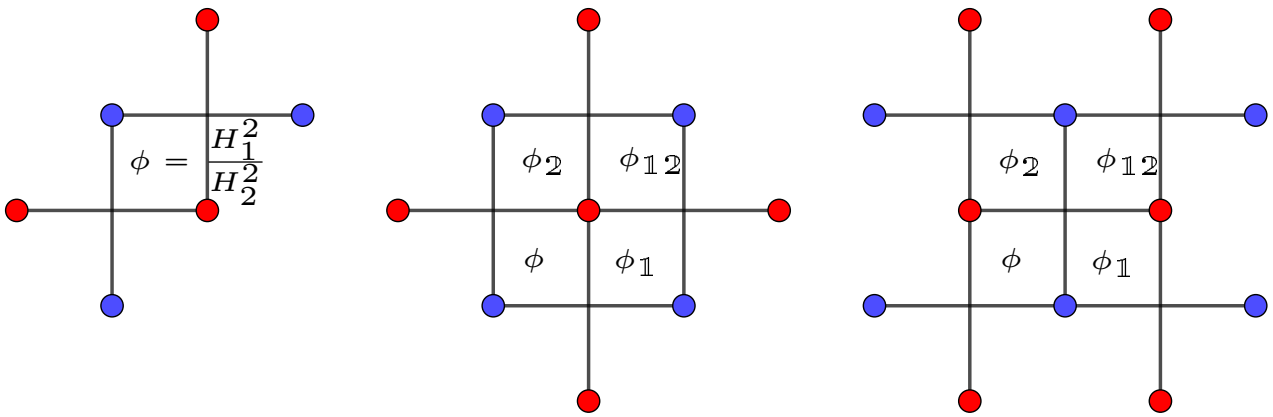
A net $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is called **isothermic** if it is a curvature line parametrization and

$$\frac{H_1^2}{H_2^2} = \frac{\alpha(s_1)}{\beta(s_2)},$$

for some functions α and β only depending on s_1 and s_2 respectively.

- definition can immediately be carried over to orthogonal pairs of discrete nets
- discrete isothermicity condition:

$$\phi\phi_{\mathbb{1}2} = \phi_{\mathbb{1}}\phi_2$$



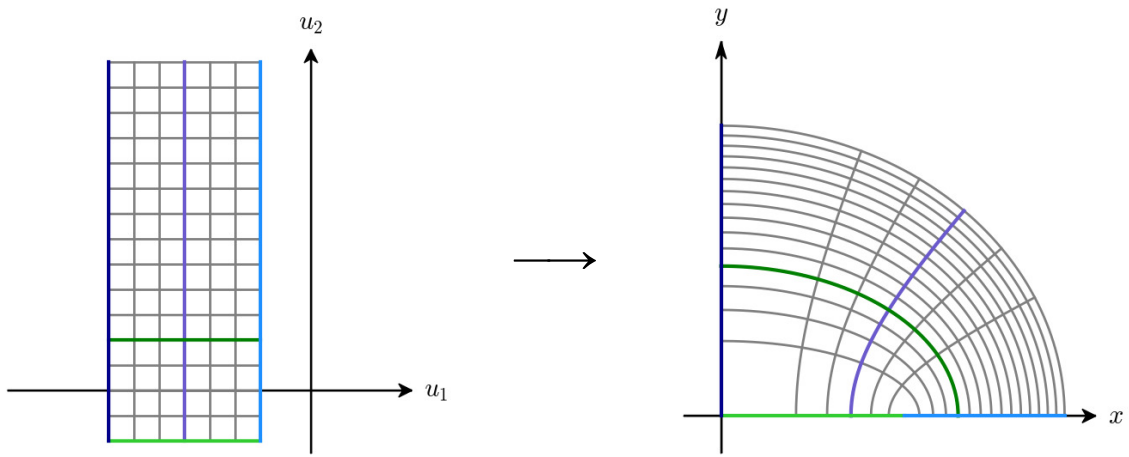
Darboux classified all triply orthogonal systems with isothermic coordinate surfaces [Da]:

