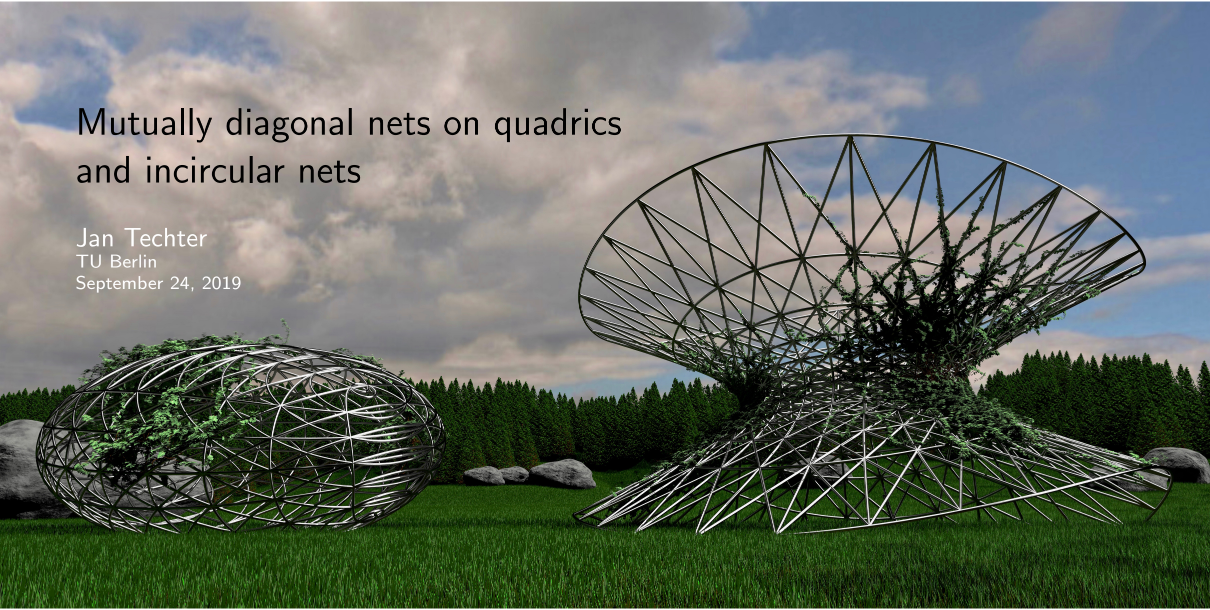
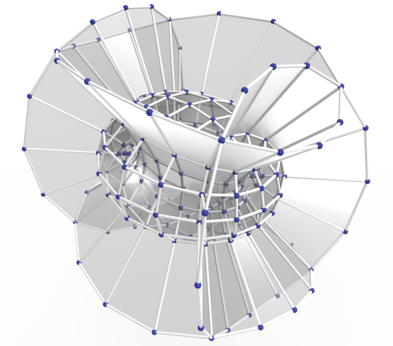
Mutually diagonal nets on quadrics and incircular nets

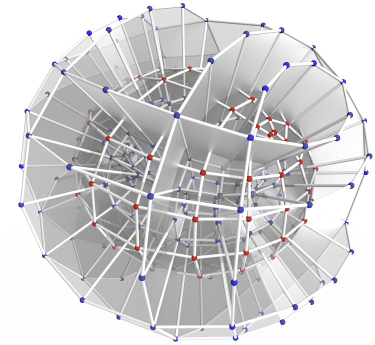
Jan Techter
TU Berlin
September 24, 2019



A.V. Akopyan, A.I. Bobenko, W.K. Schief, J. Techter
On mutually diagonal nets on (confocal) quadrics and 3-dimensional webs,
preprint (2019)

Discrete confocal quadrics

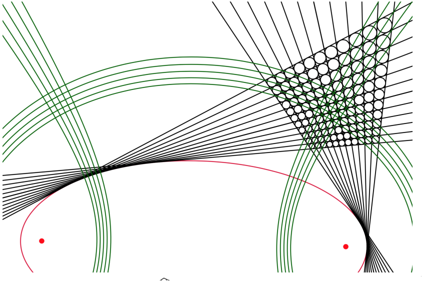




A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics. I. An integrable systems approach,
Journal of Integrable Systems (2016) Volume 1:1

A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics. II. A geometric approach to general parametrization,
IMRN (2018)

(Checkerboard) incircular nets



W. Böhm, *Verwandte Sätze über Kreisvierseitnetze*,
Arch. Math. (Basel) 21 (1970) 326–330

A. Akopyan, A.I. Bobenko, *Incircular nets and confocal conics*,
Trans. AMS 370:4 (2018)

A.I. Bobenko, W.K. Schief, J. Techter. *Checkerboard incircular nets. Laguerre geometry and parametrization*,
Geometriae Dedicata (2019)

