Geometry 3

Confocal quadrics, their discretization, and related topics

Jan Techter

February 22, 2022

Lecture notes for Geometry 3, Winter 2021/2022.

Contents

1 Orthogonal nets

We start with the definition of *(regular) nets*, which represent parametrizations of submanifolds of $\mathbb{R}^N$, in particular,

- *parametrized curves* in the case $M = 1$,
- *parametrized surfaces* in the case $M = 2$, and
- *coordinate systems* of (some region of) $\mathbb{R}^N$ in the case $M = N$.

Definition 1.1.

(i) Let $U \subset \mathbb{R}^M$ be open and connected. Then a smooth map

$$\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N, \qquad (s_1, \ldots, s_M) \mapsto \boldsymbol{x}(s_1, \ldots, s_M)$$

is called an *M-dimensional (smooth) net*.

(ii) A net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ is called *regular* if the M tangent vectors

$$\partial_1 \boldsymbol{x}, \ldots, \partial_M \boldsymbol{x} \in \mathbb{R}^N$$

are linearly independent at every point in U, where $\partial_i = \frac{\partial}{\partial s_i}$ denotes the i-th partial derivative.

(iii) Let $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ be a net and $\{i_1, \ldots, i_n\} \subset \{1, \ldots, M\}$ some indices. Then the map

$$(s_{i_1}, \ldots, s_{i_n}) \mapsto \boldsymbol{x}(s_1, \ldots, s_M)$$

for fixed s_j with complementary indices is called an *n-dimensional subnet of $\boldsymbol{x}$*. In particular, 1-dimensional subnets are called *coordinate lines* and 2-dimensional subnets are called *coordinate surfaces*.

Remark 1.1. Concerning the "smoothness" of a net, we follow the tradition of classical differential geometry assuming that all required partial derivatives exist without explicitly stating. Furthermore, we assume all appearing nets to be regular unless stated otherwise.

Our main object of interest are *orthogonal nets*.

Definition 1.2.

(i) A net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ is called *orthogonal* if

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0, \qquad i, j = 1, \ldots, M, \ i \neq j. \tag{1}$$

(ii) For an orthogonal net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$, the functions $H_i : U \to \mathbb{R}_+$,

$$H_i^2 = \langle \partial_i \boldsymbol{x}, \partial_i \boldsymbol{x} \rangle, \qquad i = 1, \ldots, M$$

are called its *Lamé coefficients*.

Remark 1.2.

(i) The notion of orthogonal nets is invariant under Möbius transformations of the codomain.

(ii) The metric of an orthogonal net, or its *first fundamental form*, is diagonal and entirely determined by its Lamé coefficients,

$$\mathrm{I} = H_1^2 \mathrm{d}s_1^2 + \ldots + H_M^2 \mathrm{d}s_M^2.$$

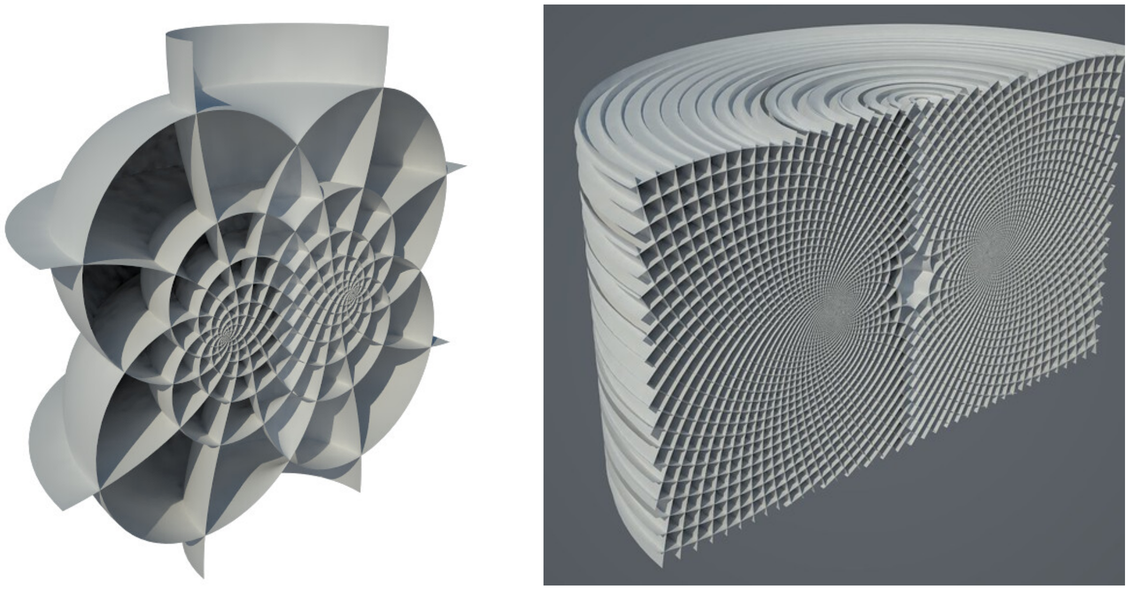

Figure 1. Coordinate surfaces of triply orthogonal coordinate systems.

Definition 1.3. A net $\boldsymbol{x} : \mathbb{R}^M \supset U \to \mathbb{R}^N$ is called *conjugate* if for every $i, j = 1, \ldots,\ i \neq j$ the three vectors

$$\partial_i \partial_j \boldsymbol{x}, \quad \partial_i \boldsymbol{x}, \quad \partial_j \boldsymbol{x}$$

are linearly dependent.

Remark 1.3.

(i) The condition of being a conjugate net is a condition on every two-dimensional subnet, and invariant under projective transformations.

(ii) Conjugate nets are governed by partial differential equations of the form

$$\partial_i \partial_j \boldsymbol{x} = a_{ji} \partial_i \boldsymbol{x} + a_{ij} \partial_j \boldsymbol{x}$$

with functions $a_{ij}, a_{ji} : U \to \mathbb{R}$ satisfying some consistency conditions if $M \geqslant 3$.

Theorem 1.1 (Dupin)**.** *For $N \geqslant 3$ every orthogonal coordinate system $\boldsymbol{x} : \mathbb{R}^N \supset U \to \mathbb{R}^N$ is conjugate.*

Proof. For three distinct $i, j, k = 1, \ldots, N$ differentiating (1) with respect to s_k leads to

$$\langle \partial_i \boldsymbol{x}, \partial_j \partial_k \boldsymbol{x} \rangle + \langle \partial_j \boldsymbol{x}, \partial_k \partial_i \boldsymbol{x} \rangle = 0.$$

By permutation of the indices, these are three equations which sum up to

$$2 \left(\langle \partial_i \boldsymbol{x}, \partial_j \partial_k \boldsymbol{x} \rangle + \langle \partial_j \boldsymbol{x}, \partial_k \partial_i \boldsymbol{x} \rangle + \langle \partial_k \boldsymbol{x}, \partial_i \partial_j \boldsymbol{x} \rangle \right) = 0.$$

Dividing by 2 and subtracting one of the first three equations again leads to

$$\langle \partial_i \boldsymbol{x}, \partial_j \partial_k \boldsymbol{x} \rangle = 0.$$

Thus, for $j, k = 1, \ldots, N$, $j \neq k$,

$$\partial_j \partial_k \boldsymbol{x} \in \operatorname{span}\left\{\partial_i \boldsymbol{x} \mid i = 1, \ldots, N,\ i \neq j, k\right\}^{\perp} = \operatorname{span}\{\partial_j \boldsymbol{x}, \partial_k \boldsymbol{x}\},$$

due to the regularity and orthogonality of $\boldsymbol{x}$. $\square$

Example 1.1 (Cylindrical coordinates)**.** Consider the map

$$\boldsymbol{x} : [0, \infty) \times [0, 2\pi) \times (-\infty, \infty) \to \mathbb{R}^3, \qquad (r, \varphi, z) \mapsto \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

Its partial derivatives are given by

$$\partial_r \boldsymbol{x} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \quad \partial_\varphi \boldsymbol{x} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}, \quad \partial_z \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $\boldsymbol{x}$ defines a coordinate system at all points with

$$\det(\partial_r \boldsymbol{x}, \partial_\varphi \boldsymbol{x}, \partial_z \boldsymbol{x}) = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \neq 0$$

The coordinate system is orthogonal since

$$\langle \partial_r \boldsymbol{x}, \partial_\varphi \boldsymbol{x} \rangle = \langle \partial_\varphi \boldsymbol{x}, \partial_z \boldsymbol{x} \rangle = \langle \partial_z \boldsymbol{x}, \partial_r \boldsymbol{x} \rangle = 0,$$

and its Lamé coefficients are given by

$$H_r = \|\partial_r \boldsymbol{x}\| = 1, \quad H_\varphi = \|\partial_\varphi \boldsymbol{x}\| = r, \quad H_z = \|\partial_z \boldsymbol{x}\| = 1.$$

By Theorem 1.1, all coordinate surfaces of $\boldsymbol{x}$ are conjugate nets. Indeed, the second partial derivatives are given by

$$\partial_r \partial_\varphi \boldsymbol{x} = \frac{1}{r} \partial_\varphi \boldsymbol{x}, \quad \partial_\varphi \partial_z \boldsymbol{x} = 0, \quad \partial_z \partial_r \boldsymbol{x} = 0.$$

2 Discrete orthogonal nets

In discrete differential geometry, the classical notion of a net is replaced by that of a *discrete net*, which is defined on the *square lattice* $\mathbb{Z}^M$.

Definition 2.1.

(i) A map

$$\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N, \qquad \boldsymbol{n} = (n_1, \ldots, n_M) \mapsto \boldsymbol{x}(\boldsymbol{n})$$

is called an *M-dimensional discrete net*.

(ii) Denote the forward and backward *difference operators*, or *discrete tangent vectors*, by

$$\Delta_i \boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}(\boldsymbol{n} + \boldsymbol{e}_i) - \boldsymbol{x}(\boldsymbol{n}), \qquad \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}(\boldsymbol{n}) - \boldsymbol{x}(\boldsymbol{n} - \boldsymbol{e}_i)$$

for any $\boldsymbol{n} \in \mathbb{Z}^M$ and $i = 1, \ldots, M$, where $\boldsymbol{e}_i \in \mathbb{Z}^M$ is the unit vector in the i-th coordinate direction. A discrete net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$ is called *regular* if for any $\boldsymbol{n} \in \mathbb{Z}^M$ all choices of M discrete tangent vectors, arbitrarily chosen among $\Delta_i \boldsymbol{x}(\boldsymbol{n})$ and $\bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n})$ for all $i = 1, \ldots, M$, are linearly independent.

(iii) *n-dimensional discrete subnets* are defined as for smooth nets (see Definition 1.1).

Remark 2.1. Note that for now, we assume discrete nets to be defined on the whole lattice $\mathbb{Z}^M$. In some sense, this replaces the openness condition on the domain assuring that, e.g., for every point in the domain all necessary neighbors are contained in the domain as well. Furthermore, as in the smooth case, we assume all appearing discrete nets to be regular unless stated otherwise.

For the purpose of introducing a novel discrete orthogonality condition, instead of using single lattices as our discrete domains, we consider pairs of dual lattices. For the square lattice $\mathbb{Z}^M$ we call $\left(\mathbb{Z} + \frac{1}{2}\right)^M$ its *dual square lattice* (see Figure 2, *left*), and say that any two edges

$$[\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}_i] \subset \mathbb{Z}^M, \qquad [\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}, \boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma} + \boldsymbol{e}_j] \subset \left(\mathbb{Z} + \tfrac{1}{2}\right)^M$$

are *dual edges*, where $\boldsymbol{n} \in \mathbb{Z}^M$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^M$ with $\sigma_i = 1$ and $\sigma_j = -1$. Furthermore, for a point $\boldsymbol{n} \in \mathbb{Z}^M$, we call the 2^M points $\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma} \in \left(\mathbb{Z} + \frac{1}{2}\right)^M$, $\boldsymbol{\sigma} \in \{\pm 1\}^M$, its *adjacent points from the dual lattice.*

Definition 2.2.

(i) A map

$$\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \tfrac{1}{2}\right)^M \to \mathbb{R}^N$$

is called a *pair of dual discrete nets.*

(ii) A pair of dual discrete nets is called *regular* if the two discrete subnets $\boldsymbol{x}\big|_{\mathbb{Z}^M}$ and $\boldsymbol{x}\big|_{\left(\mathbb{Z}+\frac{1}{2}\right)^M}$ are regular.

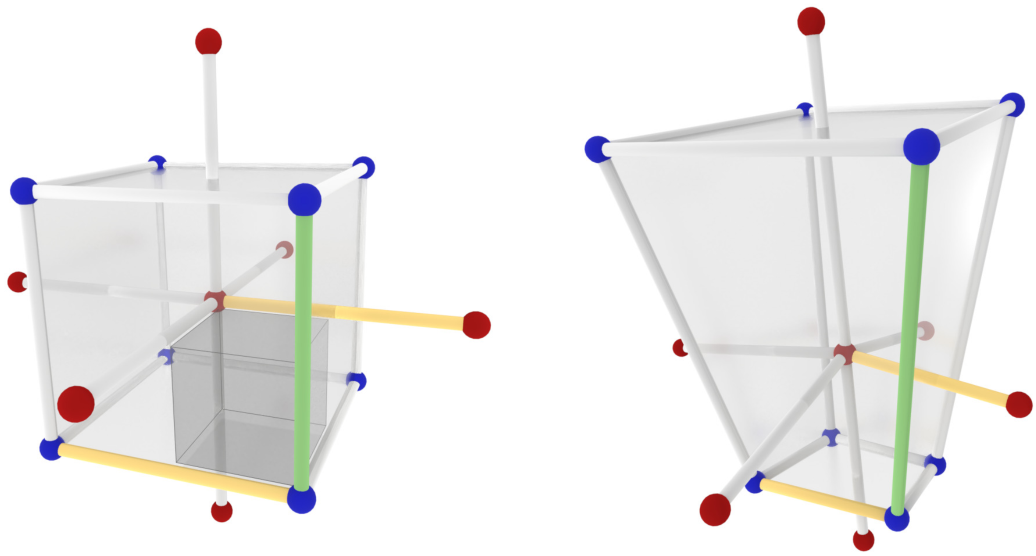

Figure 2. *Left:* Elementary cube of the square lattice $\mathbb{Z}^3$ and its dual edges from $\left(\mathbb{Z}+\frac{1}{2}\right)^3$. *Right:* Its image in $\mathbb{R}^3$ such that each pair of dual edges is orthogonal, e.g., the green and its dual yellow edge are orthogonal. The two marked yellow edges contribute to a discrete Lamé coefficient, combinatorially located at the center of the small gray cube.

For the following, we consider pairs of dual discrete nets (not just each of their two discrete subnets) as discrete analogs of smooth nets, and introduce the following discrete orthogonality condition (see Figure 2).

Definition 2.3.

(i) A pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z}+\frac{1}{2}\right)^M \to \mathbb{R}^N$ is called *orthogonal* if every pair of dual edges is orthogonal in $\mathbb{R}^N$, i.e.,

$$\langle \Delta_i \boldsymbol{x}(\boldsymbol{n}), \Delta_j \boldsymbol{x}(\boldsymbol{n}^*)\rangle = 0, \tag{2}$$

for all distinct $i, j = 1, \ldots, M$ and $\boldsymbol{n} \in \mathbb{Z}^M$, $\boldsymbol{n}^* = \boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma} \in \left(\mathbb{Z}+\frac{1}{2}\right)^M$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^M$ with $\sigma_i = 1$ and $\sigma_j = -1$.

(ii) For a pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z}+\frac{1}{2}\right)^M \to \mathbb{R}^N$ the *discrete (squared) Lamé coefficients*

$$H_i^2 : \left(\mathbb{Z}+\tfrac{1}{4}\right)^M \to \mathbb{R}, \qquad i = 1, \ldots, M$$

are defined by

$$H_i^2(\boldsymbol{n} + \tfrac{1}{4}\boldsymbol{\sigma}) = \begin{cases} \left\langle \Delta_i \boldsymbol{x}(\boldsymbol{n}), \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})\right\rangle, & \sigma_i = 1 \\ \left\langle \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n}), \Delta_i \boldsymbol{x}(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})\right\rangle, & \sigma_i = -1 \end{cases}$$

for all $\boldsymbol{n} \in \mathbb{Z}^M$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_M) \in \{\pm 1\}^M$.

Remark 2.2. The discrete orthogonality condition is invariant under similarity transformations. Furthermore, it is invariant under individual translation of each of its two discrete subnets in space.