- solutions of the **Euler-Poisson-Darboux equation**

$$\partial_i \partial_j \boldsymbol{x}(\boldsymbol{s}) = \frac{\gamma}{s_i - s_j}(\partial_j \boldsymbol{x}(\boldsymbol{s}) - \partial_i \boldsymbol{x}(\boldsymbol{s})), \quad i, j = 1, \ldots, N, \ i \neq j$$

  with $\gamma = \pm\frac{1}{2}, -1, -2$.
- the case $\gamma = \frac{1}{2}$ includes **confocal coordinates**

$$x_k(\boldsymbol{s}) = \frac{\prod_{i=1}^{k-1} \sqrt{-(s_i + a_k)} \prod_{i=k}^{N} \sqrt{s_i + a_k}}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}}, \quad k = 1, \ldots, N.$$



[Da] G. Darboux. *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. Principes de géométrie analytique*, Gauthier-Villars, Paris, (1910)

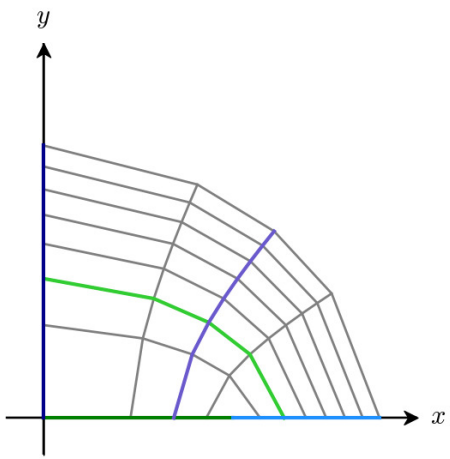
Example: Explicit solution from choosing u_i.

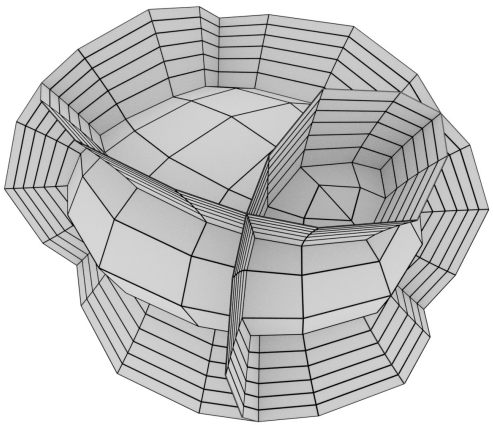
- setting $u_i(n_i + \frac{1}{4}) = n_i + \epsilon_i$ leads to

$$f_i^k(n_i) = \begin{cases} (n_i + a_k + \epsilon_i)_{1/2} & \text{for } i \geqslant k \\ (-n_i - a_k - \epsilon_i + \frac{1}{2})_{1/2} & \text{for } i < k \end{cases} \qquad \text{with } (n)_{1/2} = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)}$$

- solutions of the discrete Euler-Poisson-Darboux equation with $\gamma = \frac{1}{2}$ [1, KS]:

$$\Delta_i \Delta_j \boldsymbol{x}(\boldsymbol{n}) = \frac{\gamma}{n_i + \epsilon_i - n_j - \epsilon_j} (\Delta_j \boldsymbol{x}(\boldsymbol{n}) - \Delta_i \boldsymbol{x}(\boldsymbol{n})), \qquad i \neq j$$