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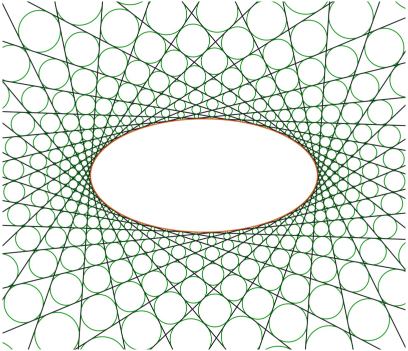

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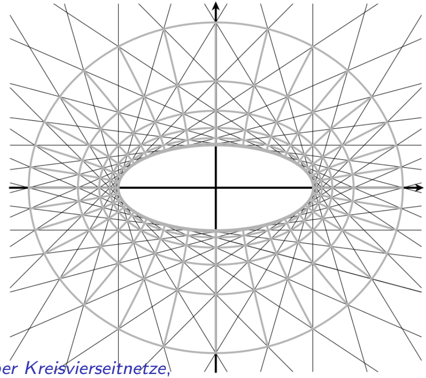

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Anschauliche Geometrie, 1932 (Geometry and the imagination)


Hilbert


Cohn-Vossen

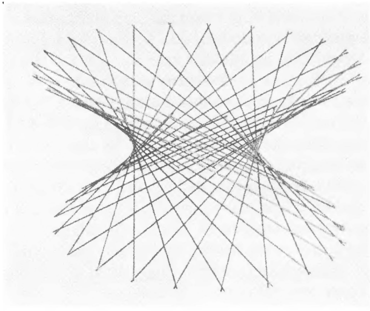
FIG. 23a

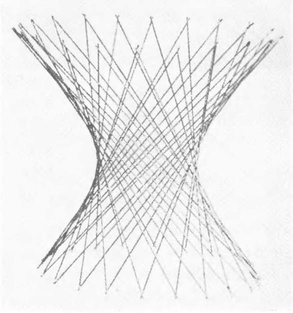
FIG. 23b

Anschauliche Geometrie, 1932 (Geometry and the imagination)

Hilbert

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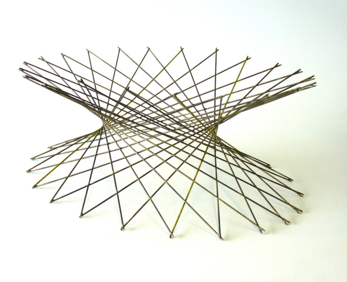
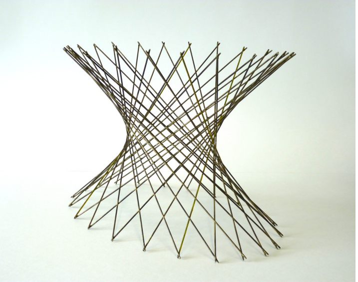


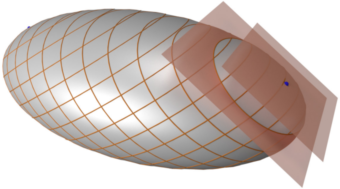


Models at TU Wien

Anschauliche Geometrie, 1932 (Geometry and the imagination)

Hilbert

Cohn-Vossen



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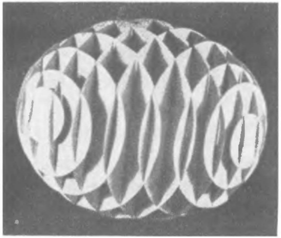

FIG. 25a

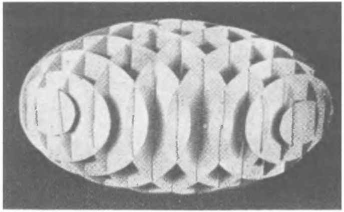

FIG. 25b

Anschauliche Geometrie, 1932 (Geometry and the imagination)

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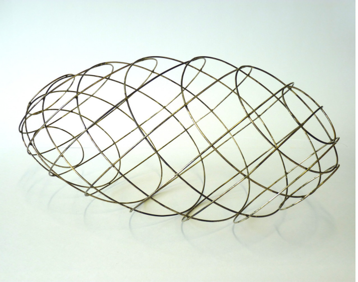

