Discrete confocal quadrics

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A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics. I. An integrable systems approach, Journal of Integrable Systems (2016) Volume 1:1

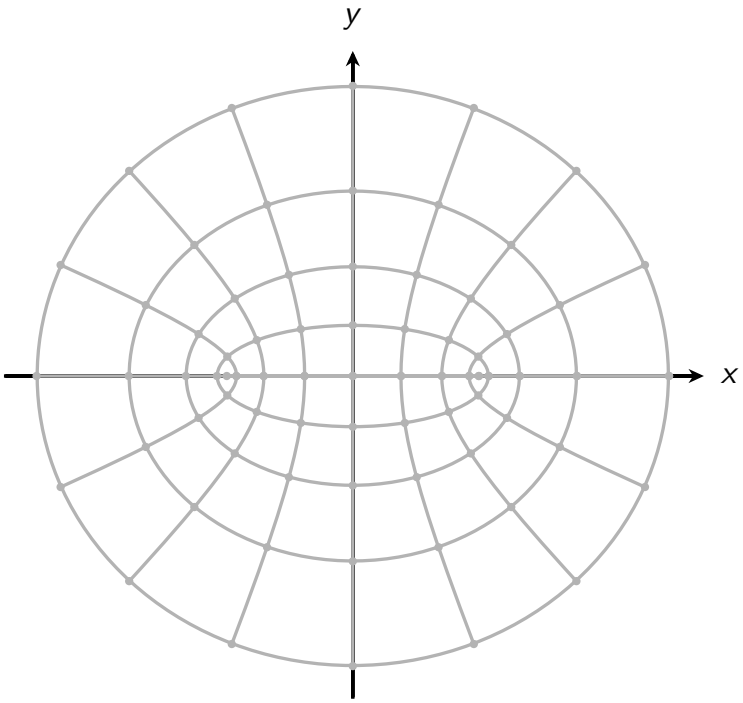
A.I. Bobenko, W.K. Schief, Y.B. Suris, J. Techter.
On a discretization of confocal quadrics. II. A geometric approach to general parametrization, to appear in IMRN

A.I. Bobenko, W.K. Schief, J. Techter.
Checkerboard incircular nets. Laguerre geometry and parametrization, submitted

Confocal conics

Given $a_1 > a_2 > 0$.
The one-parameter family of confocal conics is given by:

$$Q(\lambda) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \frac{x_1^2}{a_1 + \lambda} + \frac{x_2^2}{a_2 + \lambda} = 1 \right\}, \quad \lambda \in \mathbb{R}.$$



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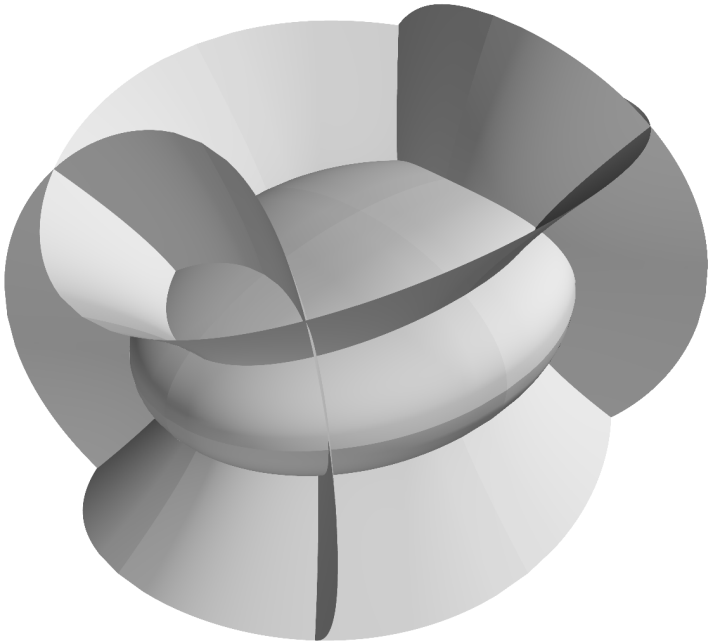
$$Q(\lambda) = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \sum_{k=1}^{N} \frac{x_k^2}{a_k + \lambda} = 1 \right\}, \quad \lambda \in \mathbb{R}.$$



Projective point of view

The confocal quadric equation may also be written as

$$\begin{pmatrix} x_1 & \dots & x_N & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{a_1+\lambda} & & & \\ & \ddots & & \\ & & \frac{1}{a_N+\lambda} & \\ & & & -1 \end{pmatrix}}_{Q_\lambda} \begin{pmatrix} x_1 \\ \vdots \\ x_N \\ 1 \end{pmatrix} = 0$$

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The dual quadrics of this family are given by

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Confocal quadrics as dual pencils

A family of confocal quadrics is a dual pencil of quadrics containing the absolute quadric $\begin{cases} x_{N+1} = 0 \\ x_1^2 + \ldots + x_N^2 = 0 \end{cases}$.

Intersecting confocal quadrics

Given $(x_1, \ldots, x_N) \in \mathbb{R}^N$ with $x_1 \cdots x_N \neq 0$ the equation

$$\sum_{k=1}^{N} \frac{x_k^2}{a_k + \lambda} = 1$$

has N roots, $-a_1 < u_1 < -a_2 < u_2 < \cdots < -a_N < u_N$.

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- The N roots correspond to N confocal quadrics $Q(u_i)$ that intersect at the point $(x_1, \ldots, x_N)$.
- The N quadrics $Q(u_i)$ all have different (affine) signature and intersect orthogonally.

Towards a parametrization

To obtain the coordinates if the intersection points solve the linear system

$$\begin{cases} \frac{x_1^2}{a_1+u_1} + \ldots + \frac{x_N^2}{a_N+u_1} = 1 \\ \vdots \\ \frac{x_1^2}{a_1+u_N} + \ldots + \frac{x_N^2}{a_N+u_N} = 1 \end{cases}$$

for $x_1^2, \ldots, x_N^2$.

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By evaluating the residues at $\lambda = -a_k$ of

$$\sum_{k=1}^{N} \frac{x_k^2}{a_k + \lambda} - 1 = -\frac{\prod_{i=1}^{N}(\lambda - u_i)}{\prod_{i=1}^{N}(a_i + \lambda)}.$$

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we obtain

$$x_k^2 = \frac{\prod_{i=1}^{N}(u_i + a_k)}{\prod_{i \neq k}(a_k - a_i)}, \quad k = 1, \ldots, N.$$

Parametrization from confocal quadrics (confocal coordinates)

Thus, for any $(u_1, \ldots, u_N) \in \mathcal{U}$ with

$$\mathcal{U} = \left\{(u_1, \ldots, u_N) \in \mathbb{R}^N \mid -a_1 < u_1 < -a_2 < u_2 < \ldots < -a_N < u_N\right\}$$

there are exactly 2^N intersection points $(x_1, \ldots, x_N) \in \mathbb{R}^N$, one in every hyperoctant of $\mathbb{R}^N$.

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We obtain a parametrization of, e.g., the first hyperoctant $\mathcal{U} \to \mathbb{R}^N_+$ by

$$x_k(u_1, \ldots, u_N) = \frac{\prod_{i=1}^{k-1} \sqrt{-(u_i + a_k)} \prod_{i=k}^{N} \sqrt{u_i + a_k}}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}}, \quad k = 1, \ldots, N.$$