The standard discretization of conjugate nets is given by discrete nets with planar quadrilaterals.

Definition 2.4. A discrete net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$ is called *conjugate*, or a *Q-net*, if for all $i, j = 1, \ldots, M,\ i \neq j$ the three vectors

$$\Delta_i \Delta_j \boldsymbol{x}, \quad \Delta_i \boldsymbol{x}, \quad \Delta_j \boldsymbol{x}$$

are linearly dependent, or equivalently, if all elementary quadrilaterals

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j))$$

are coplanar (Exercise).

With this, we obtain the following discrete version of Theorem 1.1 ("discrete Dupin's theorem").

Theorem 2.1. *Let $N \geqslant 3$ and $\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z}+\frac{1}{2}\right)^N \to \mathbb{R}^N$ be an orthogonal pair of dual discrete nets. Then its two discrete subnets $\boldsymbol{x}\big|_{\mathbb{Z}^N}$ and $\boldsymbol{x}\big|_{\left(\mathbb{Z}+\frac{1}{2}\right)^N}$ are discrete conjugate nets.*

Proof. The edge vectors of an elementary quadrilateral

$$(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j))$$

lie in the orthogonal complement of the $N-2$ linearly independent vectors

$$\bar{\Delta}_k \boldsymbol{x}(\boldsymbol{n}+\tfrac{1}{2}\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} = (1, \ldots, 1), \qquad k \neq i, j,$$

which is of dimension 2. □

Remark 2.3. For a discrete conjugate net $\boldsymbol{x} : \mathbb{Z}^M \to \mathbb{R}^N$, $M \leqslant N$, there exists a second conjugate net $\boldsymbol{x}^* : \left(\mathbb{Z}+\frac{1}{2}\right)^M \to \mathbb{R}^N$ such that $\boldsymbol{x}$ and $\boldsymbol{x}^*$ together form an orthogonal pair of dual discrete nets. Thus, from the point of view of a single discrete conjugate net, the discrete orthogonality is not a constraint. Only if we consider pairs of dual discrete nets as discretizations of *one* smooth net does the discrete orthogonality become an actual further constraint.

2.1 Discrete Möbius invariance

The discrete orthogonality constraint is not invariant under mapping each point of an orthogonal pair of dual discrete nets by a Möbius transformation. Nevertheless, one can replace the points of the pair of nets by orthogonal spheres to obtain a Möbius invariant description.

Definition 2.5. Let $\mathcal{S}$ be the space of (hyper)spheres in $\mathbb{R}^N$. We call a map $S : \mathbb{Z}^M \cup \left(\mathbb{Z}+\frac{1}{2}\right)^M \to \mathcal{S}$ an *orthogonal pair of sphere congruences* if each two adjacent spheres from the dual lattices $S(\boldsymbol{n})$ and $S(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\sigma})$, $\boldsymbol{n} \in \mathbb{Z}^M, \boldsymbol{\sigma} \in \{\pm 1\}^M$, are orthogonal (see Figure 3).

Orthogonal pairs of sphere congruences are Möbius invariant. Furthermore, given a pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z}+\frac{1}{2}\right)^M \to \mathbb{R}^N$ we can construct orthogonal spheres with centers at the points of $\boldsymbol{x}$: Choosing the radius for one sphere at $\boldsymbol{n} \in \mathbb{Z}^M$, the radii of all spheres at adjacent vertices $\boldsymbol{n}^* \in \left(\mathbb{Z}+\frac{1}{2}\right)^M$ of the dual lattice are uniquely determined by the orthogonality condition. Can this be propagated throughout the whole pair of dual lattices $\mathbb{Z}^M \cup \left(\mathbb{Z}+\frac{1}{2}\right)^M$ without contradiction?

Lemma 2.2. *Two spheres in $\mathbb{R}^N$ with centers $\boldsymbol{x}$, $\boldsymbol{x}^*$ and radii r, r^*, respectively, are orthogonal if and only if*

$$\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle = \rho + \rho^*,$$

where

$$\rho = \frac{1}{2}\left(|\boldsymbol{x}|^2 - r^2\right), \qquad \rho^* = \frac{1}{2}\left(|\boldsymbol{x}^*|^2 - (r^*)^2\right).$$

Proof. The orthogonality condition of the two spheres is equivalent to

$$|\boldsymbol{x} - \boldsymbol{x}^*|^2 = r^2 + (r^*)^2 \;\Leftrightarrow\; 2\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle = |\boldsymbol{x}|^2 - r^2 + |\boldsymbol{x}^*|^2 - (r^*)^2.$$

□

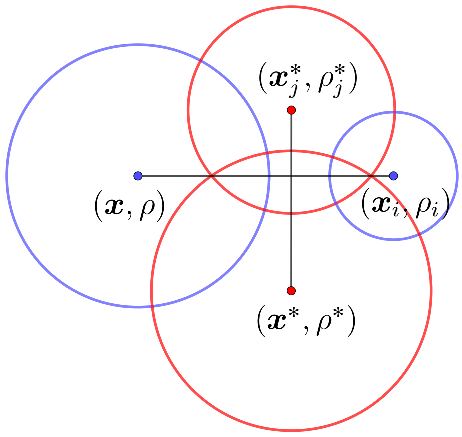

Figure 3. Two dual edges from an orthogonal pair of sphere congruences.

Proposition 2.3. *Let $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathbb{R}^N$ be a pair of dual discrete nets. Then there exists a one-parameter family of orthogonal pairs of sphere congruences with centers in the points of $\boldsymbol{x}$ if and only if the pair of discrete nets $\boldsymbol{x}$ is orthogonal.*

Moreover, let $S : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathcal{S}$ be an orthogonal pair of sphere congruences. Then the pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z} + \frac{1}{2}\right)^M \to \mathbb{R}^N$ given by the centers of S is orthogonal.

Proof. Consider a pair of dual edges of the net $\boldsymbol{x}$, and denote the involved vertices such that $\Delta_i \boldsymbol{x}(\boldsymbol{n}) = \boldsymbol{x}_i - \boldsymbol{x}$ and $\Delta_j \boldsymbol{x}(\boldsymbol{n}^*) = \boldsymbol{x}_j^* - \boldsymbol{x}^*$ (see Figure 3). Assume that the radius r at $\boldsymbol{x}$ is given by $\rho = \frac{1}{2}\left(|\boldsymbol{x}|^2 - r^2\right)$. Then the two radii at $\boldsymbol{x}^*$ and $\boldsymbol{x}_j^*$ are given by

$$\rho^* = \langle \boldsymbol{x}, \boldsymbol{x}^* \rangle - \rho, \qquad \rho_j^* = \left\langle \boldsymbol{x}, \boldsymbol{x}_j^* \right\rangle - \rho.$$

Now the radius at $\boldsymbol{x}_i$ may be obtained in two ways

$$\begin{aligned} \rho_i &= \langle \boldsymbol{x}_i, \boldsymbol{x}^* \rangle - \rho^* = \langle \boldsymbol{x}_i, \boldsymbol{x}^* \rangle - \langle \boldsymbol{x}, \boldsymbol{x}^* \rangle + \rho, \\ \tilde{\rho}_i &= \left\langle \boldsymbol{x}_i, \boldsymbol{x}_j^* \right\rangle - \rho_j^* = \left\langle \boldsymbol{x}_i, \boldsymbol{x}_j^* \right\rangle - \left\langle \boldsymbol{x}, \boldsymbol{x}_j^* \right\rangle + \rho. \end{aligned}$$

Thus,

$$\rho_i = \tilde{\rho}_i \;\Leftrightarrow\; \left\langle \boldsymbol{x}_i - \boldsymbol{x}, \boldsymbol{x}_j^* - \boldsymbol{x}^* \right\rangle = 0,$$

which is the orthogonality of the two dual edges. □

Now an orthogonal pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^M \cup \left(\mathbb{Z}^M\right)^* \to \mathbb{R}^N$ may be transformed in the following way:

- Choose an orthogonal pair of sphere congruences $S : \mathbb{Z}^M \cup \left(\mathbb{Z}^M\right)^* \to \mathcal{S}$ with centers in $\boldsymbol{x}$.
- Transform S under a Möbius transformation to obtain $\tilde{S}$.
- Take the centers $\tilde{\boldsymbol{x}}$ of the transformed pair of sphere congruences $\tilde{S}$.

3 Curvature line parametrized surfaces

Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a smooth regular parametrization of a surface in $\mathbb{R}^3$. We denote its *unit normal field* by

$$\boldsymbol{\nu}(s_1, s_2) = \frac{\partial_1 \boldsymbol{x} \times \partial_2 \boldsymbol{x}}{|\partial_1 \boldsymbol{x} \times \partial_2 \boldsymbol{x}|}, \qquad (s_1, s_2) \in U.$$

The metric on $\boldsymbol{x}$ is described by the *first fundamental form*

$$\mathrm{I}(v, w) = \langle \mathrm{d}\boldsymbol{x}(v), \mathrm{d}\boldsymbol{x}(w) \rangle = v^\mathsf{T} \left(\mathrm{d}\boldsymbol{x}^\mathsf{T} \mathrm{d}\boldsymbol{x}\right) w = v^\mathsf{T} \left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right) w$$

for all $v, w \in \mathbb{R}^2$, where

$$\begin{aligned} E &= \langle \partial_1 \boldsymbol{x}, \partial_1 \boldsymbol{x} \rangle, \\ F &= \langle \partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x} \rangle, \\ G &= \langle \partial_2 \boldsymbol{x}, \partial_2 \boldsymbol{x} \rangle. \end{aligned}$$

Definition 3.1. Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a parametrization.

(i) $\boldsymbol{x}$ is called *orthogonal* if $F = 0$, i.e., the first fundamental form is diagonal.

(ii) $\boldsymbol{x}$ is called *conformal* if $\boldsymbol{x}$ is orthogonal and $E = G$.

Remark 3.1. For an orthogonally parametrized surface, i.e., $F = 0$, its Lamé coefficients are given by $H_1^2 = E$ and $H_2^2 = G$.

The "shape" of $\boldsymbol{x}$ is described by the *second fundamental form*

$$\begin{aligned} \mathrm{II}(v, w) &= -\langle \mathrm{d}\boldsymbol{x}(v), \mathrm{d}\boldsymbol{\nu}(w) \rangle = -v^\mathsf{T} \left(\mathrm{d}\boldsymbol{x}^\mathsf{T} \mathrm{d}\boldsymbol{\nu}\right) w \\ &= -\langle \mathrm{d}\boldsymbol{\nu}(v), \mathrm{d}\boldsymbol{x}(w) \rangle = -v^\mathsf{T} \left(\mathrm{d}\boldsymbol{\nu}^\mathsf{T} \mathrm{d}\boldsymbol{x}\right) w = v^\mathsf{T} \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right) w \end{aligned}$$

for all $v, w \in \mathbb{R}^2$ where

$$\begin{aligned} e &= \left\langle \boldsymbol{\nu}, \partial_1^2 \boldsymbol{x} \right\rangle = -\langle \partial_1 \boldsymbol{\nu}, \partial_1 \boldsymbol{x} \rangle, \\ f &= \langle \boldsymbol{\nu}, \partial_1 \partial_2 \boldsymbol{x} \rangle = -\langle \partial_1 \boldsymbol{\nu}, \partial_2 \boldsymbol{x} \rangle = -\langle \partial_2 \boldsymbol{\nu}, \partial_1 \boldsymbol{x} \rangle, \\ g &= \left\langle \boldsymbol{\nu}, \partial_2^2 \boldsymbol{x} \right\rangle = -\langle \partial_2 \boldsymbol{\nu}, \partial_2 \boldsymbol{x} \rangle. \end{aligned}$$

The *shape operator* is the self-adjoint linear map[1]

$$S = (\mathrm{d}\boldsymbol{x})^{-1} \mathrm{d}\boldsymbol{\nu} : \mathbb{R}^2 \to \mathbb{R}^2.$$

It relates the first and second fundamental form in the following way

$$\mathrm{II}(v, w) = \mathrm{I}(v, Sw) = \mathrm{I}(Sv, w).$$

Definition 3.2. A parametrization $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is called *conjugate* if

$$\det(\partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x}, \partial_1 \partial_2 \boldsymbol{x}) = 0.$$

Proposition 3.1. *A parametrization $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is conjugate if $f = 0$, i.e., the second fundamental form is diagonal.*

[1] It describes the derivative of the normal field $\boldsymbol{\nu}$ in the domain (or "in coordinates").

Proof. In $\mathbb{R}^3$, the condition $\langle \boldsymbol{\nu}, \partial_1\partial_2 \boldsymbol{x}\rangle = 0$ is equivalent to $\partial_1\partial_2\boldsymbol{x} \in \operatorname{span}\{\partial_1\boldsymbol{x}, \partial_2\boldsymbol{x}\}$. $\square$

The *normal curvature* at a point $(s_1, s_2) \in U$ in direction $v \in U$ is given by

$$\kappa(v) = \frac{\mathrm{II}(v,v)}{\mathrm{I}(v,v)}$$

Thus, there exists an ortho-normal basis $e_1, e_2 \in \mathbb{R}^2$ of eigenvectors of S:

$$\mathrm{I}(e_1,e_1) = \mathrm{I}(e_2,e_2) = 1, \qquad \mathrm{I}(e_1,e_2) = 0,$$

and

$$Se_1 = \kappa_1 e_1, \qquad Se_2 = \kappa_2 e_2,$$

with some $\kappa_1, \kappa_2 \in \mathbb{R}$. The two directions $e_1, e_2 \in \mathbb{R}^2$, or $\mathrm{d}\boldsymbol{x}(e_1), \mathrm{d}\boldsymbol{x}(e_2) \in \mathbb{R}^3$ are called the *principal directions* of $\boldsymbol{x}$ at (s_1, s_2). The normal curvatures in the principal directions are called the *principal curvatures* of $\boldsymbol{x}$ at (s_1, s_2), and are given by

$$\kappa(e_1) = \frac{\mathrm{II}(e_1,e_1)}{\mathrm{I}(e_1,e_1)} = \kappa_1, \qquad \kappa(e_2) = \frac{\mathrm{II}(e_2,e_2)}{\mathrm{I}(e_2,e_2)} = \kappa_2.$$

For an arbitrary direction $v_\theta = \cos\theta e_1 + \sin\theta e_2$ the normal curvature is given by

$$\kappa(v_\theta) = \frac{\mathrm{II}(v_\theta,v_\theta)}{\mathrm{I}(v_\theta,v_\theta)} = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta.$$

A point at which the principal curvatures coincide $\kappa_1 = \kappa_2$ is called an *umbilic point.* At an umbilic point all normal curvatures coincide. Away from umbilic points the principal curvatures are the unique and distinct extrema of the normal curvatures.

Definition 3.3.

(i) A *curvature line* is curve on a surface along principal directions.

(ii) A parametrization is called a *curvature line parametrization* if its coordinate lines are curvature lines.

Proposition 3.2. *A parametrization $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is a curvature line parametrization if and only if one of the following equivalent conditions is satisfied:*

(i) $\boldsymbol{x}$ is orthogonal and conjugate.

(ii) The first and second fundamental form are diagonal.

(iii) $F = f = 0$.

Remark 3.2.

(i) Locally, and away from umbilic points, every surface in $\mathbb{R}^3$ has a unique curvature line parametrization.

(ii) The property of being a curvature line parametrization is Möbius invariant.

(iii) A parametrized surface is a two-parameter family of points in $\mathbb{R}^3$. Alternatively, it can be described as the envelope of a two-parameter family of (oriented) planes, namely its tangent planes. For a regular non-developable surface these two descriptions are equivalent. Yet the characterization of a curvature line parametrization in terms of its tangent planes is invariant under Laguerre transformations.

Definition 3.4.

(i) A parametrization $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is called *isothermic* if it is a conformal curvature line parametrization.

(ii) A surface is called *isothermic* if it admits an isothermic parametrization.

Exercise 3.1. Show that a surface is isothermic if and only if its (locally unique, see Remark 3.2) curvature line parametrization satisfies

$$\frac{E}{G} = \frac{\alpha_1(s_1)}{\alpha_2(s_2)}$$

with two functions α_1, α_2.

3.1 Triply orthogonal systems and curvature lines

An orthogonal coordinate system $\boldsymbol{x} : \mathbb{R}^3 \supset U \to \mathbb{R}^3$ in $\mathbb{R}^3$ is also called a *triply orthogonal system.*

From Dupin's Theorem 1.1 and Proposition 3.2 we obtain:

Theorem 3.3. *In a triply orthogonal system $\boldsymbol{x} : \mathbb{R}^3 \supset U \to \mathbb{R}^3$ the coordinate surfaces intersect in a curvature line.*

Remark 3.3. More generally, two surfaces in $\mathbb{R}^3$ that intersect orthogonally, intersect each other in a curvature line.