Models at TU Wien

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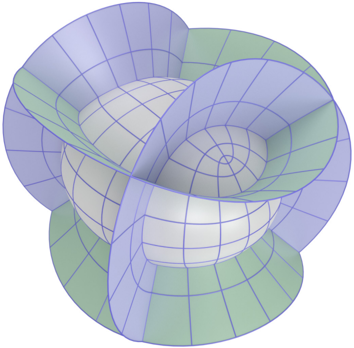
Confocal quadrics

Given $a > b > c$. Corresponding one-parameter family of confocal quadrics:

$$Q(\lambda) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1 \right\}, \quad \lambda \in \mathbb{R}.$$



Decomposition into three families:

$$\frac{x^2}{u_1+a}+\frac{y^2}{u_1+b}+\frac{z^2}{u_1+c}=1$$
$$\frac{x^2}{u_2+a}+\frac{y^2}{u_2+b}+\frac{z^2}{u_2+c}=1$$
$$\frac{x^2}{u_3+a}+\frac{y^2}{u_3+b}+\frac{z^2}{u_3+c}=1$$

with $-a < u_1 < -b < u_2 < -c < u_3$.

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with $-a < u_1 < -b < u_2 < -c < u_3$.

This leads to the system of **confocal coordinates** $(x, y, z) = \boldsymbol{r}(u_1, u_2, u_3)$:

$$x^2 = \frac{(u_1 + a)(u_2 + a)(u_3 + a)}{(a - b)(a - c)}$$
$$y^2 = \frac{(u_1 + b)(u_2 + b)(u_3 + b)}{(b - a)(b - c)}$$
$$z^2 = \frac{(u_1 + c)(u_2 + c)(u_3 + c)}{(c - a)(c - b)}$$

For any reparametrization $u_i = u_i(s_i)$ the system of confocal coordinates

- has a diagonal first fundamental form:

$$\mathrm{I} = d\boldsymbol{r} \cdot d\boldsymbol{r} = H_1^2 ds_1^2 + H_2^2 ds_2^2 + H_3^2 ds_3^2,$$

$$\text{with } \frac{H_i^2}{H_k^2} = \frac{V_i(s_i, s_l)}{V_k(s_k, s_l)},$$

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- has diagonal second fundamental forms:

$$\mathrm{II}_{ik} = -d\boldsymbol{r} \cdot d\boldsymbol{N}_{ik} = e_{ik} ds_i^2 + g_{ik} ds_k^2,$$

$$\text{with } \frac{e_{ik}}{g_{ik}} = -\frac{U_i(s_i)}{U_k(s_k)}, \quad U_i = \frac{1}{4}\frac{u_i'^2}{(u_i+a)(u_i+b)(u_i+c)}.$$

where $\boldsymbol{N}_{ik} \sim \boldsymbol{r}_{u_l}$ is the unit normal of the quadric $u_l = \text{const}$.

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In particular

Orthogonal coordinate system along **curvature lines** of its **isothermal** coordinate surfaces.

$$\mathrm{II}_{ik} = e_{ik}\, ds_i^2 + g_{ik}\, ds_k^2, \qquad \frac{e_{ik}}{g_{ik}} = -\frac{U_i(s_i)}{U_k(s_k)}, \qquad U_i = \frac{1}{4}\frac{u_i'^2}{(u_i + a)(u_i + b)(u_i + c)}.$$

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Theorem

There exists a unique parametrisation $u_i = u_i(s_i)$ of confocal coordinate lines (up to a scaling of the s_i by the same constant) such that the second fundamental forms of the confocal quadrics are "conformally flat", that is,

$$\mathrm{II}_{12} \sim ds_1^2 + ds_2^2, \quad \mathrm{II}_{13} \sim ds_1^2 - ds_3^2, \quad \mathrm{II}_{23} \sim ds_2^2 + ds_3^2.$$

$$\mathrm{II}_{ik} = e_{ik}\, ds_i^2 + g_{ik}\, ds_k^2, \qquad \frac{e_{ik}}{g_{ik}} = -\frac{U_i(s_i)}{U_k(s_k)}, \qquad U_i = \frac{1}{4}\frac{u_i'^2}{(u_i+a)(u_i+b)(u_i+c)}.$$

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Solve $U_1 = -U_2 = U_3 = \text{const} > 0$

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In terms of Jacobi elliptic functions this parametrization is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{a-c} \begin{pmatrix} \operatorname{sn}(s_1, k_1)\operatorname{dn}(s_2, k_2)\operatorname{ns}(s_3, k_3) \\ \operatorname{cn}(s_1, k_1)\operatorname{cn}(s_2, k_2)\operatorname{ds}(s_3, k_3) \\ \operatorname{dn}(s_1, k_1)\operatorname{sn}(s_2, k_2)\operatorname{cs}(s_3, k_3) \end{pmatrix},$$

with $k_1^2 = \frac{a-b}{a-c}, \quad k_2^2 = \frac{b-c}{a-c} = 1 - k_1^2, \quad k_3 = k_1.$

Net

A two-parameter family of curves on a surface, such that there exist exactly two curves of the family passing through any point on the surface.