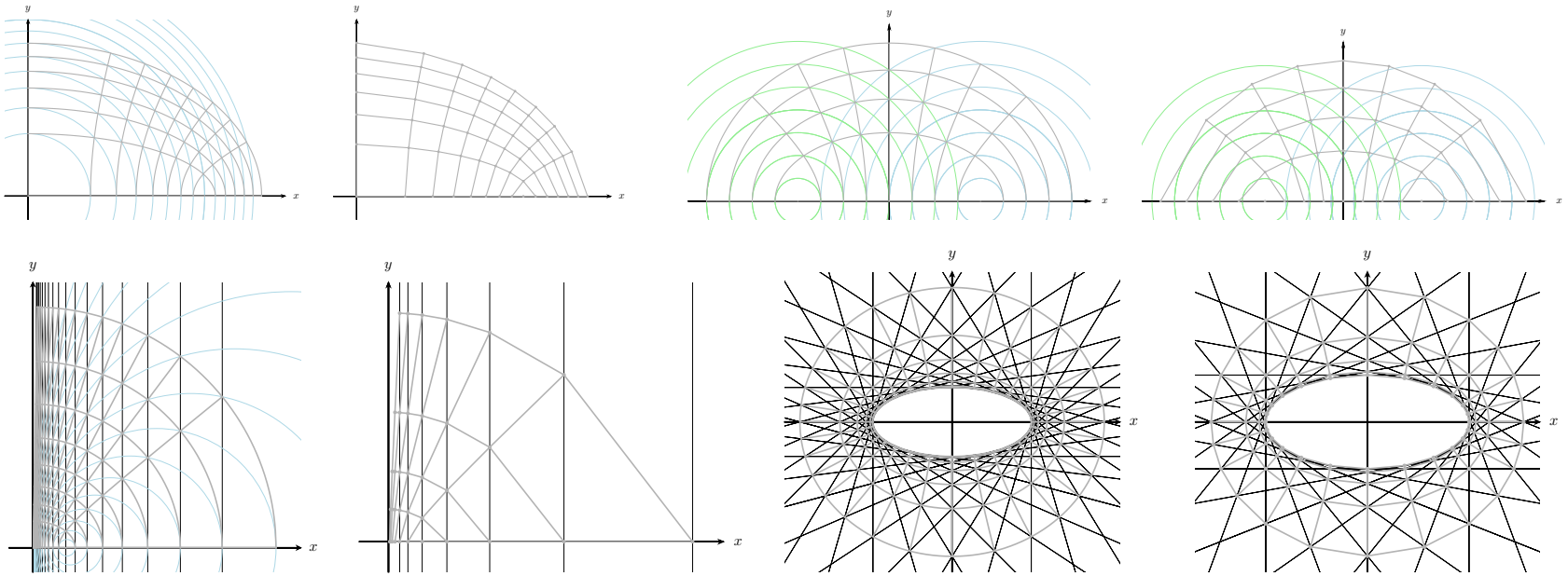
[KS] B.G. Konopelchenko, W.K. Schief. Integrable discretization of hodograph-type systems, hyperelliptic integrals and Whitham equations, *Proc. Royal Soc. A*, (2014), Vol. 470, No. 2172, 20140514, 21 pp.

generalized discrete Euler-Poisson-Darboux equation with $\gamma = \frac{1}{2}$

$$\Delta_i \Delta_j \boldsymbol{x} = \frac{1}{u_i - u_j} \left(\Delta^{1/2} u_i \Delta_j \boldsymbol{x} - \Delta^{1/2} u_j \Delta_i \boldsymbol{x} \right),$$

where $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{n})$, $u_i = u_i(n_i + \frac{1}{4})$, and $\Delta^{1/2} u_i = u_i(n_i + \frac{3}{4}) - u_i(n_i + \frac{1}{4})$

Examples related to 3-webs and 4-webs [Ak, Ag, 2]



Similar constructions of webs also exist on quadrics in 3-space [ABST].

[Ag] S.I. Agafonov. Confocal conics and 4-webs of maximal rank, arxiv:1912.01817v1 (2019).

[Ak] A.V. Akopyan. 3-Webs generated by confocal conics and circles. *Geometriae Dedicata*, (2017).

[ABST] A.V. Akopyan, A.I. Bobenko, W.K. Schief, J. Techter. On mutually diagonal nets on (confocal) quadrics and 3-dimensional webs, arXiv:1908.00856, (2019).