For $i \neq j$ we denote by $\boldsymbol{x}_{ij}$ the family of two-dimensional subnets of $\boldsymbol{x}$ in ij-direction, i.e., its coordinate surfaces. The first fundamental forms of the coordinate surfaces are given by

$$E_{ij} = H_i^2, \qquad F_{ij} = 0, \qquad G_{ij} = H_j^2.$$

Let us assume that $\det(\partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x}, \partial_3 \boldsymbol{x}) > 0$. Then the normal field of $\boldsymbol{x}_{ij}$ is given by

$$\boldsymbol{\nu}_{ij} = \frac{\partial_k \boldsymbol{x}}{\|\partial_k \boldsymbol{x}\|} = \frac{\partial_k \boldsymbol{x}}{H_k}.$$

with (ijk) cyclic permutation of (123). From this, we obtain

$$\begin{aligned}
\langle \partial_i \boldsymbol{\nu}_{ij}, \partial_i \boldsymbol{x}_{ij} \rangle &= -\left\langle \boldsymbol{\nu}_{ij}, \partial_i^2 \boldsymbol{x}_{ij} \right\rangle = -\frac{1}{H_k} \left\langle \partial_k \boldsymbol{x}, \partial_i^2 \boldsymbol{x} \right\rangle \\
&= \frac{1}{H_k} \langle \partial_i \partial_k \boldsymbol{x}, \partial_i \boldsymbol{x} \rangle = \frac{1}{2H_k} \partial_k \langle \partial_i \boldsymbol{x}, \partial_i \boldsymbol{x} \rangle \\
&= \frac{\partial_k (H_i^2)}{2H_k} = \frac{H_i \partial_k H_i}{H_k}.
\end{aligned}$$

Thus, the second fundamental forms of the coordinate surfaces are given by

$$e_{ij} = -\frac{H_i \partial_k H_i}{H_k}, \qquad f_{ij} = 0, \qquad g_{ij} = -\frac{H_j \partial_k H_j}{H_k}.$$

3.2 Focal nets

Another characterization of a curvature line parametrization is given in the following proposition:

Proposition 3.4. *Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a conjugate net. Then $\boldsymbol{x}$ is orthogonal, i.e., a curvature line parametrization, if and only if*

$$\det(\boldsymbol{\nu}, \partial_1\boldsymbol{\nu}, \partial_1\boldsymbol{x}) = 0 \quad \textit{and} \quad \det(\boldsymbol{\nu}, \partial_2\boldsymbol{\nu}, \partial_2\boldsymbol{x}) = 0. \tag{3}$$

In particular, in a curvature line parametrization the tangent vectors $\partial_i\boldsymbol{x}$ and $\partial_i\boldsymbol{\nu}$ are linearly dependent:

$$\begin{aligned} \partial_1\boldsymbol{\nu} &= -\kappa_1\partial_1\boldsymbol{x} \\ \partial_2\boldsymbol{\nu} &= -\kappa_2\partial_2\boldsymbol{x}. \end{aligned}$$

The normal direction $\boldsymbol{\nu}$ defines a line

$$\lambda \mapsto \boldsymbol{x}(s_1, s_2) + \lambda\boldsymbol{\nu}(s_1, s_2), \qquad \lambda \in \mathbb{R}$$

at every point $(s_1, s_2) \in U$, together constituting the *normal congruence* of the net $\boldsymbol{x}$. Condition (3) means that the two families of ruled surfaces contained in the normal congruence along the coordinate lines of $\boldsymbol{x}$

$$(s_i, \lambda) \mapsto \boldsymbol{x}(s_1, s_2) + \lambda\boldsymbol{\nu}(s_1, s_2), \qquad i = 1, 2, \tag{4}$$

are developable. Or more intuitively, that infinitesimally close normal lines along the principal directions intersect. Along a curvature line the points of intersection are given by the centers of the osculating circles, or curvature spheres, which have radii $\frac{1}{\kappa_i}$, $i = 1, 2$. Together they form the curve of striction of the developable surface (4).

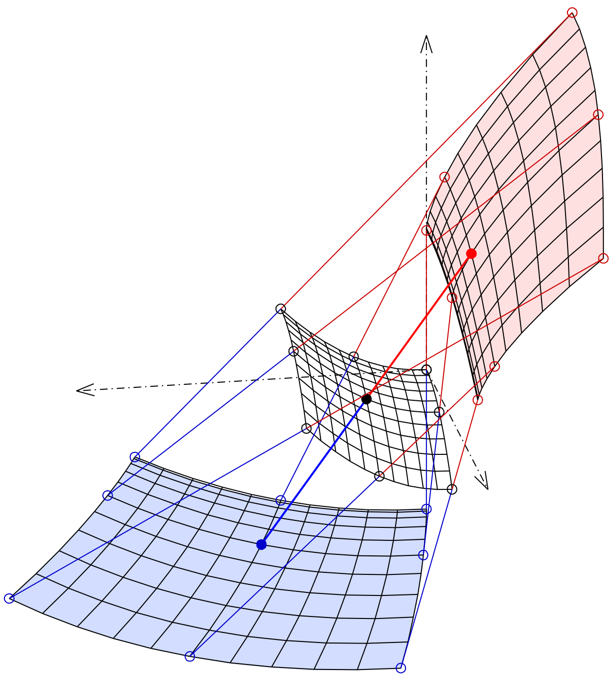

Figure 4. A curvature line parametrized surface (white) and its two focal nets (red and blue). [Image by Ag2gaeh, CC BY-SA 4.0]

Definition 3.5. Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization. Then its two *focal nets* are given by (see Figure 4),

$$\boldsymbol{f}_i : U \to \mathbb{R}^3, \qquad (s_1, s_2) \mapsto \boldsymbol{x}(s_1, s_2) + \frac{1}{\kappa_i(s_1, s_2)}\boldsymbol{\nu}(s_1, s_2), \qquad i = 1, 2.$$

Proposition 3.5. *The focal net $\boldsymbol{f}_i$ is regular at each point $(s_1, s_2) \in U$ with $\partial_i \kappa_i(s_1, s_2) \neq 0$, and the normal lines of $\boldsymbol{x}$ are the tangent lines of $\boldsymbol{f}_i$ in direction i. Furthermore,*

$$\partial_i \kappa_i(s_1, s_2) = 0 \;\Leftrightarrow\; \partial_i \boldsymbol{f}_i(s_1, s_2) = 0.$$

Proposition 3.6. *The two focal nets $\boldsymbol{f}_i$ are semi-geodesic conjugate nets.*

Remark 3.4. The envelope of a one-parameter family of spheres in $\mathbb{R}^3$ is called a *channel surface*. The curvature lines in one direction of a channel surface are circles, and thus one of its focal nets degenerates to a curve, i.e., $\partial_i \boldsymbol{f}_i = 0$ for one $i = 1, 2$ (cf. Remark 3.5). In fact, this property characterizes channel surfaces. A *Dupin cyclide* is a channel surface in both directions, i.e., the envelope of two distinct one-parameter families of spheres, and therefore characterized by the condition that both of its focal nets degenerate to curves.

3.3 Parallel nets

A *parallel surface* is a surface of constant offset in normal direction to a given surface. A net $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ can be extended to a three-dimensional net by a family of parallel nets, given by

$$\tilde{\boldsymbol{x}} : U \times I \to \mathbb{R}^3, \qquad (s_1, s_2, s_3) \mapsto \boldsymbol{x}(s_1, s_2) + \rho(s_3)\boldsymbol{\nu}(s_1, s_2), \tag{5}$$

with some smooth function $\rho : I \to \mathbb{R}$ on an open interval $I \subset \mathbb{R}$.

Proposition 3.7. *Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization. Away from the focal points ($\rho = \frac{1}{\kappa_i}$) and points with $\rho' = 0$ the three-dimensional net of parallel surfaces $\tilde{\boldsymbol{x}}$ is regular.*

By Theorem 1.1 a two-dimensional net can be a subnet of a three-dimensional orthogonal net, i.e., a triply orthogonal system, only if it is a curvature line parametrization. Yet every curvature line parametrization can be extended to a triply orthogonal system by its parallel nets.

Proposition 3.8. *Let $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ be a curvature line parametrization. Then the three-dimensional net of parallel surfaces $\tilde{\boldsymbol{x}}$ given by* (5) *is orthogonal with the third Lamé coefficient given by $H_3^2 = (\rho')^2$, which only depends on s_3.*

Remark 3.5. In particular, by Dupin's Theorem 3.3, all parallel surfaces in (5) are curvature line parametrizations. Furthermore, Proposition 3.8 implies that, generally, curvature line parametrizations are Möbius invariant (cf. Remark 3.2). Indeed, by Proposition 3.8, a curvature line parametrization $\boldsymbol{x} : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ can be extended to a triply orthogonal systems $\tilde{\boldsymbol{x}}$. Application of a Möbius transformation maps $\tilde{\boldsymbol{x}}$ to another triply orthogonal system, and thus, by Theorem 3.3, it maps $\boldsymbol{x}$ to a curvature line parametrization.

4 Discrete curvature line parametrized surfaces

Two well-established discretizations of curvature line parametrizations are given by *circular nets* and *conical nets*.

Definition 4.1. Let $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a discrete conjugate net.

(i) The net $\boldsymbol{x}$ is called a *circular net* if all its elementary quadrilaterals are circular, i.e., each four points $(\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_j))$ lie on a circle.

(ii) The net $\boldsymbol{x}$ is called a *conical net* if all four planes corresponding to any elementary quadrilateral containing a common vertex touch a common cone.

Remark 4.1.

(i) The notion of circular nets is invariant under Möbius transformations.

(ii) Conical nets are more naturally described as maps from the dual lattice into the set of (oriented) planes of $\mathbb{R}^3$. Thus, they correspond to the description of a net in terms of its tangent planes (cf. Remark 3.2 (iii)). The notion of conical nets is invariant under Laguerre transformations.

Circular nets and conical nets are intimately related.

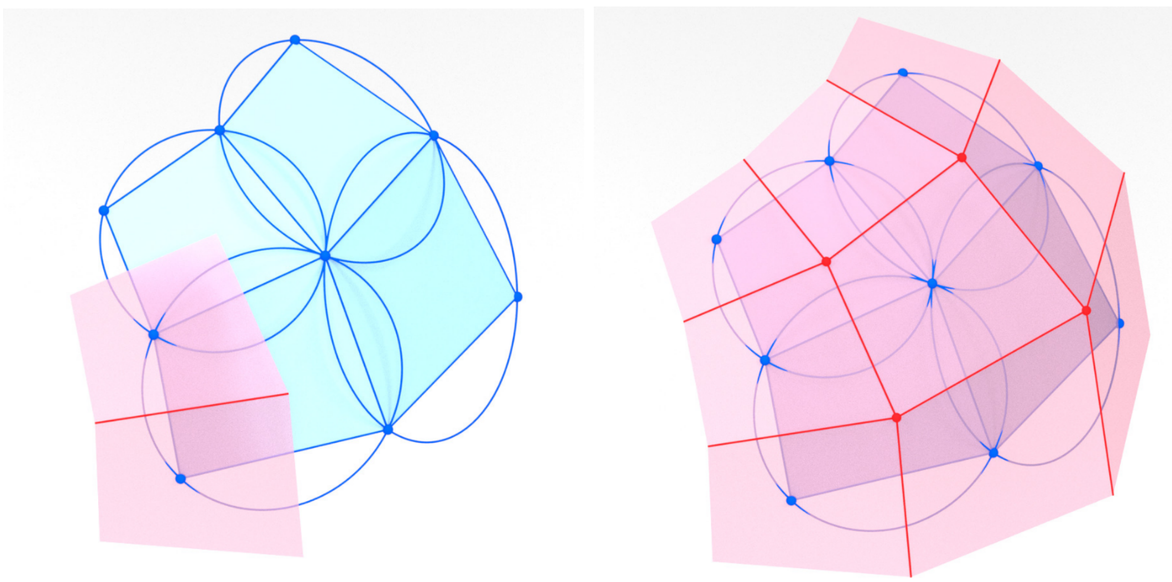

Figure 5. Generating a conical net from a circular net.

Given a circular net there exists a canonical three-parameter family of corresponding conical nets:

- Let $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ be a circular net.
- Associate to each edge $[\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_i]$ of $\mathbb{Z}^2$ the length bisecting plane of the segment $[\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n}+\boldsymbol{e}_i)]$.
- Choose a plane at some $\boldsymbol{n} \in \mathbb{Z}^2$ and reflect it in all bisecting planes.

This process is well-defined on $\mathbb{Z}^2$ in the sense that it closes along every cycle. Every four planes associated to an elementary quadrilateral of $\mathbb{Z}^2$ intersect in a point, constituting an *associated conical net* $\boldsymbol{x}^* : (\mathbb{Z} + \frac{1}{2})^2 \to \mathbb{R}^3$ on the dual lattice.

Vice versa, given a conical net there exists a canonical three-parameter family of corresponding circular nets:

- Let $\boldsymbol{x} : \left(\mathbb{Z}^2 + \frac{1}{2}\right)^2 \to \mathbb{R}^3$ be a conical net.
- Associate to each edge $[\boldsymbol{n}^*, \boldsymbol{n}^* + \boldsymbol{e}_i]$ of $\left(\mathbb{Z} + \frac{1}{2}\right)^2$ the angle bisecting plane of the two adjacent face planes.
- Choose a point at some $\boldsymbol{n} \in \mathbb{Z}^2$ and reflect it in all bisecting planes.

This process is well-defined on $\mathbb{Z}^2$ and constitutes an *associated circular net* $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ on the dual lattice.

We call two nets $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ and $\boldsymbol{x}^* : (\mathbb{Z} + \frac{1}{2})^2 \to \mathbb{R}^3$ obtained by either of the previously described procedures a *pair of associated circular and conical nets.*

Proposition 4.1. *A pair of associated circular and conical nets constitutes an orthogonal pair of dual discrete nets (in the sense of Definition 2.3).*

Proof. Consider one of the bisecting planes Π. A plane and its reflection in Π intersect in Π. On the other hand, the line through a point and its reflection in Π is orthogonal to Π. $\square$

Thus, orthogonal pairs of dual discrete conjugate nets are generalizations of pairs of associated circular and conical nets, and we view them as *discrete curvature line parametrizations.*

Remark 4.2. While circular nets are invariant under Möbius transformations and conical nets are invariant under Laguerre transformations, the associated pairs of such nets are invariant under the intersection of these transformation groups, i.e., similarity transformations (on the other hand cf. Section 2.1).

4.1 Discrete focal nets

For a pair of dual discrete conjugate nets $\boldsymbol{x} : \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2 \to \mathbb{R}^3$ we associate to every vertex $\boldsymbol{n} \in \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2$ the unit normal vector of the corresponding dual face plane, i.e.,

$$\boldsymbol{\nu}(\boldsymbol{n}) = \frac{\Delta_1 \boldsymbol{x}(\boldsymbol{n}^*) \times \Delta_2 \boldsymbol{x}(\boldsymbol{n}^*)}{|\Delta_1 \boldsymbol{x}(\boldsymbol{n}^*) \times \Delta_2 \boldsymbol{x}(\boldsymbol{n}^*)|},$$

where $\boldsymbol{n}^* = \boldsymbol{n} - \left(\frac{1}{2}, \frac{1}{2}\right)$. The corresponding normal lines

$$\ell(\boldsymbol{n}) : \lambda \mapsto \boldsymbol{x}(\boldsymbol{n}) + \lambda \boldsymbol{\nu}(\boldsymbol{n}), \qquad \lambda \in \mathbb{R}.$$

together constitute the *discrete normal congruence* of the pair of discrete nets $\boldsymbol{x}$, for which we immediately obtain a discrete version of Proposition 3.4.