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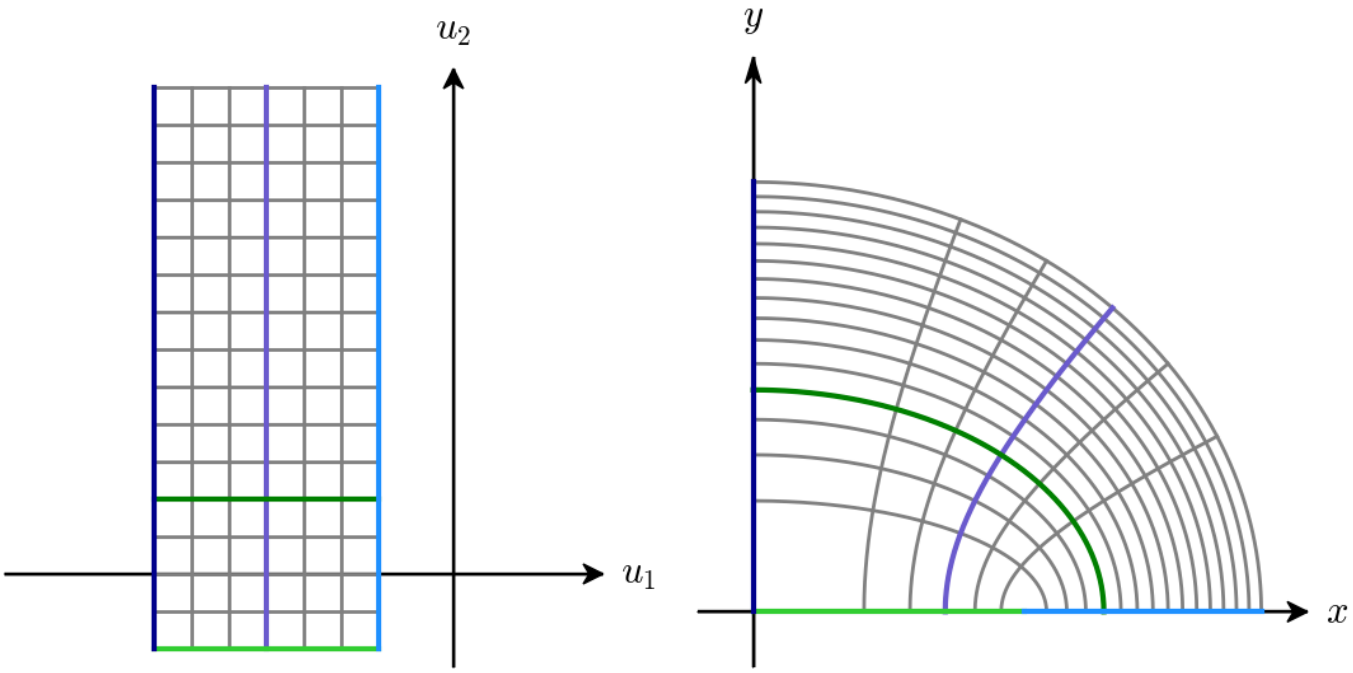
This parametrization is uniquely determined by the family of confocal quadrics up to replacing $u_i = u_i(s_i)$ (reparametrization along the coordinate lines).

Example 2D

$$x_1(u_1, u_2) = \frac{\sqrt{u_1 + a_1}\sqrt{u_2 + a_1}}{\sqrt{a_1 - a_2}}, \quad x_2(u_1, u_2) = \frac{\sqrt{-(u_1 + a_2)}\sqrt{u_2 + a_2}}{\sqrt{a_1 - a_2}},$$



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Introduce a reparametrization according to

$$u_1(s_1) + a_1 = f_1(s_1)^2, \quad u_2(s_2) + a_1 = f_2(s_2)^2$$
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This is a consistent reparametrization, if and only if

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uniformizing the square roots and leading to the parametrization

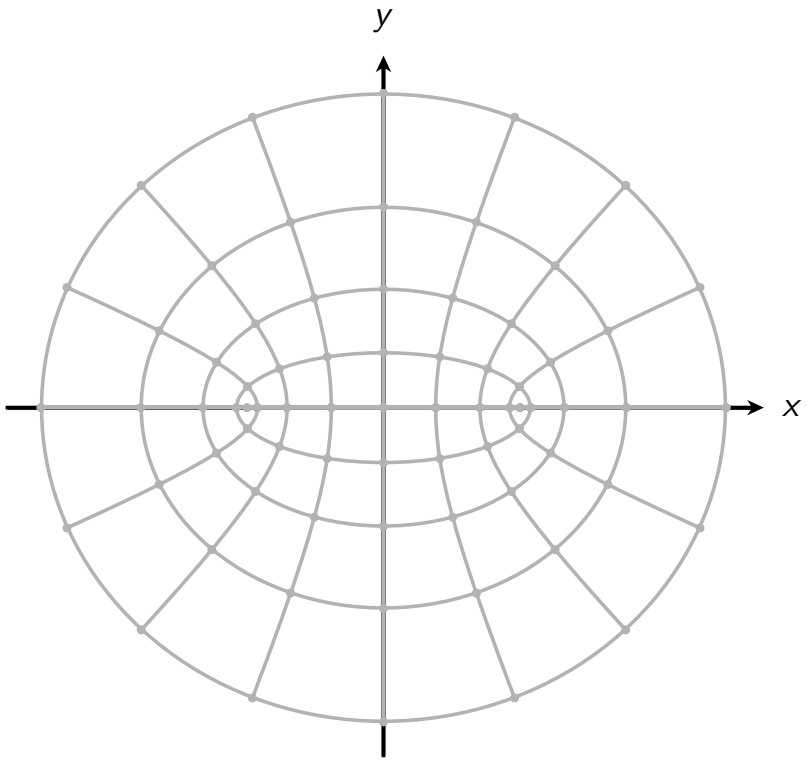
$$x_1(s_1,s_2) = \sqrt{a_1-a_2}\cos s_1\cosh s_2, \quad x_2(s_1,s_2) = \sqrt{a_1-a_2}\sin s_1\sinh s_2.$$

Example 2D: parametrization by trigonometric functions

Thus,

$$x_1(s_1, s_2) = \sqrt{a_1 - a_2} \cos s_1 \cosh s_2, \quad x_2(s_1, s_2) = \sqrt{a_1 - a_2} \sin s_1 \sinh s_2.$$

parametrizes all quadrants by confocal conics at once (periodically in s_1).



This parametrization is conformal (complex cosine function).

Example 3D

Parametrization of the first octant by square roots:

$$x_1(u_1, u_2, u_3) = \frac{\sqrt{u_1 + a_1}\sqrt{u_2 + a_1}\sqrt{u_3 + a_1}}{\sqrt{a_1 - a_2}\sqrt{a_1 - a_3}},$$

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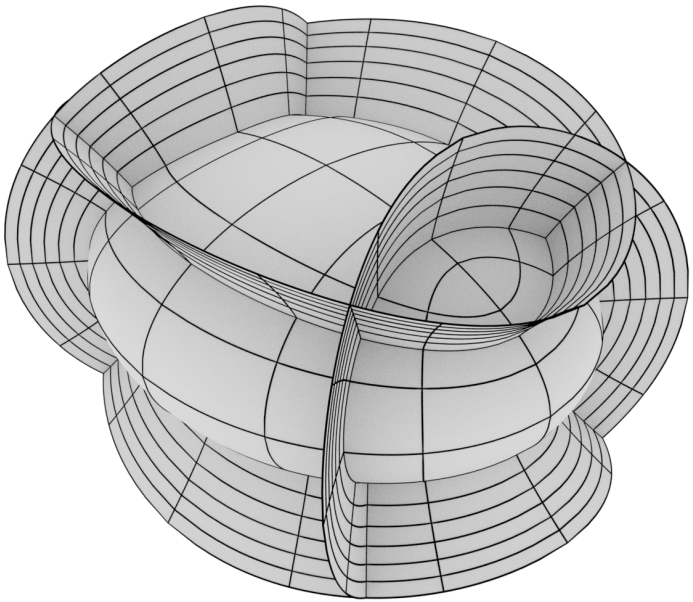

Example 3D: parametrization by elliptic functions

Reparametrization by elliptic functions allows to parametrize all octants simultaneously:

$$
\begin{aligned}
x_1(s_1, s_2, s_3) &= \sqrt{a_1 - a_3}\,\mathrm{sn}(s_1, k_1)\,\mathrm{dn}(s_2, k_2)\,\mathrm{ns}(s_3, k_3) \\
x_2(s_1, s_2, s_3) &= \sqrt{a_1 - a_3}\,\mathrm{cn}(s_1, k_1)\,\mathrm{cn}(s_2, k_2)\,\mathrm{ds}(s_3, k_3) \\
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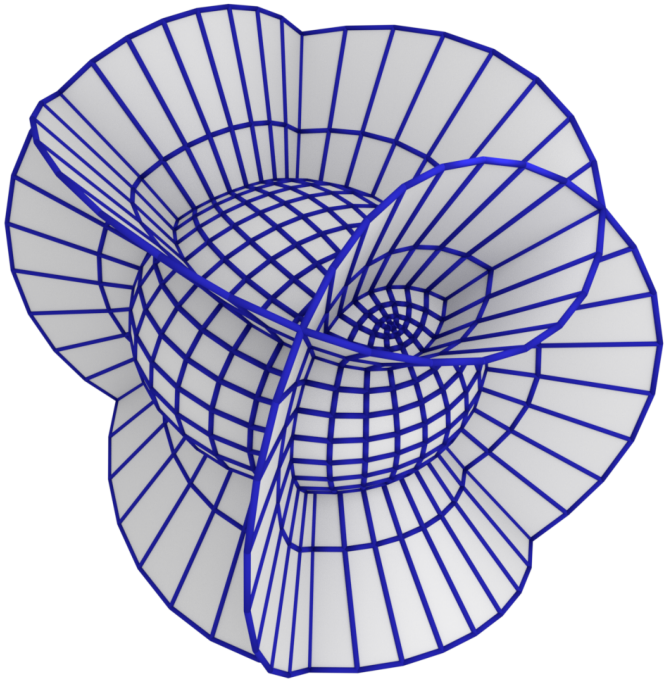
\end{aligned}
$$

with $k_1^2 = \frac{a_1 - a_2}{a_1 - a_3}$, $k_2^2 = 1 - k_2^2$, $k_3^2 = k_1^2$.