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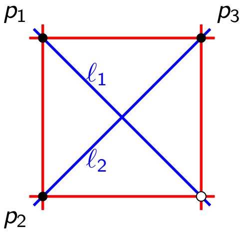
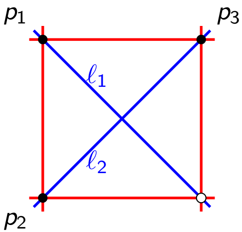


Two diagonally related nets $\mathcal{N}_1$, $\mathcal{N}_2$

For any quadrilateral of $\mathcal{N}_1$ with one pair of opposite vertices connected by a curve of $\mathcal{N}_2$, the other pair of opposite vertices is also connected by a curve of $\mathcal{N}_2$.



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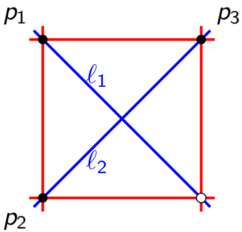

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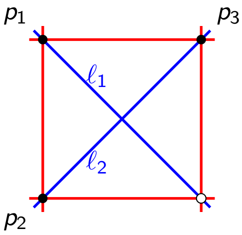
- symmetric relation



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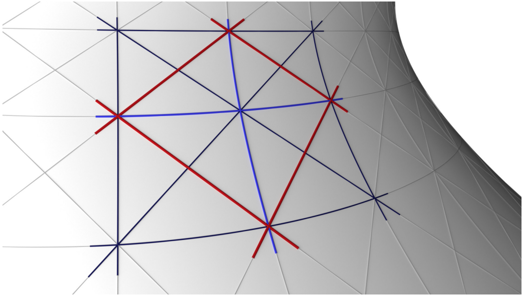
- symmetric relation
- If $\boldsymbol{r}(u, v)$ is a parametrization, then the two nets given by

$$\begin{cases} u = \text{const} \\ v = \text{const} \end{cases} \quad \text{and} \quad \begin{cases} u + v = \text{const} \\ u - v = \text{const} \end{cases}$$

are diagonally related.

Theorem

On any one-sheeted hyperboloid, the lines of curvature and the (straight) asymptotic lines form mutually diagonal nets.



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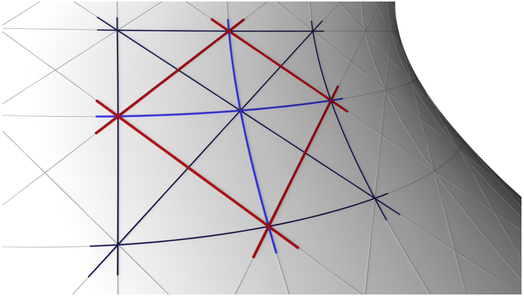

Every one-sheeted hyperboloid is part of a confocal family.

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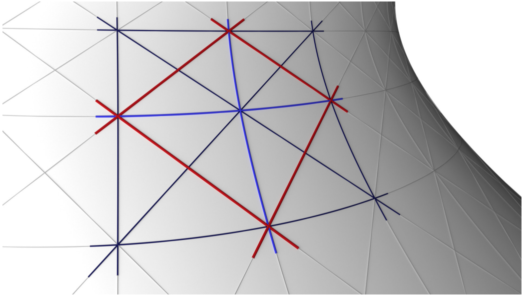

Every one-sheeted hyperboloid is part of a confocal family.
Second fundamental form: $\mathrm{II}_{13} \sim ds_1^2 - ds_3^2 = (ds_1 + ds_3)(ds_1 - ds_3)$.