Proposition 4.2. *Let* $\boldsymbol{x} : \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2 \to \mathbb{R}^3$ *be a pair of dual discrete conjugate nets. Then* $\boldsymbol{x}$ *is orthogonal, i.e., a discrete curvature line parametrization, if and only if one of the following two equivalent conditions is satisfied:*

(i) $\det(\boldsymbol{\nu}, \Delta_1\boldsymbol{\nu}, \Delta_1\boldsymbol{x}) = 0$ *and* $\det(\boldsymbol{\nu}, \Delta_2\boldsymbol{\nu}, \Delta_2\boldsymbol{x}) = 0$

(ii) Any two adjacent normals $\ell(\boldsymbol{n})$ *and* $\ell(\boldsymbol{n} + \boldsymbol{e}_i)$, $i = 1, 2$, *intersect (see Figure 6).*

Proof. Let $\boldsymbol{n} \in \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2$ and $\boldsymbol{n}^* = \boldsymbol{n} + \left(\frac{1}{2}, -\frac{1}{2}\right)$ so that $\Delta_1\boldsymbol{x}(\boldsymbol{n})$ and $\Delta_2\boldsymbol{x}(\boldsymbol{n}^*)$ are dual edges and therefore

$$\boldsymbol{\nu}(\boldsymbol{n}), \boldsymbol{\nu}(\boldsymbol{n} + \boldsymbol{e}_1) \perp \Delta_2\boldsymbol{x}(\boldsymbol{n}^*).$$

Thus, under the assumption $\boldsymbol{\nu}(\boldsymbol{n}) \neq \boldsymbol{\nu}(\boldsymbol{n} + \boldsymbol{e}_1)$, we obtain

$$\det\left(\boldsymbol{\nu}, \Delta_1\boldsymbol{\nu}, \Delta_1\boldsymbol{x}\right)(\boldsymbol{n}) = \det\left(\boldsymbol{\nu}(\boldsymbol{n}), \boldsymbol{\nu}(\boldsymbol{n} + \boldsymbol{e}_1), \Delta_1\boldsymbol{x}(\boldsymbol{n})\right) = 0 \;\Leftrightarrow\; \Delta_1\boldsymbol{x}(\boldsymbol{n}) \perp \Delta_2\boldsymbol{x}(\boldsymbol{n}^*).$$

□

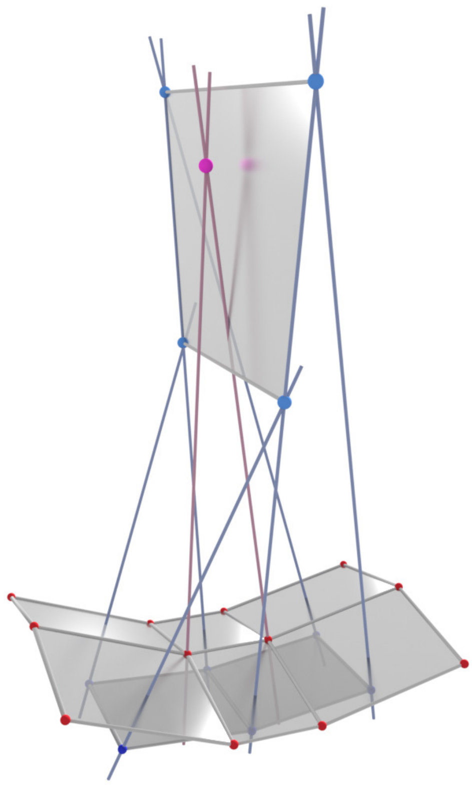

Figure 6. Patch of an orthogonal pair of dual discrete conjugate nets, its normal congruence, and one quadrilateral of one of its two focal nets.

Remark 4.3. Condition (ii) of Proposition 4.2 may be interpreted in the sense that the two families of “discrete ruled surfaces” contained in the discrete normal congruence along the coordinate lines of $\boldsymbol{x}$ are “discrete developable surfaces”.

Definition 4.2. For an orthogonal pair of dual discrete conjugate nets $\boldsymbol{x} : \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2 \to \mathbb{R}^3$ we define their *discrete focal nets* $\boldsymbol{f}_i$, $i = 1, 2$, by the points of intersection of neighboring normal lines (see Figure 6)

$$\boldsymbol{f}_i : \boldsymbol{n} \mapsto \ell(\boldsymbol{n}) \cap \ell(\boldsymbol{n} + \boldsymbol{e}_i). \tag{6}$$

Proposition 4.3. *The two discrete focal nets* (6) *are discrete conjugate nets.*

Proof. The two points $\boldsymbol{f}_i(\boldsymbol{n})$ and $\boldsymbol{f}_i(\boldsymbol{n}+\boldsymbol{e}_i)$ lie on the line $\ell(\boldsymbol{n}+\boldsymbol{e}_i)$, while the two points $\boldsymbol{f}_i(\boldsymbol{n}+\boldsymbol{e}_j)$ and $\boldsymbol{f}_i(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j)$ lie on the line $\ell(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j)$. By Proposition 4.2 these two lines intersect. □

Remark 4.4. Comparing with Remark 3.4, we obtain natural definitions for *discrete channel surfaces* and *discrete Dupin cyclides*.

4.2 Discrete parallel nets

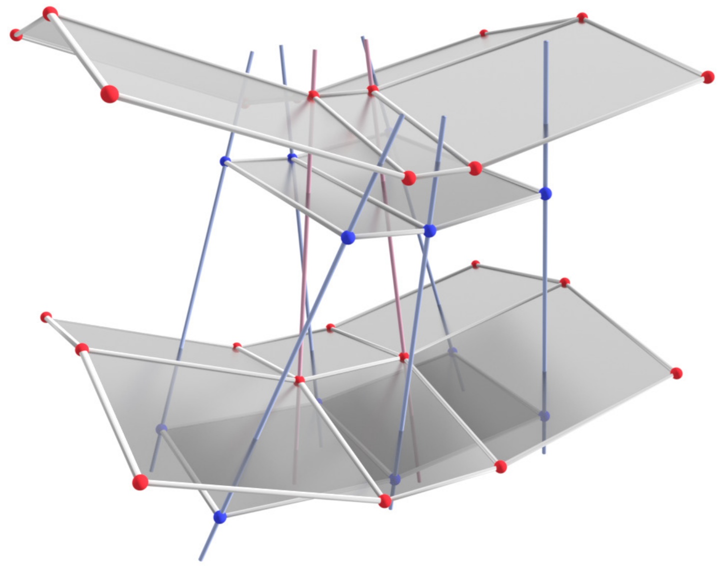

Figure 7. Patch of an orthogonal pair of dual discrete conjugate nets, and one layer of a discrete parallel pair of nets.

For an orthogonal pair of dual discrete conjugate nets $\boldsymbol{x} : \mathbb{Z}^2 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^2 \to \mathbb{R}^3$, a one-parameter family of *discrete parallel surfaces* is defined by (see Figure 7)

$$\tilde{\boldsymbol{x}} : \mathbb{Z}^3 \cup \left(\mathbb{Z}+\tfrac{1}{2}\right)^3 \to \mathbb{R}^3, \qquad (n_1,n_2,n_3) \mapsto \boldsymbol{x}(n_1,n_2) + \rho(n_1,n_2,n_3)\boldsymbol{\nu}(n_1,n_2), \tag{7}$$

where $\rho : \mathbb{Z}^2 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^3 \to \mathbb{R}$ is chosen such that for $i = 1,2$ the edges $\Delta_i\boldsymbol{x}(n_1,n_2,n_3)$ are parallel for all values of n_3. This is always possible due to the fact that neighboring normal lines of $\boldsymbol{x}$ intersect. Thus, the function ρ may only be chosen at one point for each layer $n_3 = const.$, and each two coordinate surfaces $\tilde{\boldsymbol{x}}(n_1,n_2,n_3 = const.)$ are discrete conjugate nets with parallel faces.

Similar to the smooth case, $\boldsymbol{x}$ can be extended to a discrete triply orthogonal system by its parallel surfaces.

Proposition 4.4. *Let $\boldsymbol{x} : \mathbb{Z}^2 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^2 \to \mathbb{R}^3$ be an orthogonal pair of dual discrete conjugate nets. Then the pair of discrete three-dimensional nets of parallel surfaces $\tilde{\boldsymbol{x}}$ given by* (7) *is orthogonal with the third discrete Lamé coefficient H_3^2 only depending on n_3.*

Proof. The orthogonality of two dual edges $\Delta_1\tilde{\boldsymbol{x}}(\boldsymbol{n})$ and $\Delta_2\tilde{\boldsymbol{x}}(\boldsymbol{n}^*)$ follows from parallelity to the corresponding edges of $\boldsymbol{x}$. An edge $\Delta_3\tilde{\boldsymbol{x}}(\boldsymbol{n})$ is always parallel to the discrete normal vector $\boldsymbol{\nu}(n_1, n_2)$, which in turn is orthogonal to any dual edge $\Delta_1\tilde{\boldsymbol{x}}(\boldsymbol{n}^*)$ and $\Delta_2\tilde{\boldsymbol{x}}(\boldsymbol{n}^*)$.

Let $\boldsymbol{n} \in \mathbb{Z}^3 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^3$, $\boldsymbol{\sigma}_1 = (-1, 1, 1)$, $\boldsymbol{\sigma}_2 = (1, 1, 1)$, $\boldsymbol{n}_1^* = \boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}_1$, $\boldsymbol{n}_2^* = \boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}_2$, and consider the two corresponding adjacent values of H_3^2. Then

$$\begin{aligned}
H_3^2(\boldsymbol{n} + \tfrac{1}{4}\boldsymbol{\sigma}_2) - H_3^2(\boldsymbol{n} + \tfrac{1}{4}\boldsymbol{\sigma}_1) &= \left\langle \Delta_3\boldsymbol{x}(\boldsymbol{n}), \bar{\Delta}_3\boldsymbol{x}(\boldsymbol{n}_2^*)\right\rangle - \left\langle \Delta_3\boldsymbol{x}(\boldsymbol{n}), \bar{\Delta}_3\boldsymbol{x}(\boldsymbol{n}_1^*)\right\rangle \\
&= \left\langle \Delta_3(\boldsymbol{n}), \Delta_1\bar{\Delta}_3\boldsymbol{x}(\boldsymbol{n}_1^*)\right\rangle \\
&= \left\langle \Delta_3(\boldsymbol{n}), \bar{\Delta}_3\boldsymbol{x}\Delta_1(\boldsymbol{n}_1^*)\right\rangle \\
&= \langle \Delta_3(\boldsymbol{n}), \Delta_1(\boldsymbol{n}_1^*)\rangle - \langle \Delta_3(\boldsymbol{n}), \Delta_1(\boldsymbol{n}_1^* - \boldsymbol{e}_3)\rangle = 0.
\end{aligned}$$

□

5 Confocal conics

Let $F_1, F_2 \in \mathbb{R}^2$ be two points in the Euclidean plane.

- An *ellipse* with foci F_1, F_2 is a set of points in the plane, such that for every of its points $X \in \mathbb{R}^2$ the sum of distances to F_1 and F_2 is constant:

$$d(X, F_1) + d(X, F_2) = \text{const.} \tag{8}$$

- A *hyperbola* with foci F_1, F_2 is a set of points in the plane, such that for every of its points $X \in \mathbb{R}^2$ the absolute value of the difference of distances to F_1 and F_2 is constant:

$$|d(X, F_1) - d(X, F_2)| = \text{const.}$$

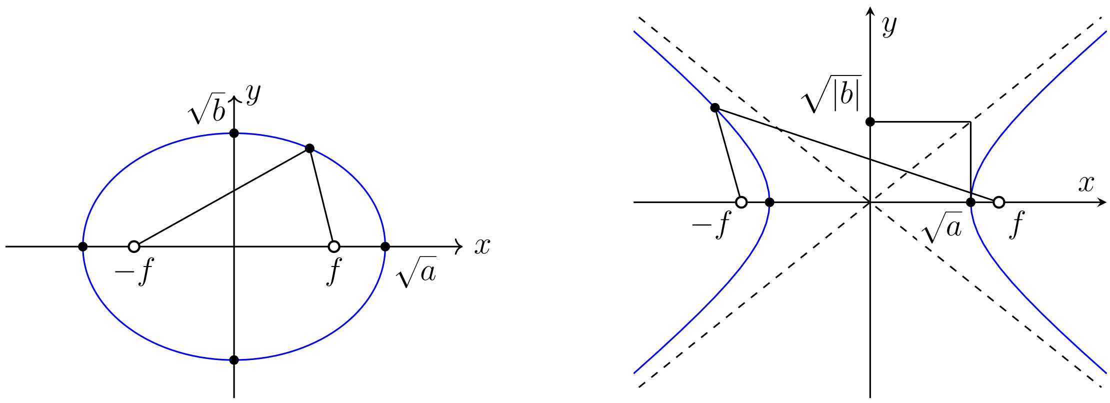

Figure 8. Ellipse, hyperbola, and their foci.

Ellipses and hyperbolas are *conics* and thus can be described by quadratic equations:

Proposition 5.1. *Let $a, b \in \mathbb{R}$, $a > 0$, $b \neq 0$, and $a > b$. Let*

$$\mathcal{C} := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\}$$

and

$$F_1 := (-f, 0), \qquad F_2 := (f, 0), \qquad \textit{where } f := \sqrt{a - b}.$$

- *If $b > 0$, then $\mathcal{C}$ is an ellipse with foci F_1 and F_1.*
- *If $b < 0$, then $\mathcal{C}$ is a hyperbola with foci F_1 and F_1.*

Vice versa, every ellipse or hyperbola can be brought into this form by a Euclidean transformation.

Proof. Exercise. □

Definition 5.1. Two conics (two ellipses, two hyperbolas, or an ellipse and a hyperbola) are called *confocal* if they have the same foci.

Consider an ellipse or hyperbola in normal form

$$\frac{x^2}{a} + \frac{y^2}{b} = 1, \qquad a > b.$$

Then its two foci lie symmetrically on the x-axis, and therefore, any confocal ellipse or hyperbola must necessarily also be in normal form

$$\frac{x^2}{\tilde{a}} + \frac{y^2}{\tilde{b}} = 1, \qquad \tilde{a} > \tilde{b}.$$

In particular this means that confocal conics have common principal axes. Now these two conics in normal form are confocal, if and only if

$$a - b = \tilde{a} - \tilde{b},$$

and we arrive at the following algebraic description of families of confocal conics:

Theorem 5.2. *Let $a > b$. The family of* ***confocal conics*** *with foci*

$$F_1 := (-f, 0), \qquad F_2 := (f, 0), \qquad \text{where } f := \sqrt{a-b}.$$

is given by

$$\mathcal{C}_\lambda = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} = 1 \right\}, \qquad \lambda \in \mathbb{R}.$$

Remark 5.1. Note that the family $\mathcal{C}_\lambda$ (up to a shift of the parameter λ) only depends on the difference $a - b$ (and not independently on a, b).

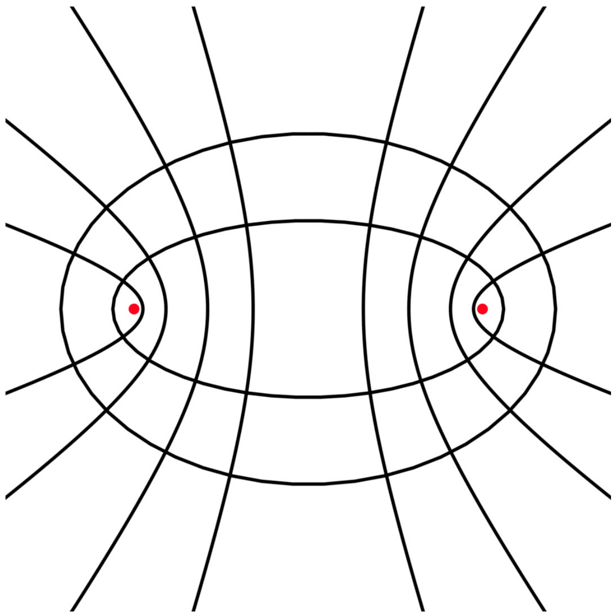

Figure 9. Confocal ellipses and hyperbolas.

The family consists of ellipses and hyperbolas, each of these two subfamilies filling the entire Euclidean plane, respectively.

- $\mathcal{C}_\lambda$ is empty (or "purely imaginary") for $\lambda < -a$.

- $\mathcal{C}_\lambda$ is a hyperbola for $-a < \lambda < -b$.
- $\mathcal{C}_\lambda$ is an ellipse for $\lambda > -b$.

The cases $\lambda = -a, -b$, can be considered as limiting cases:

- $\mathcal{C}_{\lambda \to -a}$ as the line on the y-axis ("degenerate hyperbola").
- $\mathcal{C}_{\lambda \to -b}$ as the line on the x-axis. Or the line segment between the two foci ("degenerate ellipse") if $\lambda \searrow -b$, and the two rays outside the two foci ("degenerate hyperbola") if $\lambda \nearrow -b$.

Note that for $\lambda \nearrow +\infty$ the ellipses become infinitely large.