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- All two-dimensional coordinate surfaces are isothermic.
  (Though in general not conformally parametrized.)
- Satisfies the Euler-Poisson-Darboux equation for $\gamma = \frac{1}{2}$
  (up to reparametrization)

$$\partial_{u_i} \partial_{u_j} \boldsymbol{x} = \frac{\gamma}{u_i - u_j} (\partial_{u_j} \boldsymbol{x} - \partial_{u_i} \boldsymbol{x}), \quad i, j \in \{1, \ldots, N\}$$

Characterization of confocal coordinates

Theorem

If a coordinate system $\boldsymbol{x} : \mathbb{R}^N \supset U \to \mathbb{R}^N$ *satisfies two conditions:*

i) $\boldsymbol{x}(s_1, \ldots, s_N)$ factorizes, *in the sense that*

$$\begin{cases} x_1(s_1, \ldots, s_N) = f_1^1(s_1) f_2^1(s_2) \cdots f_N^1(s_N), \\ x_2(s_1, \ldots, s_N) = f_1^2(s_1) f_2^2(s_2) \cdots f_N^2(s_N), \\ \vdots \\ x_N(s_1, \ldots, s_N) = f_1^N(s_1) f_2^N(s_2) \cdots f_N^N(s_N), \end{cases}$$

with all $f_i^k(s_i) \neq 0$ *and* $\left(f_i^k\right)'(s_i) \neq 0$*;*

ii) $\boldsymbol{x}$ *is* orthogonal, *that is,*

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0 \quad \textit{for} \quad i \neq j,$$

then all coordinate hypersurfaces are confocal quadrics.

Discrete orthogonal nets

Definition

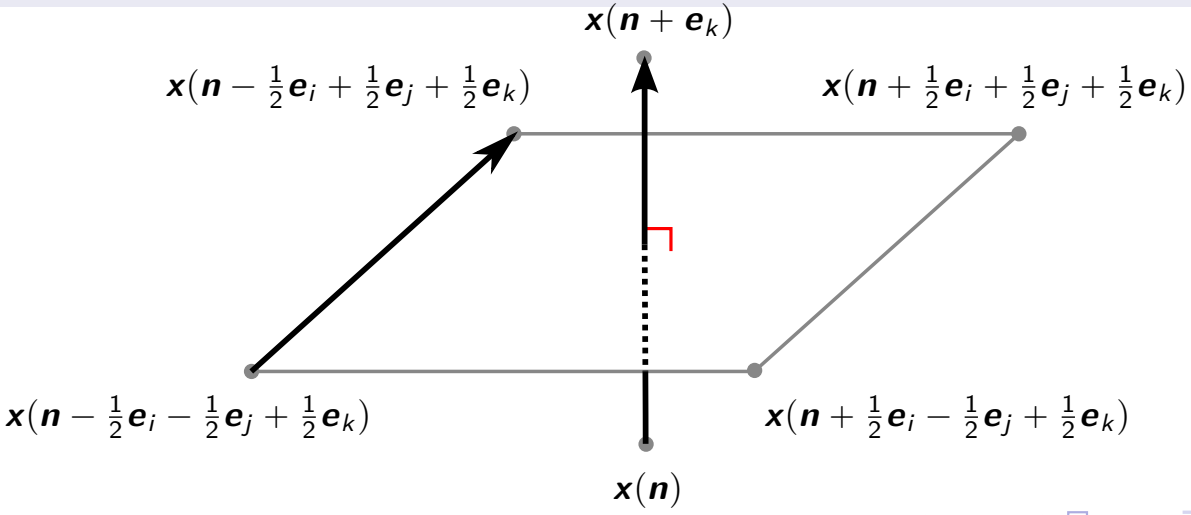
A discrete net (on a stepsize 1/2 square lattice)

$$\boldsymbol{x} : (\tfrac{1}{2}\mathbb{Z})^N \supset \mathcal{U} \to \mathbb{R}^N.$$

is called *orthogonal* if any pair of dual stepsize 1 edges is orthogonal:

$$\big(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i)\big) \perp \big(\boldsymbol{x}(\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{\sigma}), \boldsymbol{x}(\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{\sigma}+\boldsymbol{e}_j)\big),$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \{\pm 1\}^N$ *with* $\sigma_i = 1, \sigma_j = -1$.

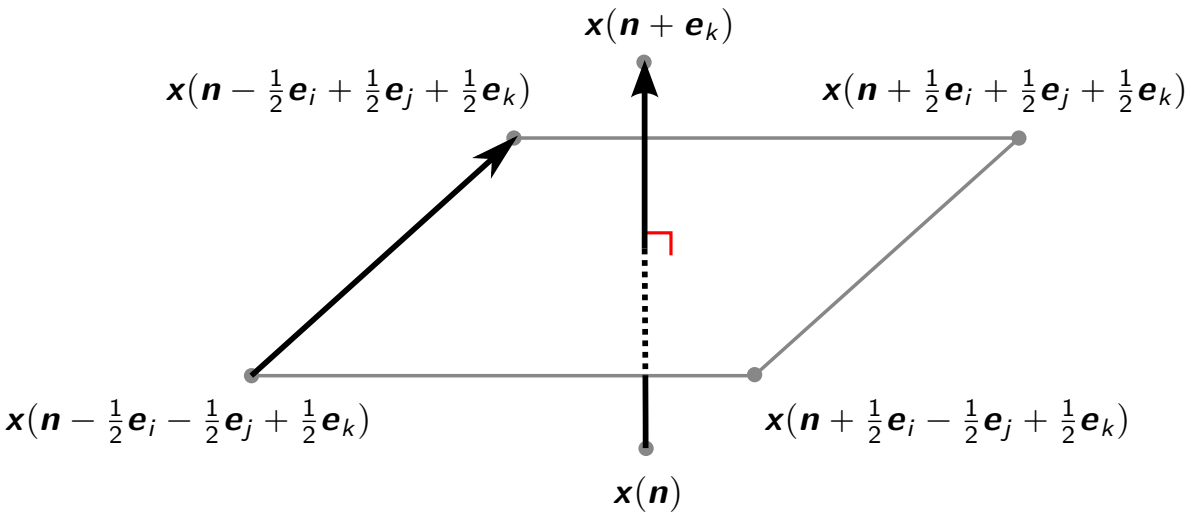


Remark

Each stepsize 1/2 discrete orthogonal net $\boldsymbol{x} : (\frac{1}{2}\mathbb{Z})^N \to \mathbb{R}^N$, contains 2^{N-1} pairs of combinatorially dual stepsize 1 nets, e.g.

$$\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{R}^N \quad \text{and} \quad \boldsymbol{x}^* : \mathbb{Z}^N + \tfrac{1}{2}\boldsymbol{\sigma} \to \mathbb{R}^N.$$

with orthogonal dual edges.
We call any such pair, a pair of discrete orthogonal nets.



Theorem ((classical) Dupin's theorem)

The coordinate surfaces of a triply orthogonal coordinate system intersect each other in curvature lines.