What is the meaning of the curvature lines on confocal ellipsoids being parametrized such that

$$\mathrm{II}_{12} \sim ds_1^2 + ds_2^2, \qquad \mathrm{II}_{23} \sim ds_2^2 + ds_3^2?$$

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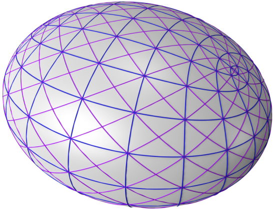

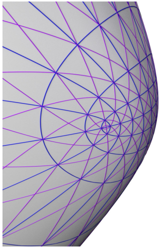

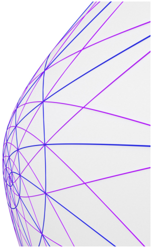

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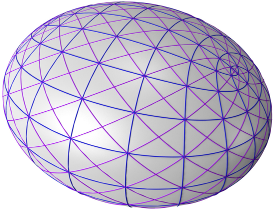

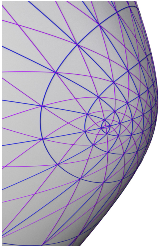

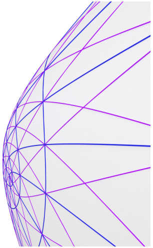

These lines are **characteristic conjugate** lines:

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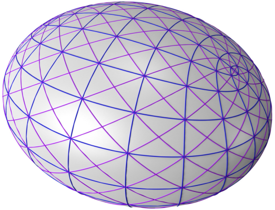

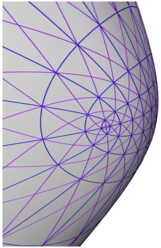

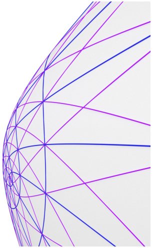

These lines are **characteristic conjugate** lines:

- analogue of asymptotic lines on positively curved surfaces

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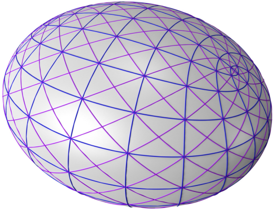
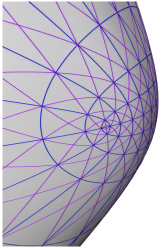
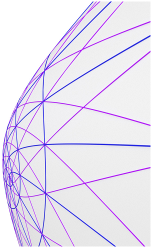




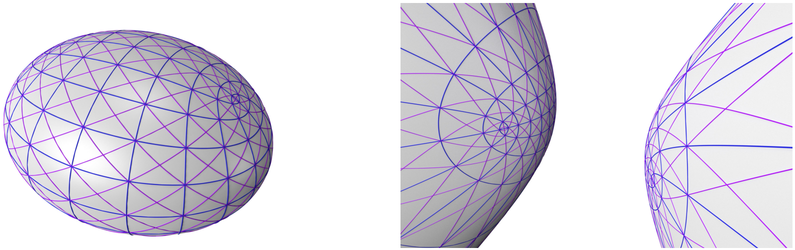


These lines are **characteristic conjugate** lines:

- analogue of asymptotic lines on positively curved surfaces
- conjugate directions that are bisected by curvature lines

What is the meaning of the curvature lines on confocal ellipsoids being parametrized such that

$$\mathrm{II}_{12} \sim ds_1^2 + ds_2^2, \qquad \mathrm{II}_{23} \sim ds_2^2 + ds_3^2?$$



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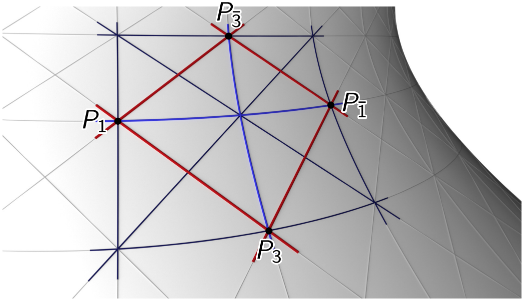
- analogue of asymptotic lines on positively curved surfaces
- conjugate directions that are bisected by curvature lines

Theorem

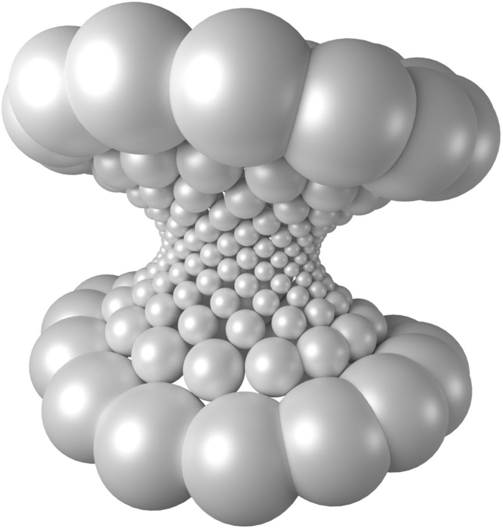
On any ellipsoid / two-sheeted hyperboloid, the lines of curvature and the characteristic conjugate lines form mutually diagonal nets.