Theorem 5.3. *Through every point $(x, y) \in \mathbb{R}^2$ not on the coordinate axes $(x \cdot y \neq 0)$, there passes exactly one ellipse and one hyperbola from the confocal family $\mathcal{C}_\lambda$.*

Proof. Given the point (x, y), and clearing the denominators, the confocal conic equation

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} = 1$$

is a quadratic equation in λ. Its two roots u_1, u_2 are real and lie in the intervals

$$-a < u_1 < -b < u_2,$$

which becomes immediately evident from the qualitative behavior of the function

$$\lambda \mapsto \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda}.$$

□

Remark 5.2. The claim remains true for points on the coordinate axes if we include the "degenerate ellipses" and "degenerate hyperbolas".

Theorem 5.4. *A confocal ellipse and hyperbola intersect in exactly 4 points, which lie mirror symmetric with respect to the common principal axes.*

Proof. Let $-a < u_1 < -b < u_2$. Then

$$\begin{aligned} \frac{x^2}{a+u_1} + \frac{y^2}{b+u_1} &= 1, \\ \frac{x^2}{a+u_2} + \frac{y^2}{b+u_2} &= 1 \end{aligned} \tag{9}$$

is an inhomogeneous linear system of two equations in the variables (x^2, y^2). Its solution is given by

$$\begin{aligned} x^2 &= \frac{(a+u_1)(a+u_2)}{a-b}, \\ y^2 &= \frac{(b+u_1)(b+u_2)}{b-a}. \end{aligned} \tag{10}$$

Since the right-hand side of both equations is positive, this yields 4 solutions for (x, y):

$$x = \pm\frac{\sqrt{a+u_1}\sqrt{a+u_2}}{\sqrt{a-b}},$$
$$y = \pm\frac{\sqrt{-(b+u_1)}\sqrt{b+u_2}}{\sqrt{a-b}}.$$

Each solution is contained in one quadrant of $\mathbb{R}^2$, mirror symmetric with respect to the coordinate axes. $\square$

Exercise 5.1. How to obtain the solution to the linear system (9)?

(i) The linear system (9) may be written as

$$A\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \text{with } A := \begin{pmatrix} \frac{1}{a+u_1} & \frac{1}{b+u_1} \\ \frac{1}{a+u_2} & \frac{1}{b+u_2} \end{pmatrix}$$

Then we compute

$$\det A = \tfrac{1}{(a+u_1)(b+u_2)} - \tfrac{1}{(a+u_2)(b+u_1)} = \tfrac{(u_1-u_2)(a-b)}{(a+u_1)(a+u_2)(b+u_1)(b+u_2)},$$

$$A^{-1} = \tfrac{1}{\det A}\begin{pmatrix} \frac{1}{b+u_2} & \frac{-1}{b+u_1} \\ \frac{-1}{a+u_2} & \frac{1}{a+u_1} \end{pmatrix} = \tfrac{1}{(u_1-u_2)(a-b)}\begin{pmatrix} (a+u_1)(a+u_2)(b+u_1) & -(a+u_1)(a+u_2)(b+u_2) \\ -(a+u_1)(b+u_1)(b+u_2) & (a+u_2)(b+u_1)(b+u_2) \end{pmatrix}$$

and thus

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{1}{(u_1-u_2)(a-b)}\begin{pmatrix} (a+u_1)(a+u_2)((b+u_1)-(b+u_2)) \\ (b+u_1)(b+u_2)((a+u_1)-(a+u_1)) \end{pmatrix} = \tfrac{1}{a-b}\begin{pmatrix} (a+u_1)(a+u_2) \\ -(b+u_1)(b+u_2) \end{pmatrix}.$$

(ii) Alternatively, define

$$g(\lambda) := \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1.$$

Then $g(u_1) = g(u_2) = 0$ and $g(\lambda)(a+\lambda)(b+\lambda)$ is a polynomial in λ of degree 2 of which the highest order coefficient is -1. Thus,

$$g(\lambda)(a+\lambda)(b+\lambda) = -(\lambda-u_1)(\lambda-u_2),$$

or equivalently,

$$g(\lambda) = \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} - 1 = -\frac{(\lambda-u_1)(\lambda-u_2)}{(a+\lambda)(b+\lambda)}.$$

Evaluating the residues of g at $-a$ and $-b$, we obtain

$$x^2 = \mathrm{res}_{\lambda=-a} g(\lambda) = \frac{(a+u_1)(a+u_2)}{a-b},$$
$$y^2 = \mathrm{res}_{\lambda=-b} g(\lambda) = \frac{(b+u_1)(b+u_2)}{b-a}.$$

Theorem 5.5. *A confocal ellipse and hyperbola intersect orthogonally.*

Proof. Confocal conics are given by level sets of the function

$$f_\lambda(x, y) = \frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda}.$$

The gradient of f_λ is given by

$$\operatorname{grad} f_\lambda(x, y) = \left(\frac{2x}{a + \lambda}, \frac{2y}{b + \lambda}\right).$$

Now let $-a < u_1 < -b < u_2$, and $(x, y) \in \mathbb{R}^2$ one of the 4 points of intersection of the corresponding hyperbola and ellipse, i.e. satisfying (9). Then

$$\begin{aligned}
\langle \operatorname{grad} f_{u_1}(x, y), \operatorname{grad} f_{u_2}(x, y)\rangle &= \left\langle \left(\frac{2x}{a + u_1}, \frac{2y}{b + u_1}\right), \left(\frac{2x}{a + u_2}, \frac{2y}{b + u_2}\right) \right\rangle \\
&= 2\left(\frac{x^2}{\underbrace{(a + u_1)(a + u_2)}_{\underset{(10)}{=} a-b}} + \frac{y^2}{\underbrace{(b + u_1)(b + u_2)}_{\underset{(10)}{=} b-a}}\right) = 0
\end{aligned}$$

□

5.1 Confocal conics as dual pencils

By embedding $\mathbb{R}^2 \subset \mathbb{RP}^2$, we now look at a description of confocal conics in terms of projective geometry (cf. Appendix A).

Homogenizing the equation for confocal conics

$$\frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} = 1$$

by introducing homogeneous coordinates (x_1, x_2, x_3) with $x = \frac{x_1}{x_3}, y = \frac{x_2}{x_3}$ we obtain

$$\frac{x_1^2}{a + \lambda} + \frac{x_2^2}{b + \lambda} - x_3^2 = 0.$$

Theorem 5.6. *The family of confocal conics is a dual pencil of conics.*

Proof. The Gram matrices of a confocal family of conics is given by

$$Q_\lambda = \begin{pmatrix} \frac{1}{a+\lambda} & & \\ & \frac{1}{b+\lambda} & \\ & & -1 \end{pmatrix},$$

and thus, the family of dual conics is given by

$$Q_\lambda^{-1} = \begin{pmatrix} a+\lambda & & \\ & b+\lambda & \\ & & -1 \end{pmatrix},$$

which and corresponds to the family of equations

$$(a + \lambda)\tilde{x}_1^2 + (b + \lambda)\tilde{x}_2^2 - \tilde{x}_3^2 = 0.$$

□

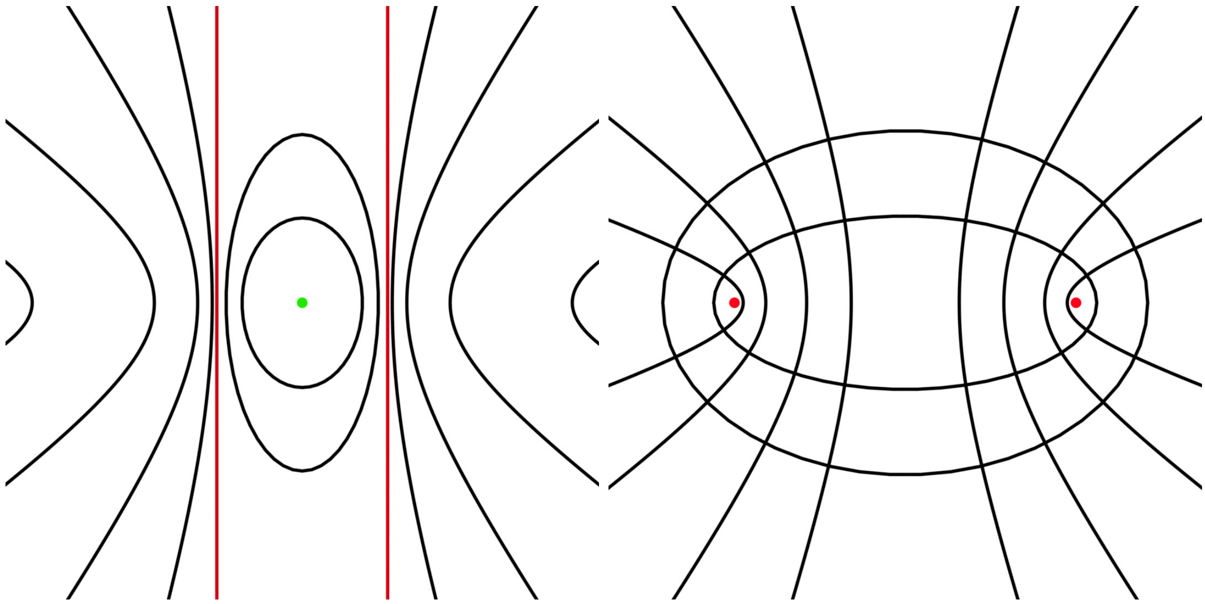

Figure 10. A pencil of conics that dualizes to a confocal family of conics.

We now determine the three degenerate conics in this pencil of conics, and dually, in the confocal family: The three corresponding roots of

$$\det Q_\lambda^{-1} = -(a+\lambda)(b+\lambda) = 0$$

are given by $\lambda = -a, -b, \infty$.

- $\lambda = -a$: The equation of the degenerate conic is given by

  $$(a-b)\tilde{x}_2^2 + \tilde{x}_3^2 = 0.$$

  Since $a-b>0$ these are two complex conjugate imaginary lines $\tilde{x}_3 = \pm i\sqrt{a-b}\,\tilde{x}_2$, or dually, two complex conjugate imaginary points,

  $$G_\pm = \begin{bmatrix} 0 \\ \pm i\sqrt{a-b} \\ 1 \end{bmatrix}$$

  which lie on the y-axis, which is the minor principal axis of the confocal family.

- $\lambda = -b$: The equation of the degenerate conic is given by

  $$(a-b)\tilde{x}_1^2 - \tilde{x}_3^3 = 0.$$

  These are two real lines $x_3 = \pm\sqrt{a-b}\,x_1$, or dually the two real points

  $$F_\pm = \begin{bmatrix} \pm\sqrt{a-b} \\ 0 \\ 1 \end{bmatrix}.$$

  They are the two foci of the confocal family, and lie on the x-axis, which is the major principal axis.

- $\lambda = \infty$: The equation of the degenerate conic is given by

  $$\tilde{x}_1^2 + \tilde{x}_2^2 = 0.$$

  These are two complex conjugate imaginary lines $x_1 = \pm i x_2$, or dually two imaginary points at infinity

  $$Z_\pm = \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}.$$

  These two points are also called the *circle points* of similarity geometry.

Note that the degenerate dual conic consisting of the pair of circle points is independent of the confocal family, and can be used to characterize confocal conics in the space of dual pencils: Let

$$\mathcal{Z} := Z_+ \cup Z_-$$

denote the degenerate dual conic consisting of the two circle points.

Theorem 5.7. *A dual pencil of conics is a family of confocal conics (including confocal parabolas and concentric circles) if and only if it contains the circle points $\mathcal{Z}$ (as a dual degenerate conic).*

As a final note, we sketch how to obtain the orthogonality of two intersecting confocal conics (already proven in Theorem 5.5) in this projective setup:

Proposition 5.8. *Two Euclidean lines ℓ_1 and ℓ_2 are orthogonal if and only if its two dual points $\ell_1^\star$ and $\ell_1^\star$ are conjugate with respect to the degenerate conic $\mathcal{Z}^\star$.*

Thus, the orthogonality of confocal conics is closely related to the corresponding dual pencil containg the degenerate conic $\mathcal{Z}$ by dualizing the following statement:

Lemma 5.9. *Let $\mathcal{C}_\lambda$ be a pencil of conics. Let ℓ be a common tangent line of two distinct conics $\mathcal{C}_{\lambda_1}$ and $\mathcal{C}_{\lambda_2}$ from the pencil touching them in the points X_1 and X_2, respectively. and let ℓ be a line tangent to two conics and in the points X_1 and X_2. Then X_1 and X_2 are conjugate with respect to every conic in the pencil.*

Proof. Exercise. □

Theorem 5.10. *Two intersecting confocal conics intersect orthogonally.*

Proof. The two tangents at a point of intersection dually correspond to the two touching points of a common tangent. By Lemma 5.9, these two points are conjugate with respect to every conic in the pencil, in particular, to the degenerate conic corresponding to the two circle points $\mathcal{Z}$. Thus, by Proposition 5.8, the two tangent lines intersect orthogonally. □

5.2 Confocal coordinates

In Theorem 5.3 we have seen that every point $(x, y) \in \mathbb{R}^2$ with $x \cdot y \neq 0$ is the intersection of two confocal conics:

$$\begin{aligned} \frac{x^2}{a+u_1} + \frac{y^2}{b+u_1} &= 1, \\ \frac{x^2}{a+u_2} + \frac{y^2}{b+u_2} &= 1, \end{aligned} \tag{11}$$

which are given by $(u_1, u_2) \in \mathcal{U}$, where

$$\mathcal{U} := \mathcal{I}_1 \times \mathcal{I}_2 := \left\{ (u_1, u_2) \in \mathbb{R}^2 \;\middle|\; -a < u_1 < -b < u_2 \right\}.$$

In Theorem 5.4 we have seen that (11) is equivalent to

$$\begin{aligned} x^2 &= \frac{(a+u_1)(a+u_2)}{a-b}, \\ y^2 &= \frac{(b+u_1)(b+u_2)}{b-a}. \end{aligned} \tag{12}$$

and thus, vice versa, for each $(u_1, u_2) \in \mathcal{U}$ there are exactly 4 solutions $(x, y) \in \mathbb{R}^2$, one in each quadrant of $\mathbb{R}^2$. This means, that one obtains a coordinate system in (or a parametrization of) the first quadrant $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ by

$$\mathcal{U} \to \mathbb{R}^2_+, \qquad (u_1, u_2) \mapsto \begin{pmatrix} x(u_1, u_2) \\ y(u_1, u_2) \end{pmatrix}, \qquad \begin{cases} x(u_1, u_2) = \dfrac{\sqrt{a + u_1}\sqrt{a + u_2}}{\sqrt{a - b}} \\ y(u_1, u_2) = \dfrac{\sqrt{-(b + u_1)}\sqrt{b + u_2}}{\sqrt{a - b}} \end{cases} \tag{13}$$

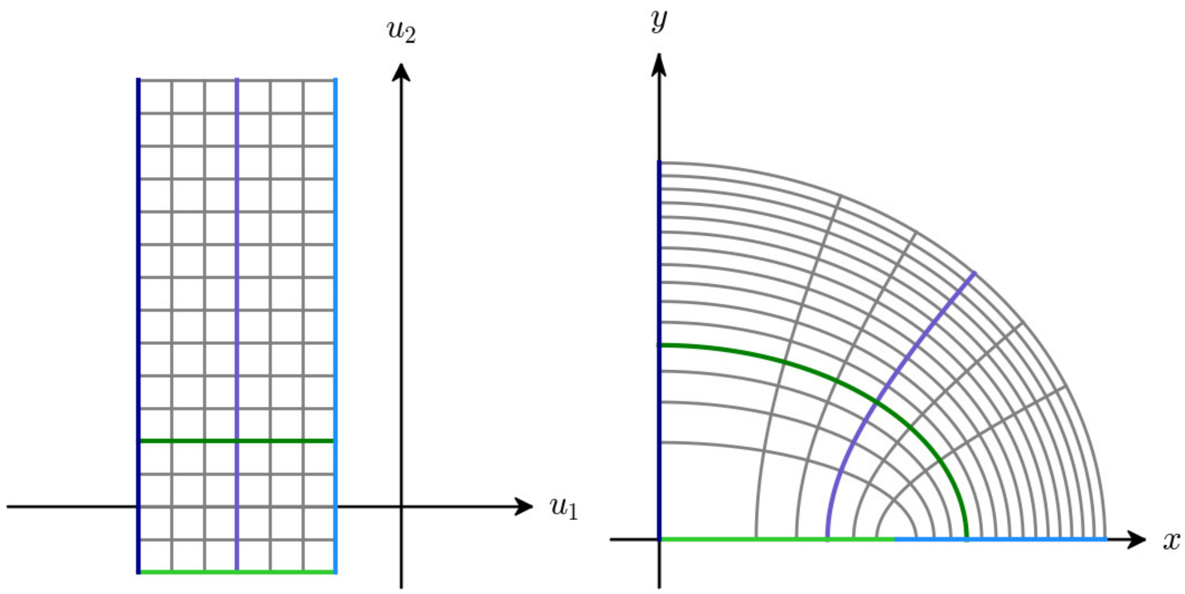

Figure 11. Confocal coordinate system in one quadrant using a "square root parametrization".