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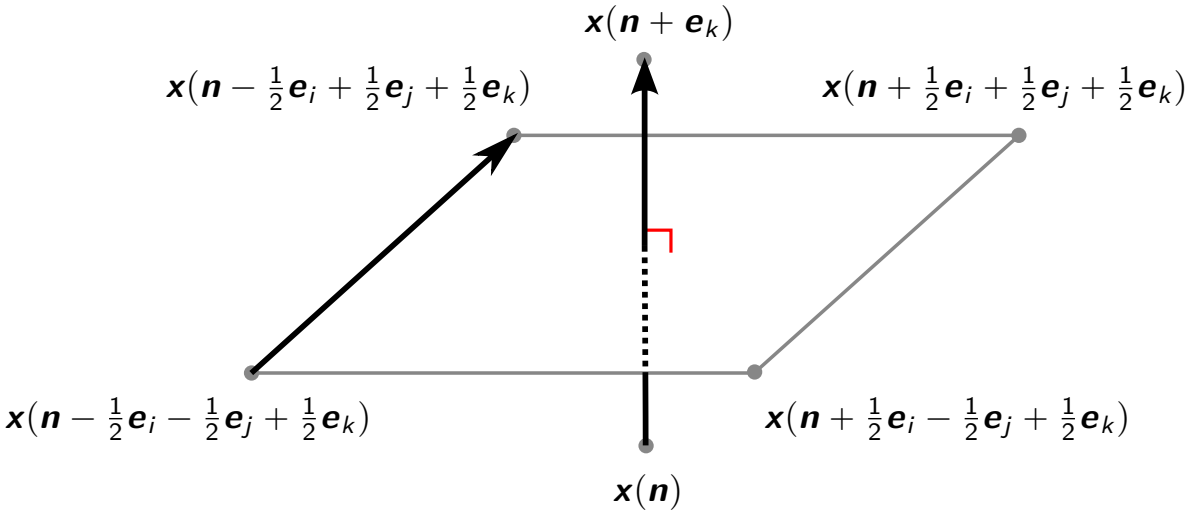
Theorem (discrete Dupin's theorem)

All elementary quadrilaterals

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_k)) \tag{1}$$

of a generic orthogonal net are planar.



Möbius invariant formulation

Given a pair of two combinatorially dual stepsize 1 nets $\boldsymbol{x}$, $\boldsymbol{x}^*$, introduce circles / spheres with centers $\boldsymbol{x}$, $\boldsymbol{x}^*$ and radii r, r^* respectively.

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In this sense, discrete orthogonal nets (plus a choice of orthogonal spheres / 1 global parameter) are Möbius invariant.

Discrete confocal quadrics: definition

Definition

A discrete coordinate system $\boldsymbol{x} : \left(\frac{1}{2}\mathbb{Z}\right)^N \supset \mathcal{U} \to \mathbb{R}^N$ is called *a discrete confocal coordinate system* if it satisfies two conditions:

i) $\boldsymbol{x}(\boldsymbol{n})$ *factorizes*, in the sense that for any $\boldsymbol{n} \in \mathcal{U}$

$$\begin{cases} x_1(\boldsymbol{n}) = f_1^1(n_1) f_2^1(n_2) \cdots f_N^1(n_N), \\ x_2(\boldsymbol{n}) = f_1^2(n_1) f_2^2(n_2) \cdots f_N^2(n_N), \\ \cdots \\ x_N(\boldsymbol{n}) = f_1^N(n_1) f_2^N(n_2) \cdots f_N^N(n_N), \end{cases}$$

with $f_i^k(n_i) \neq 0$ and $\bar{\Delta} f_i^k(n_i) = f_i^k(n_i) - f_i^k(n_i - 1) \neq 0$;

ii) $\boldsymbol{x}$ is *orthogonal*.

Discrete confocal quadrics: main theorem

Theorem

For a discrete confocal coordinate system, there exist N real numbers a_k, $1 \leqslant k \leqslant N$, and N sequences $u_i : \frac{1}{2}\mathbb{Z} + \frac{1}{4} \to \mathbb{R}$ such that the following equations are satisfied for any $\boldsymbol{n} \in \mathcal{U}$ and for any $\boldsymbol{\sigma} \in \{\pm 1\}^N$:

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n}) x_k(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})}{a_k + u_i} = 1, \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i), \quad i = 1, \ldots, N.$$

Equivalently,

$$x_k(\boldsymbol{n}) x_k(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}) = \frac{\prod_{j=1}^{N}(u_j + a_k)}{\prod_{j \neq k}(a_k - a_j)}, \quad u_j = u_j(n_j + \tfrac{1}{4}\sigma_j), \quad k = 1, \ldots, N.$$

Geometric interpretation

The *discrete confocal quadric equation*

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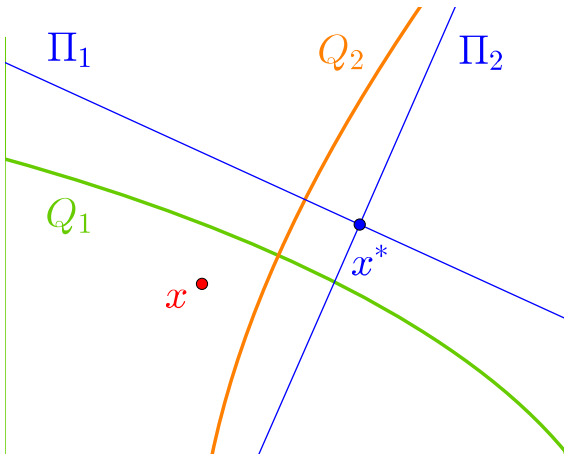
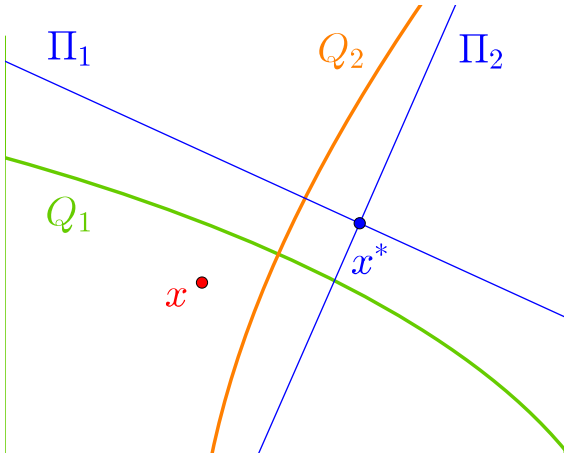
geomemtric interpretation

The point $\boldsymbol{x}(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})$ lies in the intersection of the polar hyperplanes of $\boldsymbol{x}(\boldsymbol{n})$ with respect to the confocal quadrics $Q(u_i)$, $i = 1,\dots,N$.



Geometric construction



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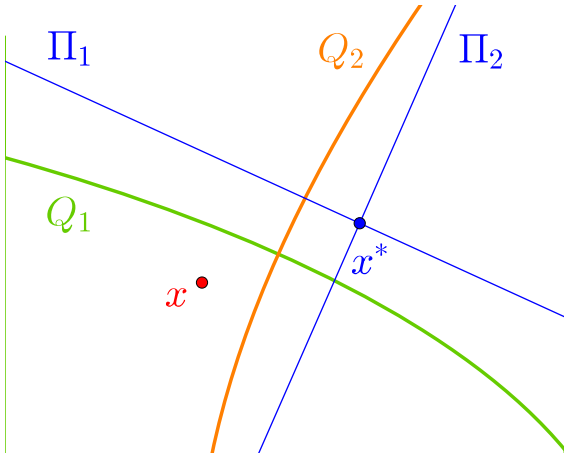

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Suppose $\boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}$ is already known. Construct a neighboring point $\boldsymbol{x}(\boldsymbol{n}^*) = \boldsymbol{x}^*$ as the intersection point of the N polar hyperplanes

$$\boldsymbol{x}^* = \bigcap_{i=1}^{N} P_{Q(u_i)}(\boldsymbol{x}), \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i).$$

Closeness and orthogonality

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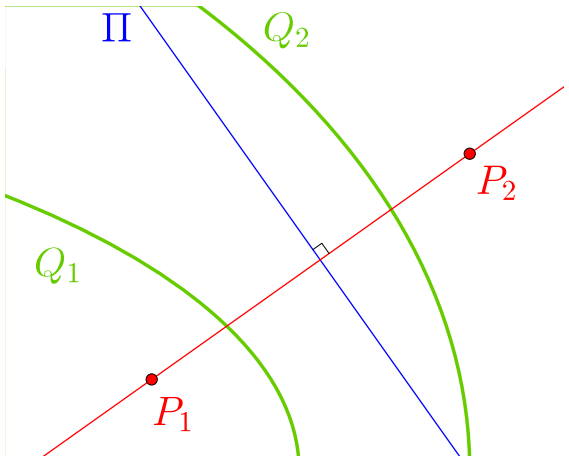
Let Π be a hyperplane. Then the poles of Π with respect to all quadrics of a dual pencil of quadrics lie on a line ℓ.