Additional property (for asymptotic lines on hyperboloids)

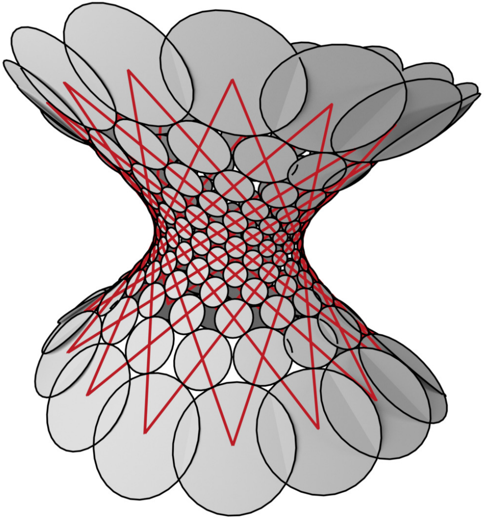
If $P_1, P_3, P_{\bar{1}}, P_{\bar{3}}$ are the vertices of a quadrilateral of asymptotic lines with curvature lines as diagonals, then $\overline{P_1 P_3} + \overline{P_{\bar{1}} P_{\bar{3}}} = \overline{P_1 P_{\bar{3}}} + \overline{P_{\bar{1}} P_3}$.



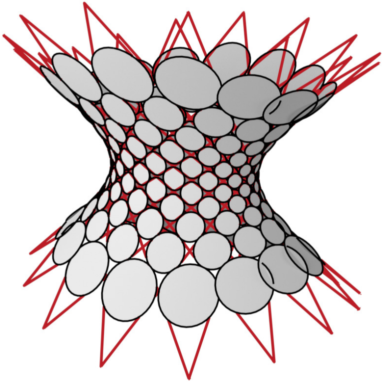
Sphere model of a one-sheeted hyperboloid:



Disk model of a one-sheeted hyperboloid:

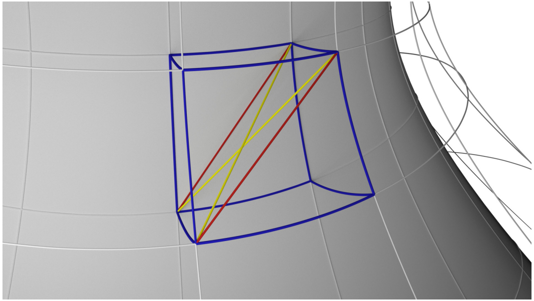


Indisk model of a one-sheeted hyperboloid:



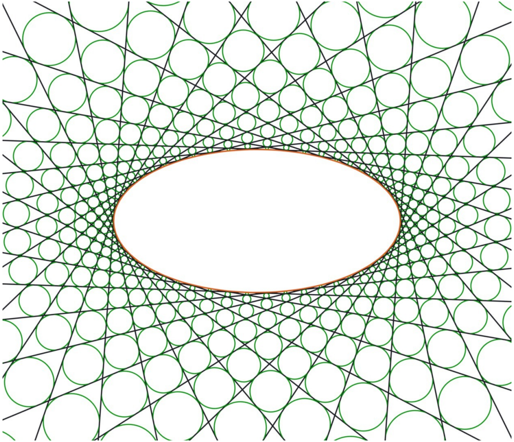
The deformation between the one-sheeted hyperboloids of a confocal family

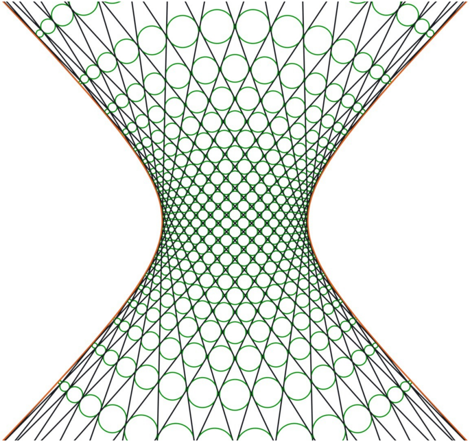
- preserves lines of curvature / asymptotic lines and their mutual diagonal relation
- is isometric along the asymptotic lines
- produces skew parallelograms