More generally, any coordinate system, whose coordinate lines are contained in confocal conics is called a confocal coordinate system.

Definition 5.2. A coordinate system $\boldsymbol{x} : U \to \mathbb{R}^2$ is called a *confocal coordinate system* if its coordinate lines $\boldsymbol{x}(s_1 = \text{const}, s_2)$ and $\boldsymbol{x}(s_1, s_2 = \text{const})$ are contained in confocal conics.

Theorem 5.11. *Confocal coordinate systems are orthogonal coordinate systems:*

$$\langle \partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x} \rangle = 0.$$

Proof. Follows from Theorem 5.5. □

Exercise 5.2. Show that the Lamé coefficients of (13) are given by

$$H_1^2 = \|\partial_1 \boldsymbol{x}\|^2 = \frac{u_1 - u_2}{4(u_1 + a)(u_1 + b)}, \quad H_2^2 = \|\partial_2 \boldsymbol{x}\|^2 = \frac{u_2 - u_1}{4(u_2 + a)(u_2 + b)},$$

and thus, in particular,

$$\frac{H_1^2}{H_2^2} = -\frac{\alpha_2(u_2)}{\alpha_1(u_1)}, \qquad \alpha_1(u_1) = (u_1 + a)(u_1 + b), \quad \alpha_2(u_2) = (u_2 + a)(u_2 + b).$$

At least locally we can assume that

$$U = I_1 \times I_2$$

with two intervals $I_1, I_2 \subset \mathbb{R}$. Then $\boldsymbol{x}$ is a confocal coordinate system if there exist two smooth functions

$$\begin{aligned} u_1 : I_1 \to \mathcal{I}_1, \quad & s_1 \mapsto u_1(s_1), \\ u_2 : I_2 \to \mathcal{I}_2, \quad & s_2 \mapsto u_2(s_2) \end{aligned} \tag{14}$$

such that (11), or equivalently, (12) is satisfied with $(x, y) = \boldsymbol{x}(s_1, s_2)$ and $u_1 = u_1(s_1), u_2 = u_2(s_2)$. Thus, all confocal coordinates are essentially reparametrizations of the "square root parametrization" (13) along the coordinate lines.

Note that the operation of reparametrization does not have a simple counterpart in the discrete context. The following reformulation turns out to be more convenient for finding certain confocal coordinates, and plays an important role in the discretization.

Exercise 5.3. How do the Lamé coefficients change under reparametrization? Derive differential equations for the functions $u_i(s_i)$ such that $\frac{H_1^2(s_1)}{H_2^2(s_2)} = 1$. Compare the result with Exercise 5.4.

Theorem 5.12. *Let*

$$\boldsymbol{x} : \mathbb{R}^2 \supset I_1 \times I_2 \to \mathbb{R}^2, \quad \boldsymbol{x}(s_1, s_2) = \begin{pmatrix} x(s_1, s_2) \\ y(s_1, s_2) \end{pmatrix}$$

be a coordinate system. Then $\boldsymbol{x}$ is a confocal coordinate system if and only if there exist functions

$$f_1, g_1 : I_1 \to \mathbb{R}, \quad f_2, g_2 : I_2 \to \mathbb{R}$$

with

$$\begin{aligned} f_1(s_1)^2 + g_1(s_1)^2 &= a - b, \\ f_2(s_2)^2 - g_2(s_2)^2 &= a - b \end{aligned} \tag{15}$$

such that

$$\begin{cases} x(s_1, s_2) = \dfrac{f_1(s_1) f_2(s_2)}{\sqrt{a-b}} \\ y(s_1, s_2) = \dfrac{g_1(s_1) g_2(s_2)}{\sqrt{a-b}} \end{cases} \tag{16}$$

Proof. Let $\boldsymbol{x}$ be a confocal coordinate system. Then there exist functions u_1, u_2 as in (14), and we can define the functions f_1, f_2, g_1, g_2 by

$$\begin{aligned} f_1(s_1) &= \sqrt{a + u_1(s_1)}, & f_2(s_2) &= \sqrt{a + u_2(s_2)}, \\ g_1(s_1) &= \sqrt{-(b + u_1(s_1)}, & g_2(s_2) &= \sqrt{b + u_2(s_2)}. \end{aligned}$$

Then (15) and (16) are satisfied.

Now assume there exist functions f_1, f_2, g_1, g_2 satisfying (15) and (16). Equations (15) are the compatibility conditions for the system

$$\begin{aligned} u_1(s_1) &= f_1(s_1)^2 - a, & u_2(s_2) &= f_2(s_2)^2 - a, \\ u_1(s_1) &= -g_1(s_1)^2 - b, & u_2(s_2) &= g_2(s_2)^2 - b, \end{aligned} \tag{17}$$

Thus, u_1, u_2 can be consistently defined satisfying (17). Then (11), or equivalently, (12) is satisfied. □

Thus, finding confocal coordinates reduces to solving the two equation (15). A distinguished solution is given by

$$f_1(s_1) = \sqrt{a-b}\cos s_1, \quad f_2(s_2) = \sqrt{a-b}\cosh s_2,$$
$$g_1(s_1) = \sqrt{a-b}\sin s_1, \quad g_2(s_2) = \sqrt{a-b}\sinh s_2,$$

leading to the confocal coordinate system

$$\begin{cases} x(s_1, s_2) = \sqrt{a-b}\ \cos s_1 \cosh s_2 \\ y(s_1, s_2) = \sqrt{a-b}\ \sin s_1 \sinh s_2 \end{cases} \tag{18}$$

which is naturally periodic in s_1 and covers the entire plane $\mathbb{R}^2$.

Exercise 5.4. Show that the Lamé coefficients of (18) are equal.

Remark 5.3. Essentially the same confocal coordinate system is obtained by considering the coordinate lines of the holomorphic functions $\mathbb{C} \to \mathbb{C}$, $z \mapsto \cos z$, or $z \mapsto \sin z$.

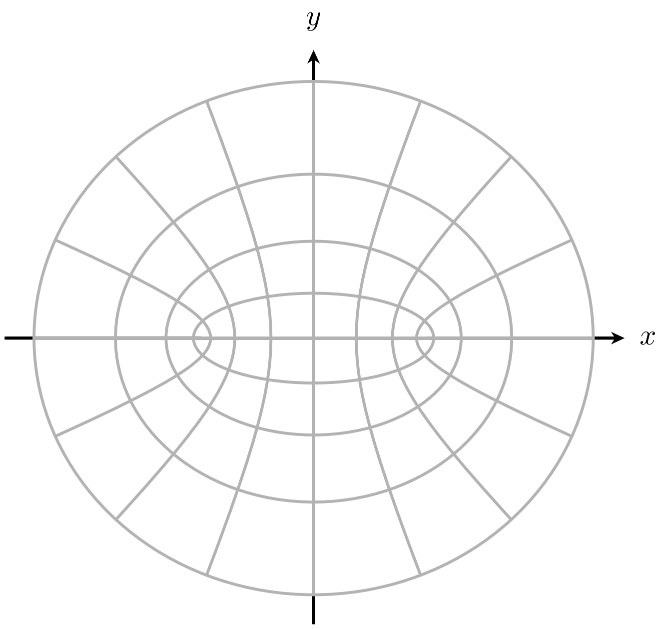

Figure 12. Confocal coordinate system using a "trigonometric functions parametrization".

Some more confocal coordinate systems are shown in Figures 13, 14, 15.

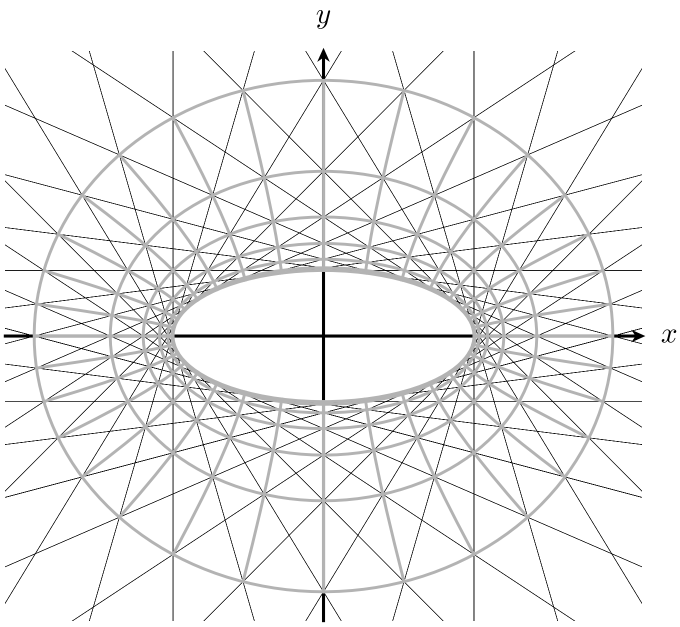

Figure 13. Confocal coordinate system, diagonally related to two families of straight lines tangent to an ellipse.

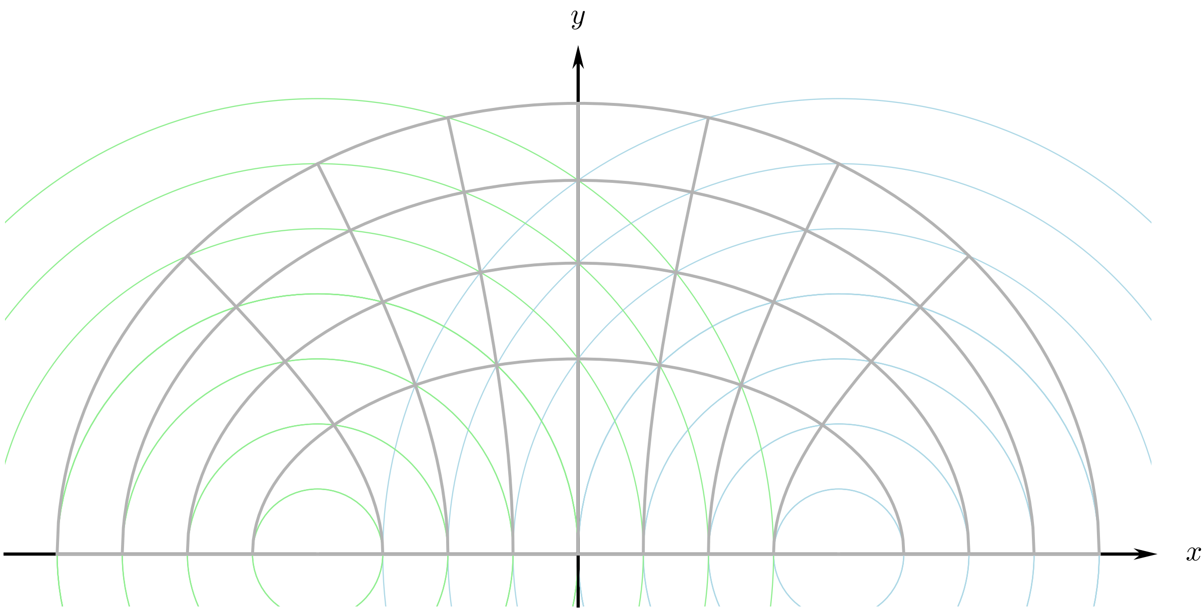

Figure 14. Confocal coordinate system, diagonally related to two families of concentric circles.

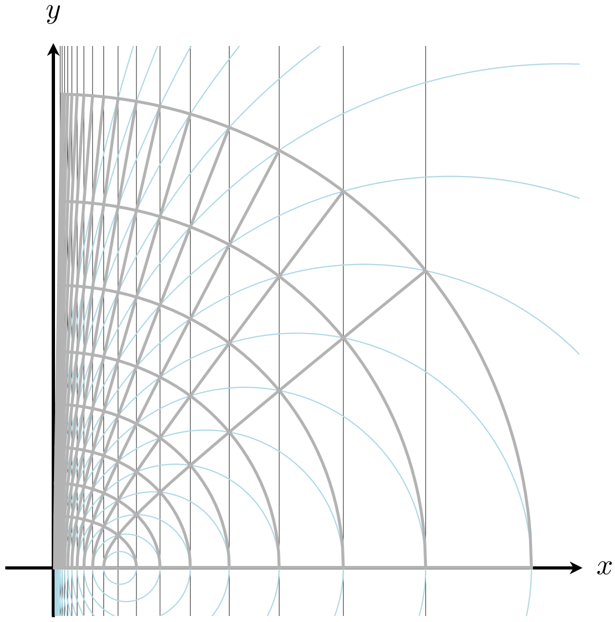

Figure 15. Confocal coordinate system, diagonally related to vertical lines and a hyperbolic pencil of circles.

In (16) we see that each component of a confocal coordinate system factorizes into the product of functions that each only depends on one of the variable.[2] Geometrically this means, that any two coordinate lines of the same family are related by an affine scaling along the coordinate axes. It turns out, that this property together with the orthogonality characterizes confocal coordinate systems.

Theorem 5.13. *Let*

$$\boldsymbol{x} : \mathbb{R}^2 \supset I_1 \times I_2 \to \mathbb{R}^2, \quad \boldsymbol{x}(s_1, s_2) = \begin{pmatrix} x(s_1, s_2) \\ y(s_1, s_2) \end{pmatrix}$$

be a coordinate system that satisfies the following two conditions:

(i) $\boldsymbol{x}$ *factorizes:*

$$\begin{cases} x(s_1, s_2) = f_1(s_1) f_2(s_2) \\ y(s_1, s_2) = g_1(s_1) g_2(s_2) \end{cases}$$

with some smooth functions $f_1, g_1 : I_1 \to \mathbb{R}, \quad f_2, g_2 : I_2 \to \mathbb{R}$ *that do not vanish:*[3]

$$\begin{aligned} f_1(s_1) \neq 0, g_1(s_1) \neq 0 \quad &\text{for any} \quad s_1 \in I_1 \\ f_2(s_2) \neq 0, g_2(s_2) \neq 0 \quad &\text{for any} \quad s_2 \in I_2 \end{aligned} \tag{19}$$

and whose derivatives do not constantly vanish:[4]

$$f_1' \neq 0, \quad g_1' \neq 0, \quad f_2' \neq 0, \quad g_2' \neq 0 \tag{20}$$

[2]This can also be expressed by $\partial_1 \partial_2 \log \boldsymbol{x} = 0$.

[3]By this, we consider the coordinate system away from the axes.

[4]By this, we exclude the case of coordinate lines that are parallel to the axes, which only leads to square grid parametrizations.

(ii) $\boldsymbol{x}$ is orthogonal:

$$\langle \partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x} \rangle = 0. \tag{21}$$

Then $\boldsymbol{x}$ is a confocal coordinate system, or, as a degenerate case, a polar coordinate system.

Remark 5.4. Note that the functions f_1, f_2, g_1, g_2 are only equal to the ones in Theorem 5.12 up to a factor each.

For the proof it is be more convenient to consider the squares of all involved functions. To this end, we introduce

$$F_1(s_1) := f_1(s_1)^2, \quad G_1(s_1) := g_1(s_1)^2, \quad F_2(s_2) := f_2(s_2)^2, \quad G_2(s_2) := g_2(s_2)^2.$$

With this we have

$$\begin{cases} x^2 = F_1 F_2 \\ y^2 = G_1 G_2. \end{cases}$$

The conditions (19) and (20) become

$$\begin{aligned} F_1(s_1) \neq 0, G_1(s_1) \neq 0 \quad &\text{for any} \quad s_1 \in I_1 \\ F_2(s_2) \neq 0, G_2(s_2) \neq 0 \quad &\text{for any} \quad s_2 \in I_2 \end{aligned}$$

and

$$F_1' \neq 0, \quad G_1' \neq 0, \quad F_2' \neq 0, \quad G_2' \neq 0$$

For the orthogonality condition (21) we obtain

$$\begin{aligned} & \langle \partial_1 \boldsymbol{x}, \partial_2 \boldsymbol{x} \rangle = 0 \\ \Leftrightarrow \quad & f_1' f_2 f_1 f_2' + g_1' g_2 g_1 g_2' = 0 \\ \Leftrightarrow \quad & F_1' F_2' + G_1' G_2' = 0. \end{aligned} \tag{22}$$

To show that $\boldsymbol{x}$ is a confocal coordinate system we need to show the existence of functions u_1 and u_2 such that (11) holds, or equivalently,

$$\begin{aligned} \frac{F_1 F_2}{\tilde{a}_1(s_1)} + \frac{G_1 G_2}{\tilde{b}_1(s_1)} &= 1, \\ \frac{F_1 F_2}{\tilde{a}_2(s_2)} + \frac{G_1 G_2}{\tilde{b}_2(s_2)} &= 1, \end{aligned} \tag{23}$$

with functions $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ such that

$$\tilde{a}_1 - \tilde{b}_1 = \tilde{a}_2 - \tilde{b}_2 = \text{const} \neq 0.$$

Proof.