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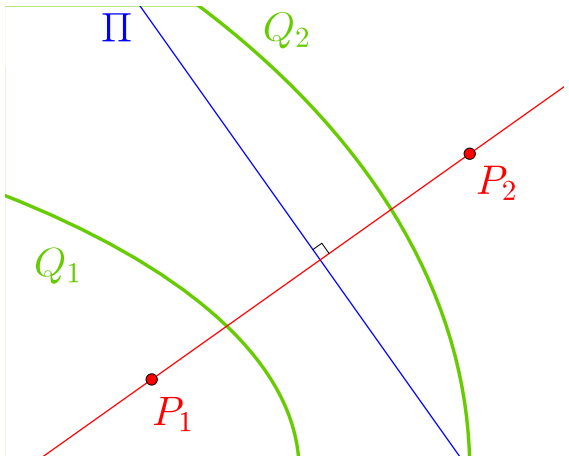

For a family of confocal quadrics this line ℓ is orthogonal to Π.

Finding explicit solutions

Looking at

$$x_k(\boldsymbol{n})x_k(\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{\sigma}) = \frac{\prod_{j=1}^{N}(u_j+a_k)}{\prod_{j\neq k}(a_k-a_j)}, \quad u_j = u_j(n_j+\tfrac{1}{4}\sigma_j), \quad k=1,\dots,N,$$

we might want to rewrite the coordinate functions as

$$x_k(\boldsymbol{n}) = \frac{\prod_{j=1}^{N} f_j^k(n_j)}{\prod_{i=1}^{k-1}\sqrt{a_i-a_k}\prod_{i=k+1}^{N}\sqrt{a_k-a_i}}, \quad k=1,\dots,N,$$

where

$$f_i^k(n_i)f_i^k(n_i+\tfrac{1}{2}) = \begin{cases} u_i(n_i+\frac{1}{4})+a_k, & k\leqslant i, \\ -\big(u_i(n_i+\frac{1}{4})+a_k\big), & k>i. \end{cases}$$

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$$x_k(\boldsymbol{n}) = \frac{\prod_{j=1}^N f_j^k(n_j)}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^N \sqrt{a_k - a_i}}, \quad k = 1, \ldots, N,$$

Example: Parametrization in terms of Γ-functions

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which may be solved by

$$f_i^k(n_i) = \begin{cases} \sqrt[\Delta]{n_i + a_k + \epsilon_i} & \text{for} \quad i \geqslant k, \\ \sqrt[\Delta]{-n_i - a_k - \epsilon_i + \frac{1}{2}} & \text{for} \quad i < k. \end{cases}$$

with the "discrete square root" function $\sqrt[\Delta]{u} = \frac{\Gamma(u+\frac{1}{2})}{\Gamma(u)}$, which satisfies

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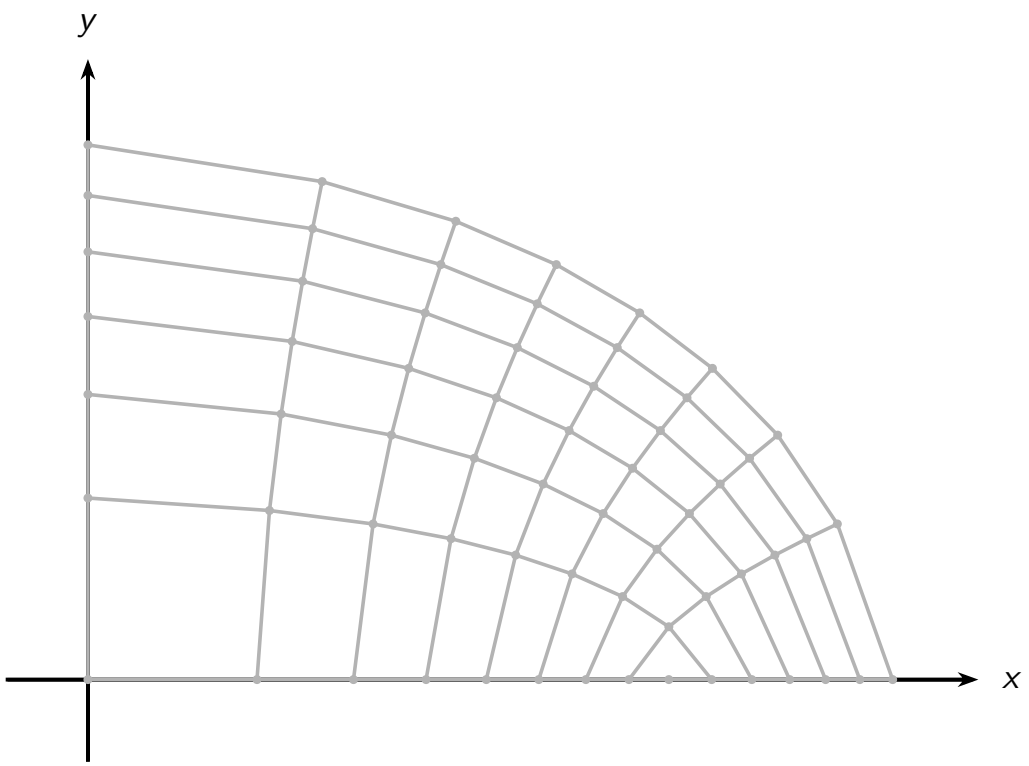
The parameters ϵ_i can be used to achieve certain boundary conditions.

Example 2D

$$x(n_1, n_2) = \frac{\sqrt[\Delta]{n_1 + a_1 - \frac{1}{2}}\ \sqrt[\Delta]{n_2 + a_1 - 1}}{\sqrt{a_1 - a_2}}, \quad y(n_1, n_2) = \frac{\sqrt[\Delta]{-n_1 - a_2 + 1}\ \sqrt[\Delta]{n_2 + a_2 - 1}}{\sqrt{a_1 - a_2}},$$

with $a_1 = \alpha_1 + \frac{1}{2}$, $a_2 = \alpha_2 + 1$ and $\alpha_1 > \alpha_2$ integers.



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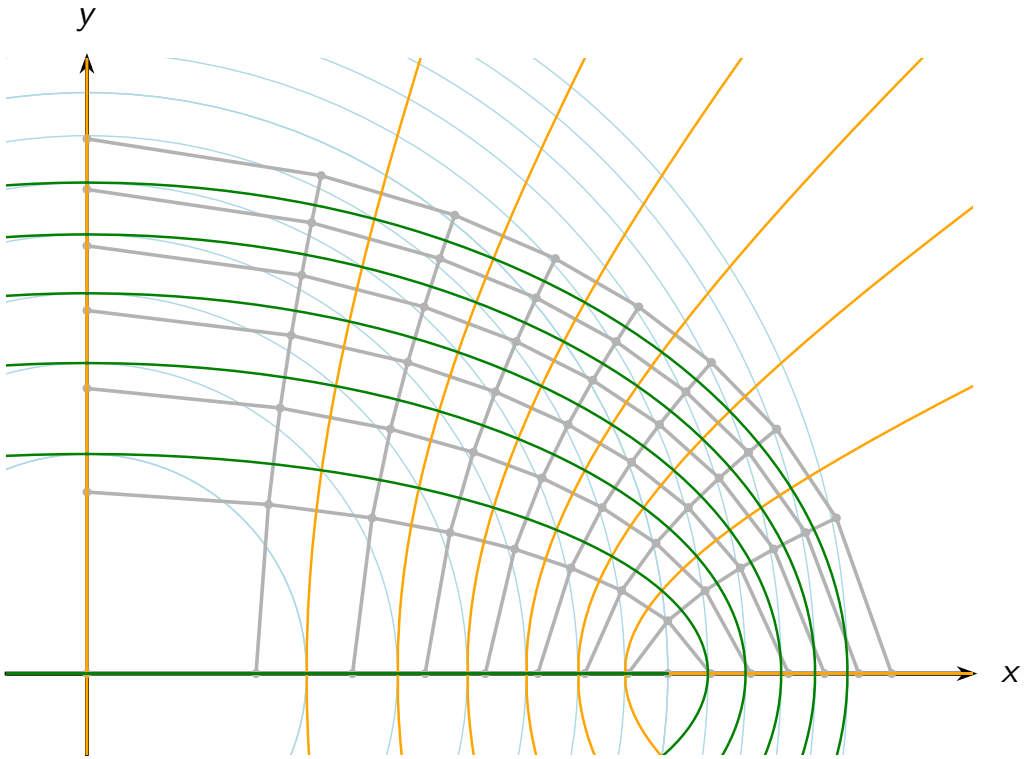