“Isometric” deformation of hyperboloids

Incircular nets

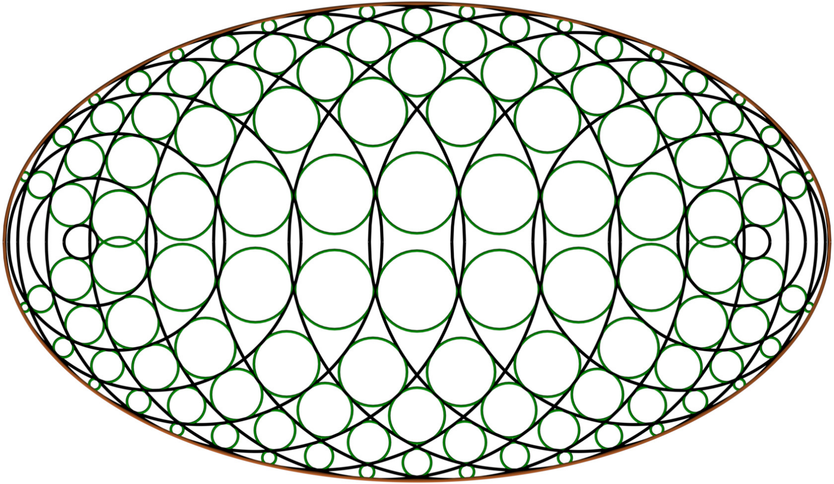




“Isometric” deformation of ellipsoids

Circular sections and “diagonal” curvature lines.

Hyperbolic incircular net



Is there a version of confocal coordinates where the curvature lines on all quadrics are diagonally related to straight lines?

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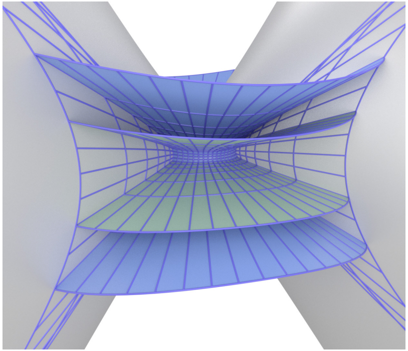

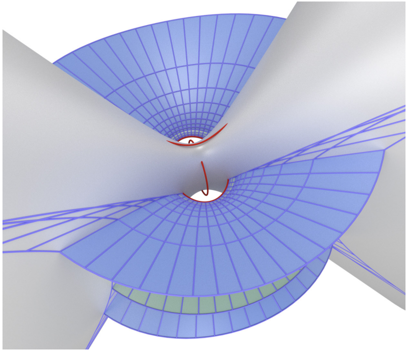

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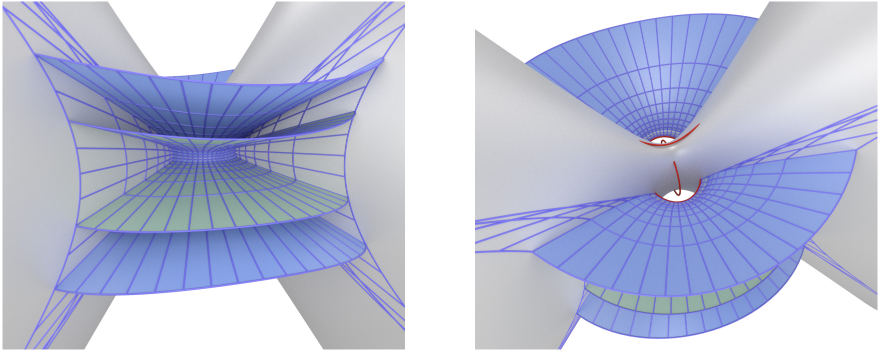

All second fundamental forms: $\mathrm{II}_{ij} \sim ds_i^2 - ds_j^2, \quad i < j$

Octahedral webs of planes

Theorem

The four "diagonal" families of planes of a system of Minkowski confocal coordinates, given by

$$s_1 + s_2 + s_3 = const, \quad s_1 + s_2 - s_3 = const$$
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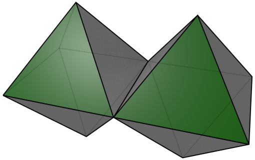
form an octahedral web of planes.



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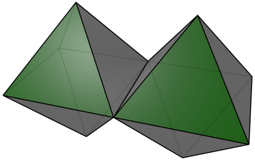
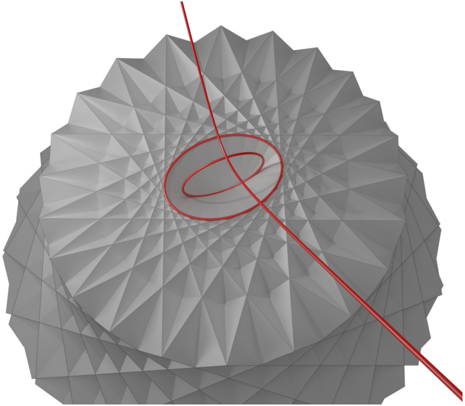
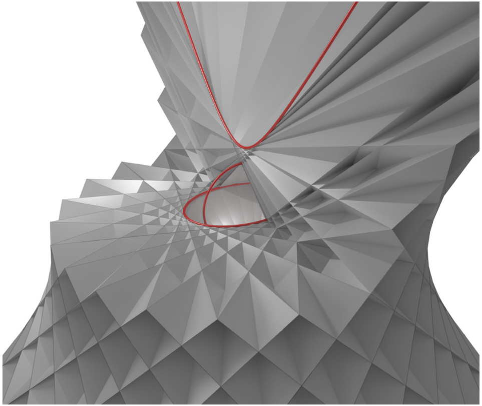