- **The orthogonality condition as orthogonality of two curves**
  The orthogonality condition (22) says that the tangent vectors of two planar curves

  $$\gamma_1(s_1) := \begin{pmatrix} F_1(s_1) \\ G_1(s_1) \end{pmatrix}, \quad \gamma_2(s_2) := \begin{pmatrix} F_2(s_2) \\ G_2(s_2) \end{pmatrix}$$

  are always orthogonal:

  $$\langle \gamma_1'(s_1), \gamma_2'(s_2) \rangle = 0 \quad \text{for all} \quad s_1 \in I_1, s_2 \in I_2.$$

Since γ_1' and γ_2' have no constantly vanishing components, this implies that both curves γ_1 and γ_2 must lie on straight lines:

$$\gamma_1(s_1) = \alpha_1(s_1)v_1 + w_1, \quad \gamma_2(s_2) = \alpha_2(s_2)v_2 + w_2,$$

with some constant vectors $v_1 = \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix}, w_1 = \begin{pmatrix} w_1^1 \\ w_1^2 \end{pmatrix}, v_2 = \begin{pmatrix} v_2^1 \\ v_2^2 \end{pmatrix}, w_2 = \begin{pmatrix} w_2^1 \\ w_2^2 \end{pmatrix} \in \mathbb{R}^3$ with

$$\langle v_1, v_2 \rangle = 0, \quad v_1^1, v_1^2, v_2^1, v_2^2 \neq 0$$

(in particular $v_1 \neq 0$, $v_2 \neq 0$) and two functions α_1, α_2.

- **The case of polar coordinates**
  We now have

$$\begin{aligned} x^2 + y^2 &= F_1G_1 + F_2G_2 \\ &= \langle \gamma_1(s_1), \gamma_2(s_2) \rangle \\ &= \alpha_1(s_1) \langle v_1, w_2 \rangle + \alpha_2(s_2) \langle v_2, w_1 \rangle + \langle w_1, w_2 \rangle . \end{aligned} \tag{24}$$

  If $\langle v_1, w_2 \rangle = 0$, this implies that the coordinate lines $s_2 =$ const are circles, and therefore, the coordinate lines $s_1 =$ const must be lines through the origin. In this case, we obtain polar coordinates. Similarly for $\langle v_2, w_1 \rangle = 0$.

  Thus, let $\langle v_1, w_2 \rangle \neq 0$ and $\langle v_2, w_1 \rangle \neq 0$. In particular, this means that $w_1 \neq 0$, $w_2 \neq 0$.

- **The conic equations**
  Aiming for (23) we attempt to construct a quadratic equation from

$$\langle v_1, \gamma_2(s_2) \rangle = \langle v_1, \alpha_2(s_2)v_2 + w_2 \rangle = \langle v_1, w_2 \rangle$$

  Using $F_1(s_1) \neq 0$ and $G_1(s_1) \neq 0$ we obtain

$$\begin{aligned} \langle v_1, w_2 \rangle &= v_1^1 F_2 + v_1^2 G_2 \\ &= \frac{v_1^1}{F_1} F_1F_2 + \frac{v_1^2}{G_1} G_1G_2 \\ &= \frac{v_1^1}{\alpha_1 v_1^1 + w_1^1} F_1F_2 + \frac{v_1^2}{\alpha_1 v_1^2 + w_1^2} G_1G_2, \end{aligned}$$

  or equivalently (using $\langle v_1, w_2 \rangle, v_1^1, v_1^2 \neq 0$)

$$\frac{F_1F_2}{\tilde{a}_1(s_1)} + \frac{G_1G_2}{\tilde{b}_1(s_1)} = 1$$

  with

$$\tilde{a}_1(s_1) = \langle v_1, w_2 \rangle \left(\alpha_1(s_1) + \frac{w_1^1}{v_1^1} \right), \quad \tilde{b}_1(s_1) = \langle v_1, w_2 \rangle \left(\alpha_1(s_1) + \frac{w_1^2}{v_1^2} \right).$$

  One similarly obtains

$$\frac{F_1F_2}{\tilde{a}_2(s_2)} + \frac{G_1G_2}{\tilde{b}_2(s_2)} = 1$$

  with

$$\tilde{a}_2(s_2) = \langle v_2, w_1 \rangle \left(\alpha_2(s_2) + \frac{w_2^1}{v_2^1} \right), \quad \tilde{b}_2(s_2) = \langle v_2, w_1 \rangle \left(\alpha_2(s_2) + \frac{w_2^2}{v_2^2} \right).$$

- **The confocality**
  The differences

$$\tilde{a}_1(s_1) - \tilde{b}_1(s_1) = \langle v_1, w_2 \rangle \left(\frac{w_1^1}{v_1^1} - \frac{w_1^2}{v_1^2} \right) = \frac{\langle v_1, w_2 \rangle \det(w_1, v_1)}{v_1^1 v_1^2}$$

and

$$\tilde{a}_2(s_2) - \tilde{b}_2(s_2) = \langle v_2, w_1 \rangle \left(\frac{w_2^1}{v_2^1} - \frac{w_2^2}{v_2^2} \right) = \frac{\langle v_2, w_1 \rangle \det(w_2, v_2)}{v_2^1 v_2^2}$$

are constant (do not depend on s_1 and s_2).

To see that $\tilde{a}_1 - \tilde{b}_1 = \tilde{a}_2 - \tilde{b}_2$, choose (w.l.o.g.) v_1, v_2, w_1, w_2 such that

$$\|v_1\| = \|v_2\| = 1, \quad w_1 = \lambda_1 v_2, w_2 = \lambda_2 v_1 \tag{25}$$

with some $\lambda_1, \lambda_2 \neq 0$ and use

$$\langle v_1, v_2 \rangle = v_1^1 v_2^1 + v_1^2 v_2^2 = 0.$$

Furthermore, $\tilde{a}_1 - \tilde{b}_1 \neq 0$ since $\langle v_1, w_2 \rangle \neq 0$ and $\det(w_1, v_1) \neq 0$, and similarly $\tilde{a}_2 - \tilde{b}_2 \neq 0$.

□

Exercise 5.5. Show that the choice (25) is indeed possible, and use this to show $\tilde{a}_1 - \tilde{b}_1 = \tilde{a}_2 - \tilde{b}_2$.

Remark 5.5. The orthogonality condition (22) may be written as

$$\partial_1 \partial_2 \left(x^2 + y^2\right) = \partial_1 \partial_2 \left(F_1 F_2 + G_1 G_2\right) = 0,$$

which is equivalent to (cf. (24))

$$x^2 + y^2 = A_1(s_1) + A_2(s_2)$$

with some functions A_1, A_2. In the "square root parametrization" (13), or more generally using the functions u_1 and u_2 defined in (14), one obtains

$$x^2 + y^2 = u_1(s_1) + u_2(s_2) + a + b.$$

Thus, the curves

$$u_1(s_1) + u_2(s_2) = \text{const}$$

are circles (see Figure 16).

Exercise 5.6. Prove all claims in Remark 5.5.

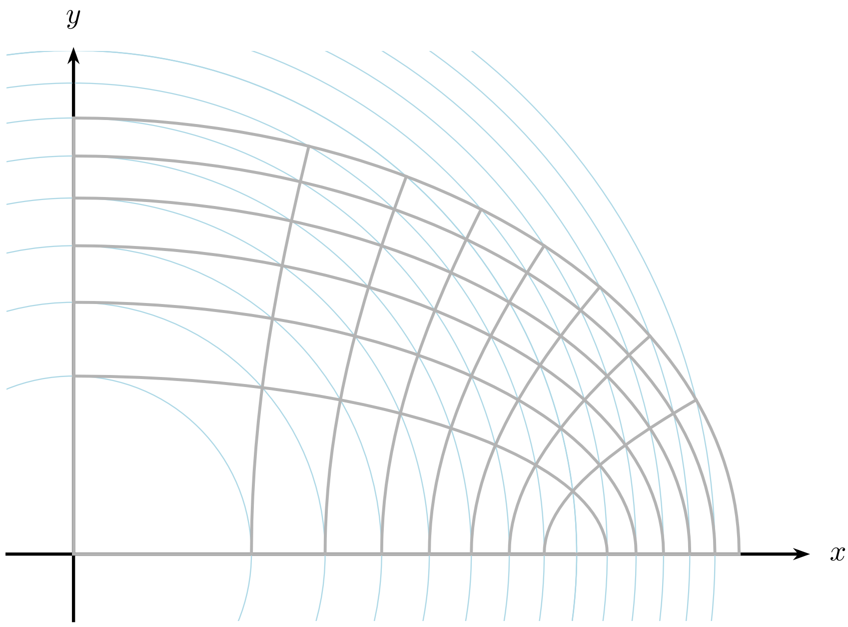

Figure 16. The confocal coordinate system with “square root parametrization” is diagonally related to concentric circles with center in the origin.

6 Confocal quadrics

We now consider confocal quadrics in $\mathbb{R}^3$.

Proposition 6.1. *Any ellipsoid, one-sheeted hyperboloid, or two-sheeted hyperboloid in $\mathbb{R}^3$ (which we collectively call non-parabolic non-degenerate quadric) can be brought into the following form by a Euclidean transformation:*

$$\mathcal{Q} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \right\}$$

with some $a > 0$, $b, c \neq 0$, $a > b > c$.

- *If $a, b, c > 0$, then $\mathcal{Q}$ is an ellipsoid.*
- *If $a, b > 0$, $c < 0$, then $\mathcal{Q}$ is a one-sheeted hyperboloid.*
- *If $a > 0$, $b, c < 0$, then $\mathcal{Q}$ is a two-sheeted hyperboloid.*

Definition 6.1. Two quadrics (non-degenerate and non-parabolic) are called *confocal* if they have the same principal planes, and the two conic sections in each of these principal planes are confocal.

Consider a non-parabolic non-degenerate quadric in normal form

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \qquad a > b > c.$$

Then any confocal quadric must necessarily also be in normal form

$$\frac{x^2}{\tilde{a}} + \frac{y^2}{\tilde{b}} + \frac{z^2}{\tilde{c}} = 1, \qquad \tilde{a} > \tilde{b} > \tilde{c}.$$

Now these two quadrics in normal form are confocal, if and only if

$$a - b = \tilde{a} - \tilde{b}, \qquad a - c = \tilde{a} - \tilde{c},$$

and therefore, also

$$b - c = \tilde{b} - \tilde{c}.$$

Theorem 6.2. *By a Euclidean transformation, any family of* ***confocal quadrics*** *can be brought into the form:*

$$\mathcal{Q}_\lambda = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1 \right\}, \qquad \lambda \in \mathbb{R}. \tag{26}$$

with some $a > b > c$.

Remark 6.1. Note that the family $\mathcal{Q}_\lambda$ (up to a shift of the parameter λ) only depends on two of the differences, say $a - b$ and $a - c$ (and not independently on a, b and c).

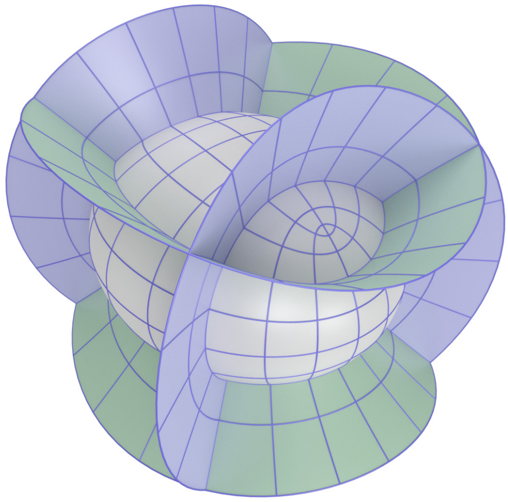

Figure 17. A confocal ellipsoid, one-sheeted hyperboloid, and two-sheeted hyperboloid.

The family consists of ellipsoids, one-sheeted hyperboloids, and two-sheeted hyperboloids each of these three subfamilies filling the entire Euclidean space, respectively.

- $\mathcal{Q}_\lambda$ is empty (or "purely imaginary") for $\lambda < -a$.
- $\mathcal{Q}_\lambda$ is a two-sheeted hyperboloid for $-a < \lambda < -b$.
- $\mathcal{Q}_\lambda$ is a one-sheeted hyperboloid for $-b < \lambda < -c$.
- $\mathcal{Q}_\lambda$ is an ellipsoid for $\lambda > -c$.

The cases $\lambda = -a, -b, -c$ can be considered as limiting cases, which we study in more detail in the projective description of the family as a dual pencil of quadrics.

Theorem 6.3. *Through every point $(x, y, z) \in \mathbb{R}^3$ not on the coordinate planes ($x \cdot y \cdot z \neq 0$), there passes exactly one ellipsoid, one one-sheeted hyperboloid, and one two-sheeted hyperboloid from the confocal family $\mathcal{Q}_\lambda$.*

Proof. Exercise (similar to Theorem 5.3). □

Theorem 6.4. *A confocal ellipsoid, one-sheeted hyperboloid, and two-sheeted hyperboloid intersect in exactly 8 points, which lie mirror symmetric with respect to the common principal planes.*

Proof. Exercise (similar to Theorem 5.4). □

Theorem 6.5. *If two confocal quadrics intersect, they intersect orthogonally.*

Proof. Exercise (similar to Theorem 5.5). □

6.1 Confocal quadrics as dual pencils

By embedding $\mathbb{R}^3 \subset \mathbb{R}\mathrm{P}^3$, we look at the projective description of confocal quadrics. Homogenizing the equation for confocal quadrics

$$\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 1$$

by introducing homogeneous coordinates (x_1, x_2, x_3, x_4) with $x = \frac{x_1}{x_4}, y = \frac{x_2}{x_4}, z = \frac{x_3}{x_4}$, we obtain

$$\frac{x_1^2}{a+\lambda} + \frac{x_2^2}{b+\lambda} + \frac{x_3^2}{c+\lambda} - x_4^2 = 0.$$

Thus, the corresponding Gram matrix and its inverse are given by

$$Q_\lambda = \begin{pmatrix} \frac{1}{a+\lambda} & & & \\ & \frac{1}{b+\lambda} & & \\ & & \frac{1}{c+\lambda} & \\ & & & -1 \end{pmatrix}, \qquad Q_\lambda^{-1} = \begin{pmatrix} a+\lambda & & & \\ & b+\lambda & & \\ & & c+\lambda & \\ & & & -1 \end{pmatrix},$$

and it holds again:

Theorem 6.6. *The family of confocal quadrics is a dual pencil of quadrics.*

We now determine the four degenerate quadrics in this pencil of quadrics, and dually, in the confocal family: The four corresponding roots of

$$\det Q_\lambda^{-1} = -(a+\lambda)(b+\lambda)(c+\lambda) = 0$$

are given by $\lambda = -a, -b, -c, \infty$.

- $\lambda = -a$: The equation of the degenerate quadric is given by

  $$(a-b)\tilde{x}_2^2 + (a-c)\tilde{x}_3^2 + x_4^2 = 0.$$

  Since $a - b > 0$ and $a - c > 0$ this is a imaginary cone with real vertex $[1, 0, 0, 0]$, or dually (and in affine coordinates), an imaginary conic in the principal plane $x = 0$:

  $$\frac{y^2}{a-b} + \frac{z^2}{a-c} + 1 = 0, \quad x = 0.$$

- $\lambda = -b$: The equation of the degenerate quadric is given by

  $$(a-b)\tilde{x}_1^2 - (b-c)\tilde{x}_3^2 - x_4^2 = 0.$$

  This is a cone with vertex $[0, 1, 0, 0]$, or dually (and in affine coordinates), a hyperbola in the principal plane $y = 0$:

  $$\frac{x^2}{a-b} - \frac{z^2}{b-c} = 1, \quad y = 0. \tag{27}$$

- $\lambda = -c$: The equation of the degenerate quadric is given by

  $$(a-c)\tilde{x}_1^2 + (b-c)\tilde{x}_2^2 - x_4^2 = 0.$$

  This is a cone with vertex $[0, 0, 1, 0]$, or dually (and in affine coordinates), an ellipse in the principal plane $z = 0$:

  $$\frac{x^2}{a-c} + \frac{y^2}{b-c} = 1, \quad z = 0. \tag{28}$$

- $\lambda = \infty$: The equation of the degenerate quadric is given by

$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 = 0.$$

This is an imaginary cone with real vertex $[0, 0, 0, 1]$, or dually, an imaginary conic in the plane at infinity:

$$x_1^1 + x_2^2 + x_3^2 = 0, \quad x_4 = 0.$$

This is also called the *absolute conic* $\mathcal{Z}$ of similarity geometry, which does not depend on the confocal family.