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Eliminating u_1 and u_2 we obtain

$$f_1(n_1)f_1(n_1 + \tfrac{1}{2}) + g_1(n_1)g_1(n_1 + \tfrac{1}{2}) = a_1 - a_2,$$
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This can be solved via

$$f_1(n_1) = \sqrt{\frac{a-b}{\cos\frac{\delta_1}{2}}}\cos(\delta_1 n_1 + c_1), \quad g_1(n_1) = \sqrt{\frac{a-b}{\cos\frac{\delta_1}{2}}}\sin(\delta_1 n_1 + c_1),$$

and

$$f_2(n_2) = \sqrt{\frac{a-b}{\cosh\frac{\delta_2}{2}}}\cosh(\delta_2 n_2 + c_2), \quad g_2(n_2) = \sqrt{\frac{a-b}{\cosh\frac{\delta_2}{2}}}\sinh(\delta_2 n_2 + c_2).$$

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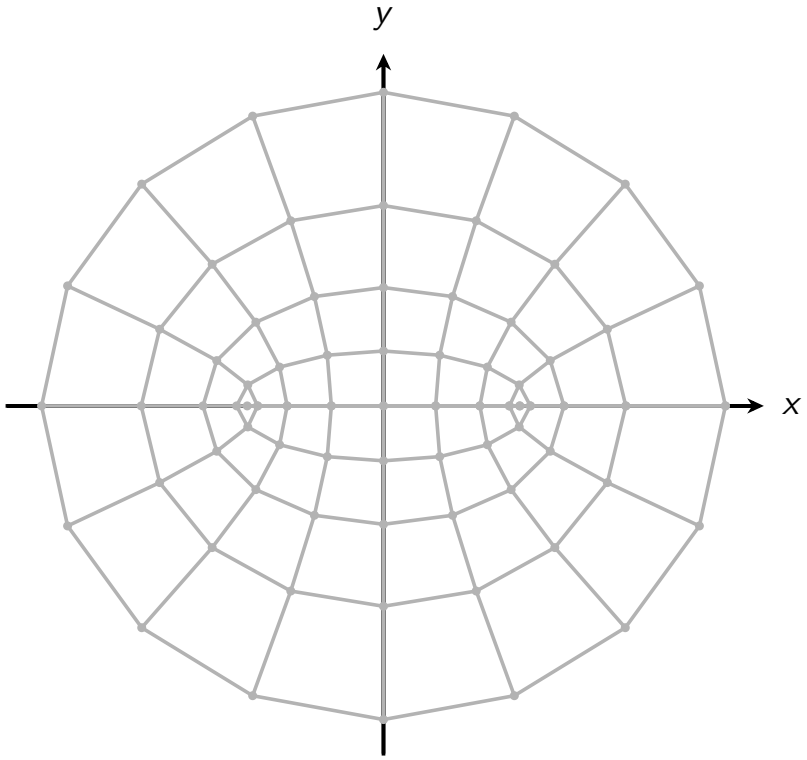

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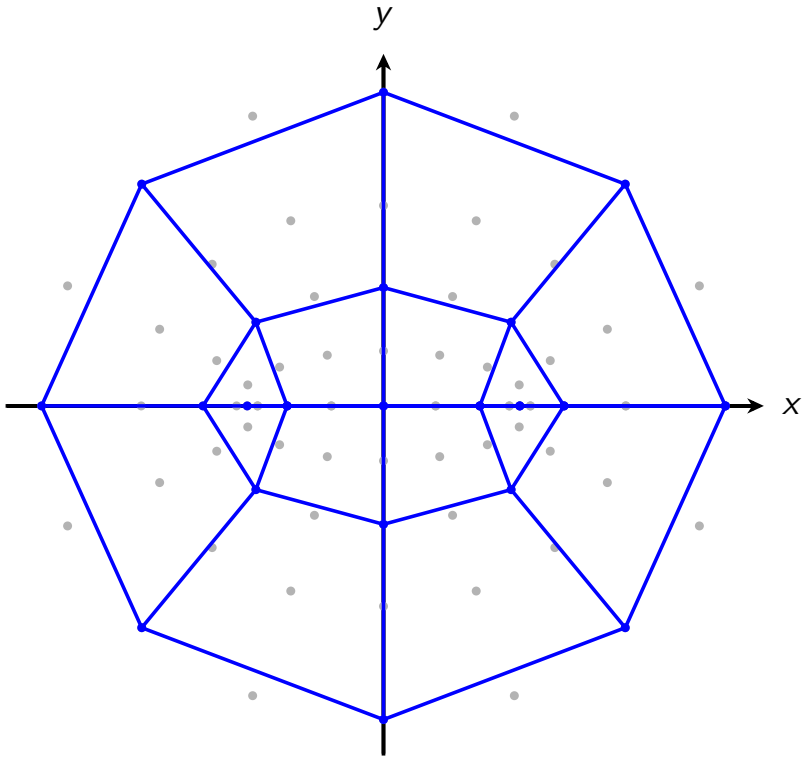

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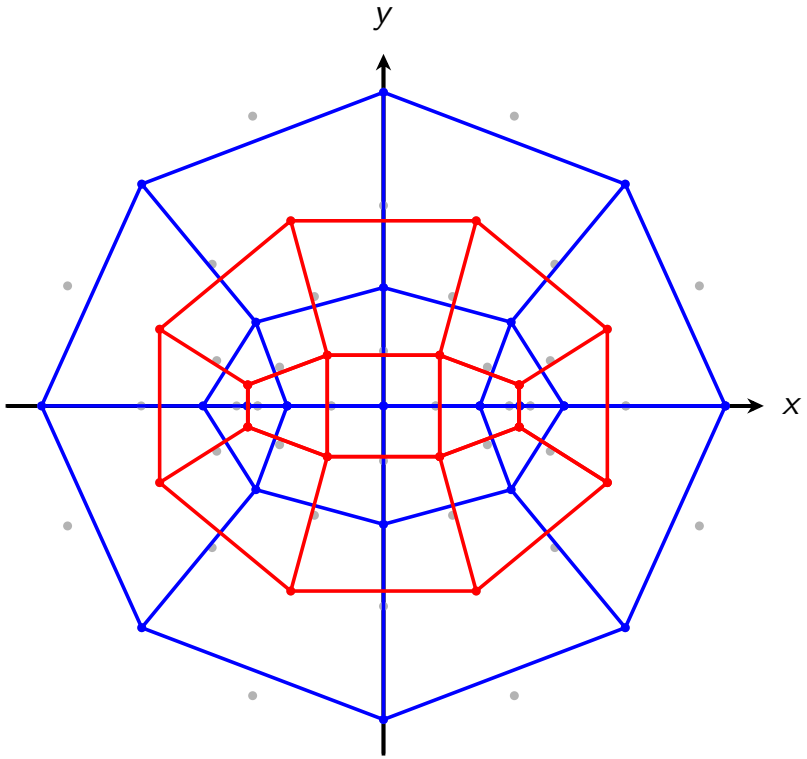