R. Sauer, *Die Raumeinteilungen, welche durch Ebenen erzeugt werden, von denen je vier sich in einem Punkt schneiden*, Sitzgsb. der bayer. Akad. der Wiss., math.-naturw. Abt. (1925) 41-56.

W. Blaschke, *Topologische Fragen der Differentialgeometrie II. Achtflachgewebe*, Math. Z. 28 (1928) 158–160.

Conical octahedral grids





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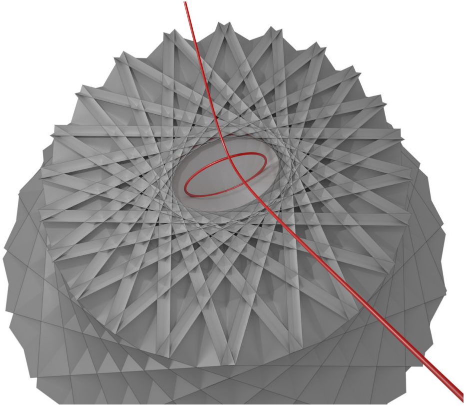
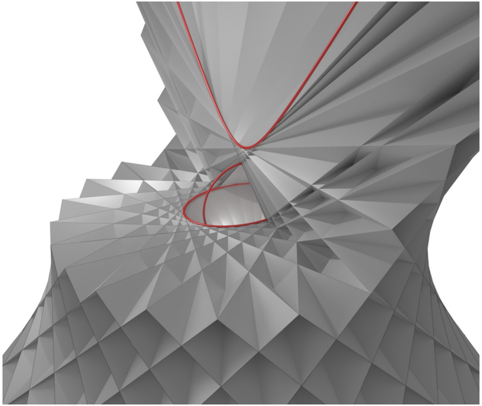
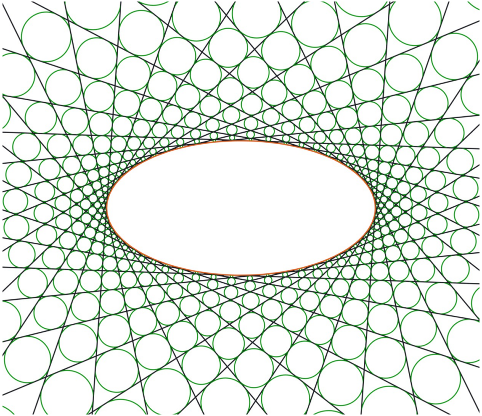
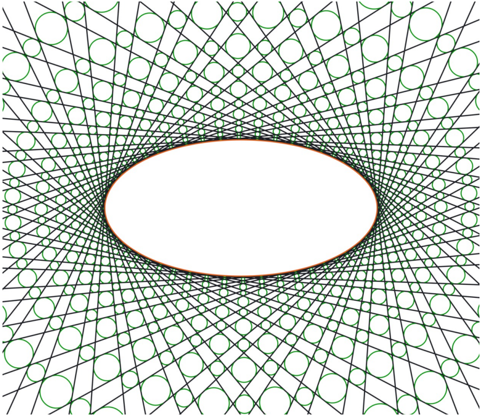




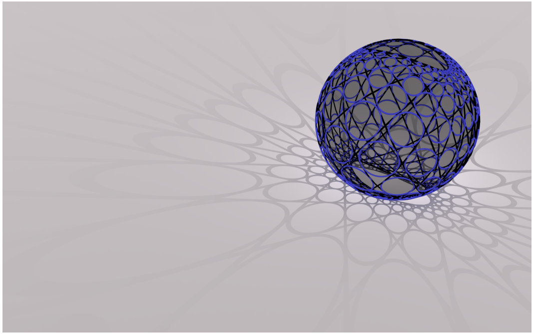
(Checkerboard) incircular nets



(Checkerboard) incircular nets



(Checkerboard) incircular nets in space forms



A.I. Bobenko, C.O.R. Lutz, H. Pottmann, J. Techter
Laguerre geometry and incircular nets in space forms,
in preparation

Hyperbolic checkerboard incircular net

Thank you!