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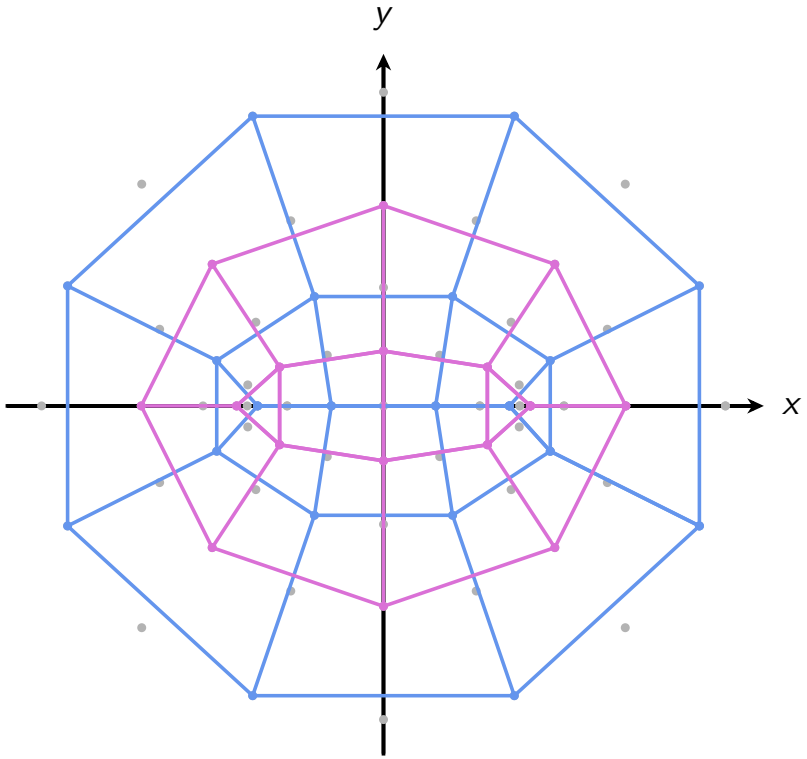
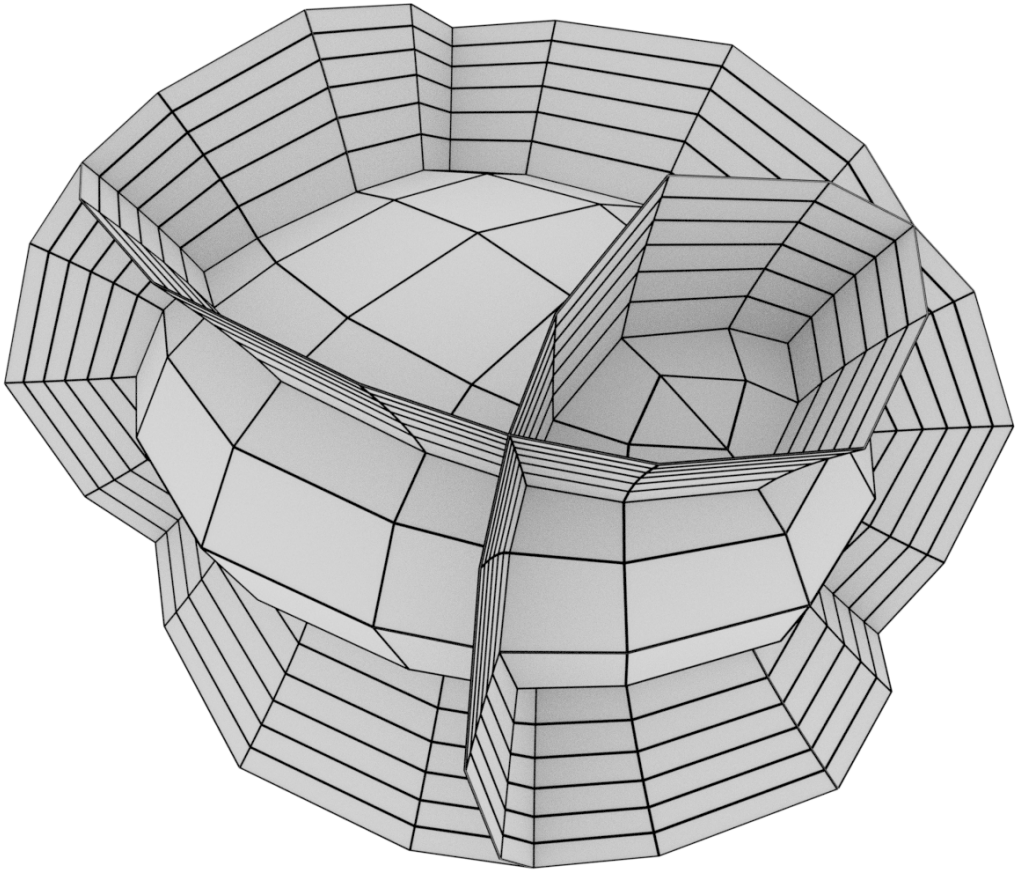


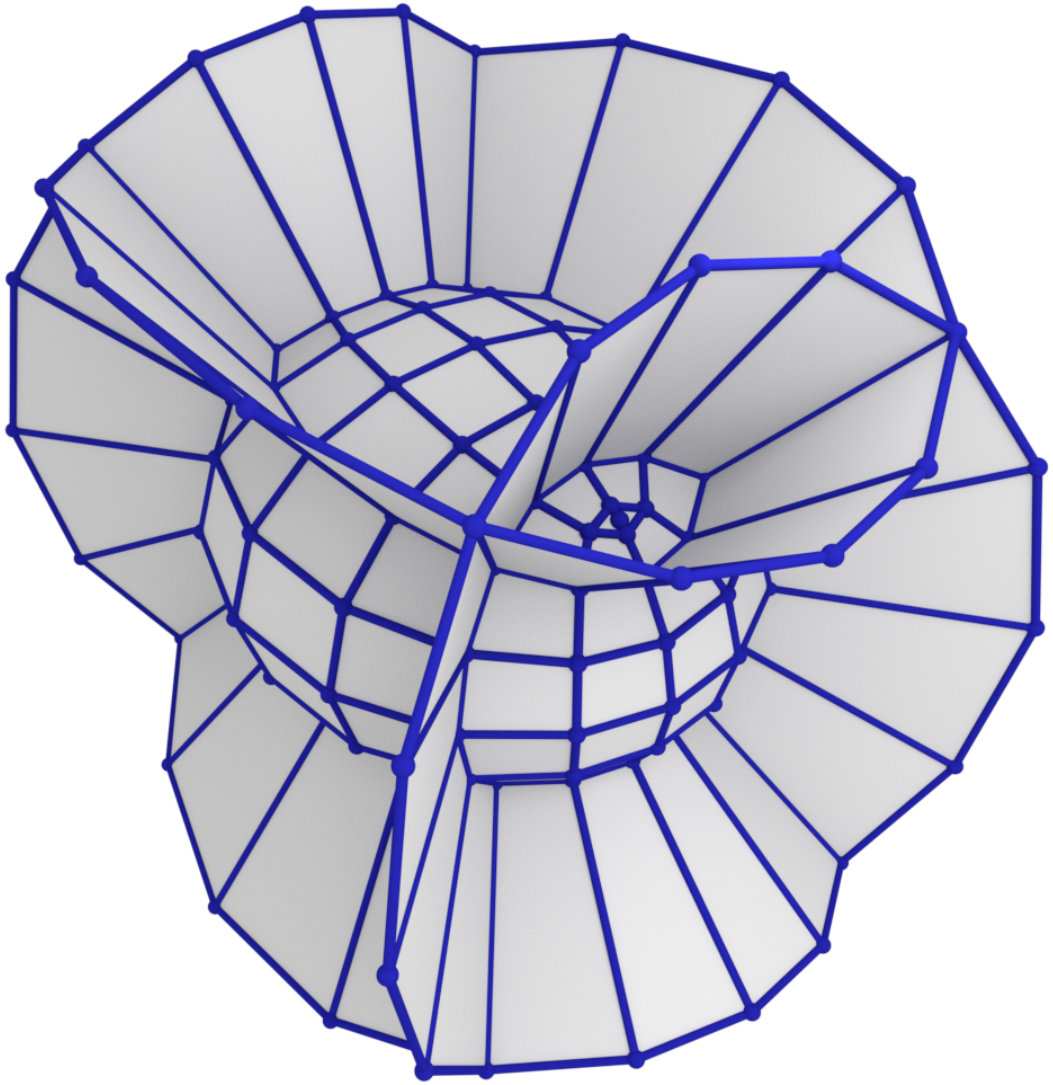
Example 3D

Discrete “square root” parametrization.



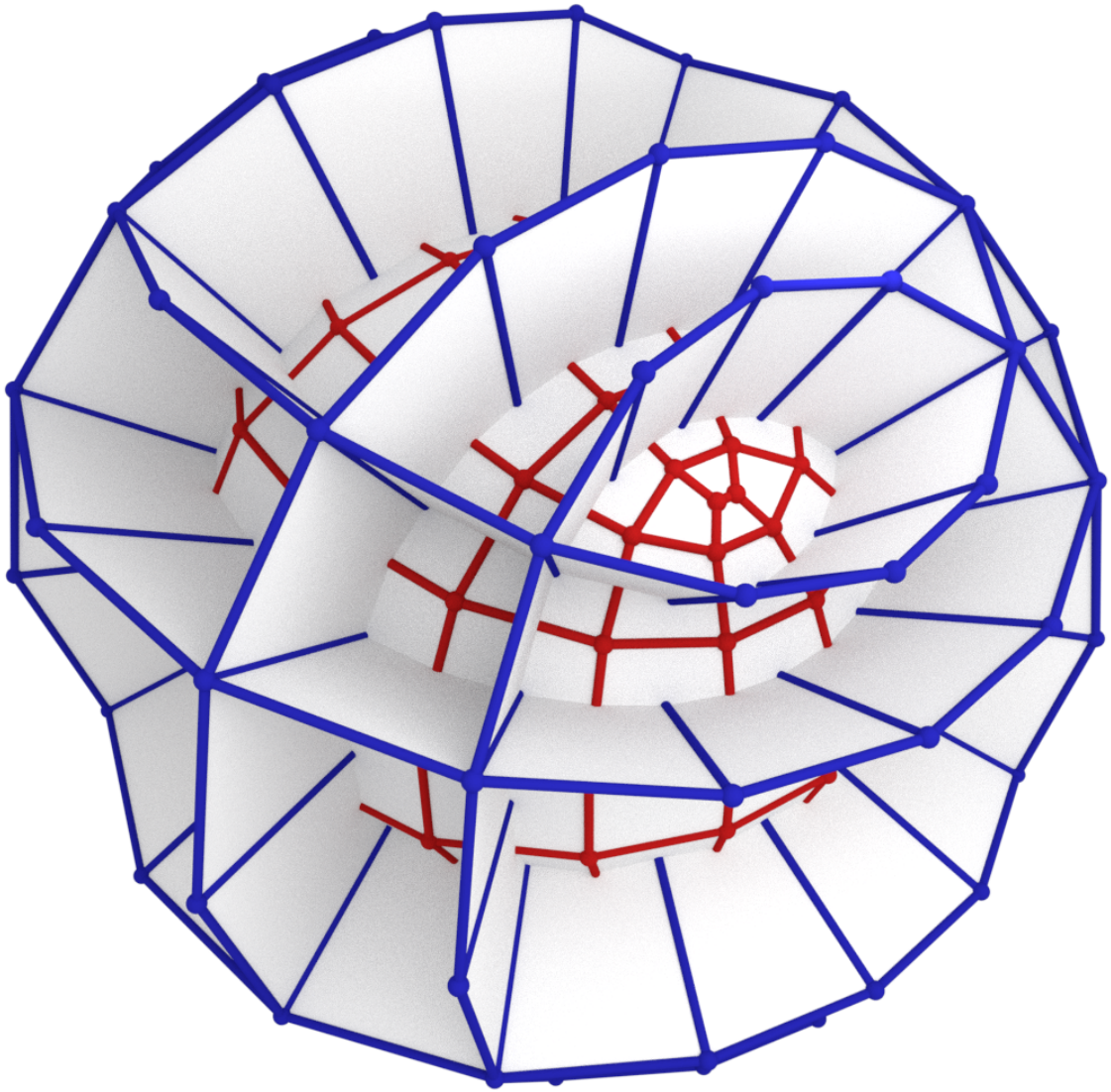
Example 3D

Discrete parametrization by elliptic functions.



Example 3D

Discrete parametrization by elliptic functions.



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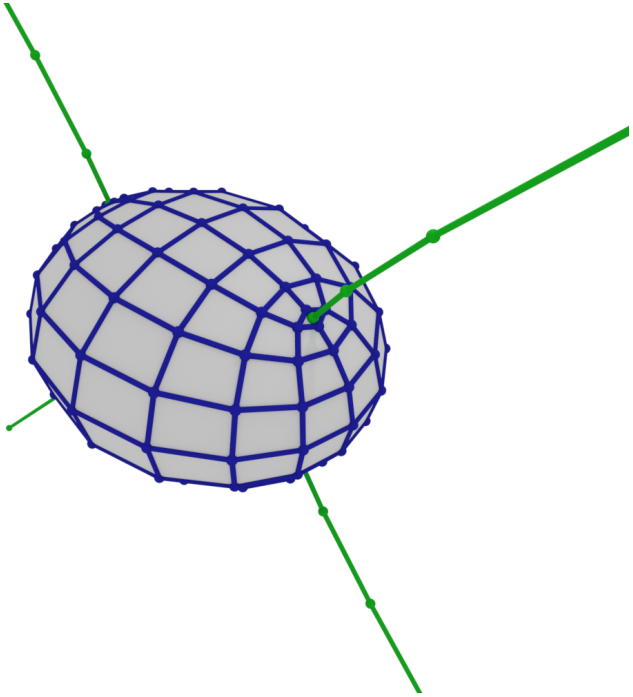
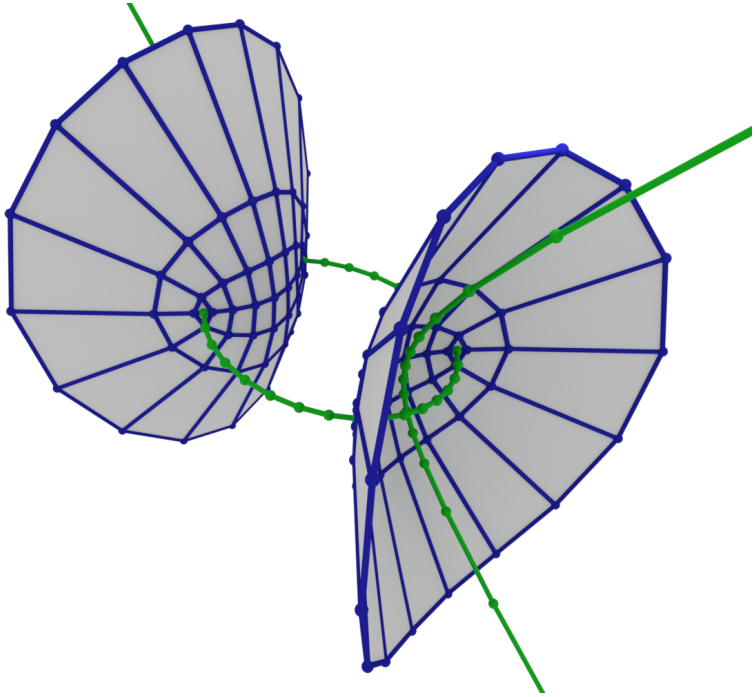
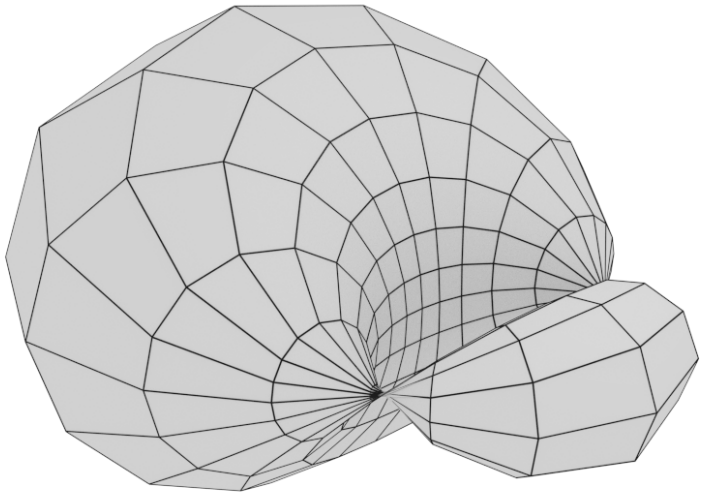
Discrete focal conics





Discrete Dupin cyclides



IC-nets as discrete confocal conics