Theorem 6.7. *A dual pencil of quadric is a family of confocal quadrics (including limiting cases such as concentric spheres) if and only if it contains the absolute conic* $\mathcal{Z}$ *(as a dual degenerate quadric).*

Remark 6.2. The projective description of confocal quadrics can be used exactly as in the two-dimensional case (Theorem 5.10) to show that two confocal quadrics intersect orthogonally (Theorem 6.5).

6.2 Focal conics

In the projective description of confocal conics, the common foci, which are located on the major axis, appeared as a degenerate conic in the dual pencil of conics. Thus, by comparison, we can say that in the three-dimensional case, the role of the foci is taken by the two real conics (28) and (27), which are located in two of the principal planes.

Remark 6.3. In the two-dimensional case the other pair of imaginary points on the minor axis may be understood as a second pair of foci, while in the three-dimensional case the imaginary conic in the remaining principal plane may be understood as a third focal conic.

A closer look reveals, that that two conics (28) and (27) are located in orthogonal planes while containing each others foci, respectively.

Definition 6.2. Two (planar) conics $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^3$ are called *focal conics* if the two planes which contain them are orthogonal, the foci of $\mathcal{C}_1$ lie on $\mathcal{C}_2$, and the foci of $\mathcal{C}_2$ lie on $\mathcal{C}_1$.

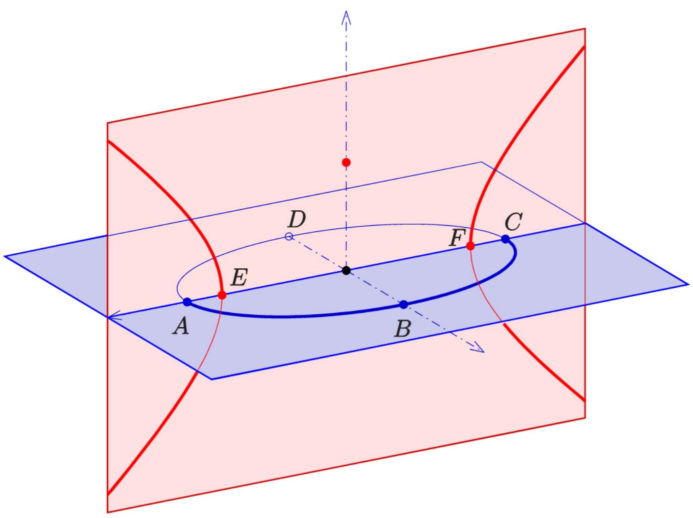

Figure 18. Two focal conics.

Theorem 6.8. *Two conics*

$$\frac{x^2}{a} + \frac{y^2}{b} = 1, \qquad a > b$$
$$\frac{x^2}{\tilde{a}} + \frac{z^2}{\tilde{b}} = 1, \qquad \tilde{a} > \tilde{b}$$

in normal form in the xy-plane and xz-plane are focal conics if and only if

$$\tilde{a} = a - b, \qquad \tilde{b} = -b.$$

Proof. Exercise. □

Proposition 6.9. *Let $\mathcal{C}_1, \mathcal{C}_2$ be a pair of focal conics in $\mathbb{R}^3$. Then $\mathcal{C}_2$ consists of all vertices of right circular cones (cones of revolution) that contain $\mathcal{C}_1$, and vice versa, $\mathcal{C}_1$ consists of all vertices of right circular cones that contain $\mathcal{C}_2$.*

Remark 6.4.

(i) This means that from a point on $\mathcal{C}_2$ the conic $\mathcal{C}_1$ looks like a circle.

(ii) Moreover, it holds that the axis of the cone is the tangent of the focal conic in its vertex.

For the proof we recall the definition of Dandelin spheres. All ellipses and hyperbolas arise as planar sections of right circular cones. Let $\mathcal{R}$ be a right circular cone and P a plane that is not parallel to a tangent plane of $\mathcal{R}$ and does not contain the vertex of $\mathcal{R}$. Then the intersection $\mathcal{C} := \mathcal{R} \cap P$ is an ellipse or hyperbola, The two spheres that touch $\mathcal{R}$ in a circle and P in a point are called *Dandelin spheres*.

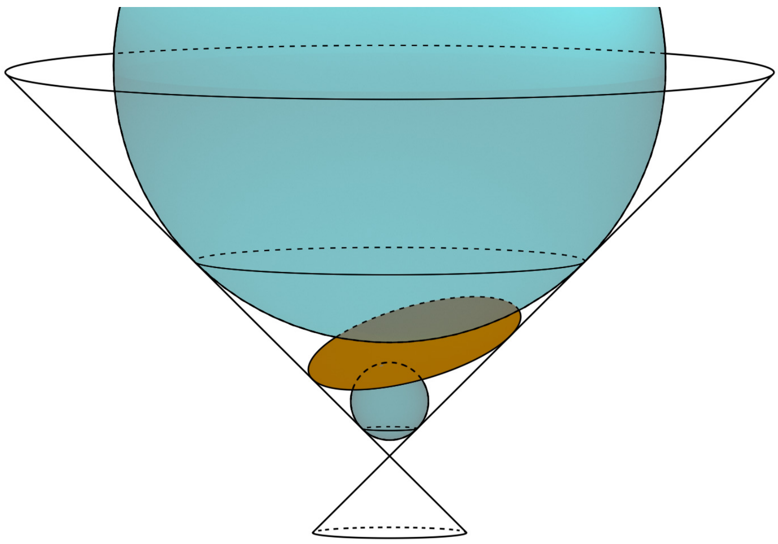

Figure 19. Dandelin spheres of an ellipse cut from a right circular cone.

Proposition 6.10. *The two Dandelin spheres touch the conic in its two foci.*

Proof. We consider the case where $\mathcal{C}$ is an ellipse. Let F_1 and F_2 be the two touching points of the Dandelin spheres, and C_1 and C_2 the two touching circles of the cone $\mathcal{R}$.

Consider a point $X \in \mathcal{C}$ and the (straight line) generator ℓ of the cone passing through X. Let $X_1 = \ell \cap C_1$ and $X_2 = \ell \cap C_2$ be the intersection points with the touching circles.

Since all touching segments to a sphere from a point have equal lengths, we obtain that

$$d(X, F_1) + d(X, F_2) = d(X, X_1) + d(X, X_2) = d(X_1, X_2)$$

is constant for all points on $\mathcal{C}$. Thus $\mathcal{C}$ is an ellipse with foci F_1 and F_2. □

Exercise 6.1. Prove the case where $\mathcal{C}$ is a hyperbola.

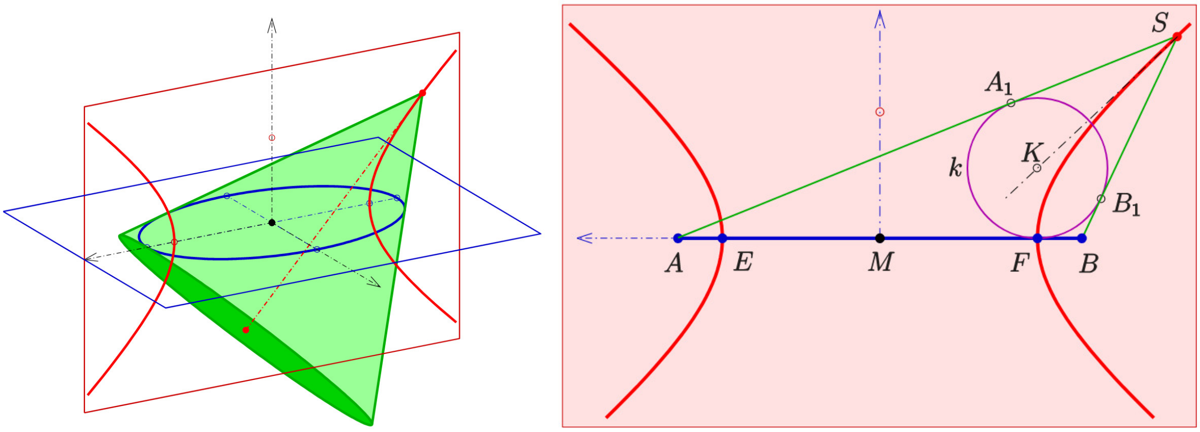

Figure 20. Right circular cone containing an ellipse. Its vertex is located on the focal conic.

Partial proof of Proposition 6.9. Let $\mathcal{C}_1$ be an ellipse with points A, B on the principal axis and foci E, F (see Figure 20). Let $\mathcal{R}$ be a right circular cone with vertex S that contains $\mathcal{C}_1$. We only show that S is contained in the focal conic $\mathcal{C}_2$.

For symmetry reasons S must lie in the plane that contains $\mathcal{C}_2$. Consider the Dandelin sphere that touches the plane that contains $\mathcal{C}_1$ in the point F. Let A_1, B_1 be the two touching points of the Dandelin sphere with the cone $\mathcal{R}$ in the plane that contains $\mathcal{C}_2$. Then

$$\begin{aligned} d(A, S) &= d(A, A_1) + d(A_1, S) = d(A, F) + d(B_1, S) \\ d(B, S) &= d(B, B_1) + d(B_1, S) = d(B, F) + d(B_1, S), \end{aligned}$$

and therefore

$$d(A, S) - d(B, S) = d(A, F) - d(B, F) = d(E, F) = \text{const},$$

which describes a hyperbola with foci A, B which contains the points E, F. □

Exercise 6.2. Complete the proof in the reverse direction, and for the case where the circular cones contain the hyperbola.

Definition 6.3. Let $\mathcal{Q}$ be a (non-degenerate non-parabolic) quadric in $\mathbb{R}^3$. Then the pair of focal conics in the family of confocal quadrics of $\mathcal{Q}$ is called the *focal conics* of $\mathcal{Q}$.

Proposition 6.11. *A tangent cone from any point of a focal conic to its quadric is a right circular cone (if it exists).*

Idea of the proof. The family of tangent cones from a fixed point to a dual pencil of quadrics is a (degenerate) dual pencil of quadrics. It contains one of the focal conics (and thus a right circular cone) and the absolute conic $\mathcal{Z}$ at infinity. Thus, the entire family consists of (coaxial) right circular cones. □

Remark 6.5. By also considering imaginary (right circular) tangent cones, the focal conics of a quadric consist of exactly all vertices of right circular tangent cones to the quadric.

Remark 6.6. The focal conics allow for a "string construction" of ellipsoids, generalizing the property (8) of ellipses (see Figure 21).

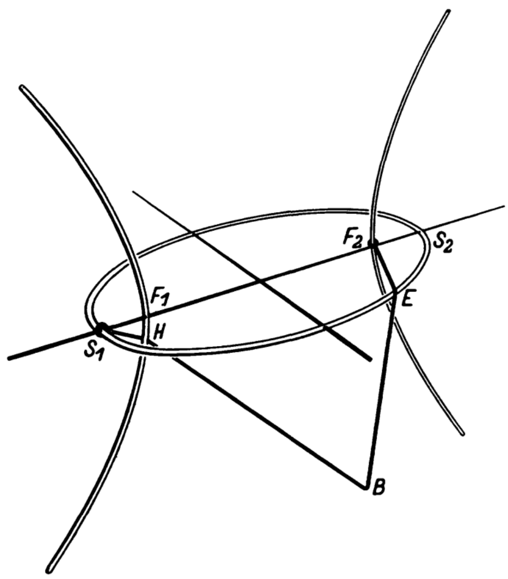

Figure 21. String construction of an ellipsoid from a pair of focal conics.

6.3 Confocal coordinates

In Theorem 6.3 we have seen that every point $(x, y, z) \in \mathbb{R}^3$ with $x \cdot y \cdot z \neq 0$ is the intersection of three confocal quadrics:

$$\begin{aligned} \frac{x^2}{a+u_1} + \frac{y^2}{b+u_1} + \frac{z^2}{b+u_1} &= 1, \\ \frac{x^2}{a+u_2} + \frac{y^2}{b+u_2} + \frac{z^2}{b+u_2} &= 1, \\ \frac{x^2}{a+u_3} + \frac{y^2}{b+u_3} + \frac{z^2}{b+u_3} &= 1, \end{aligned} \tag{29}$$

which are given by $(u_1, u_2, u_3) \in \mathcal{U}$, where

$$\mathcal{U} := \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3 := \left\{ (u_1, u_2, u_3) \in \mathbb{R}^3 \;\middle|\; -a < u_1 < -b < u_2 < -c < u_3 \right\}.$$

In Theorem 6.4, we have seen that, vice versa, for each $(u_1, u_2, u_3) \in \mathcal{U}$ there are exactly 8 solutions $(x, y, z) \in \mathbb{R}^3$, one in each octant of $\mathbb{R}^3$, and mirror symmetric with respect to the coordinate planes. This, is evident from the fact that the solution of the linear system

(11) is given by

$$
\begin{aligned}
x^2 &= \frac{(a+u_1)(a+u_2)(a+u_3)}{(a-b)(a-c)},\\
y^2 &= \frac{(b+u_1)(b+u_2)(b+u_3)}{(b-a)(b-c)},\\
z^2 &= \frac{(c+u_1)(c+u_2)(c+u_3)}{(c-a)(c-b)}.
\end{aligned}
\tag{30}
$$

This means, that one obtains a coordinate system in (or a parametrization of) the first octant $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\}$ by

$$
\mathcal{U} \to \mathbb{R}^3_+, \qquad (u_1,u_2,u_3) \mapsto \begin{pmatrix} x(u_1,u_2,u_3) \\ y(u_1,u_2,u_3) \\ z(u_1,u_2,u_3) \end{pmatrix},
$$

$$
\begin{cases}
x(u_1,u_2,u_3) = \dfrac{\sqrt{a+u_1}\sqrt{a+u_2}\sqrt{a+u_3}}{\sqrt{a-b}\sqrt{a-c}} \\
y(u_1,u_2,u_3) = \dfrac{\sqrt{-(b+u_1)}\sqrt{b+u_2}\sqrt{b+u_3}}{\sqrt{a-b}\sqrt{b-c}} \\
z(u_1,u_2,u_3) = \dfrac{\sqrt{-(c+u_1)}\sqrt{-(c+u_3)}\sqrt{c+u_3}}{\sqrt{a-c}\sqrt{b-c}}
\end{cases}
\tag{31}
$$

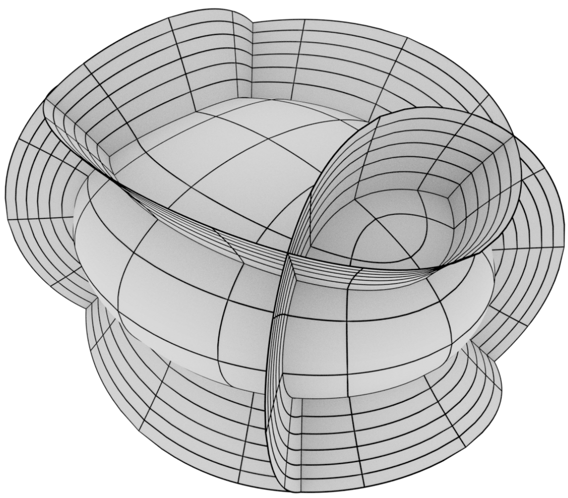

Figure 22. Three quadrics and some parameter lines from a confocal coordinate system in one octant using a "square root parametrization", and then reflected to all octants.

Definition 6.4. A coordinate system $\boldsymbol{x} : U \to \mathbb{R}^3$ is called a *confocal coordinate system* if its coordinate planes $\boldsymbol{x}(s_1 = \text{const}, s_2, s_3)$, $\boldsymbol{x}(s_1, s_2 = \text{const}, s_3)$, and $\boldsymbol{x}(s_1, s_2, s_3 = \text{const})$ are contained in confocal quadrics.

Theorem 6.12. *Confocal coordinate systems are orthogonal coordinate systems:*

$$
\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0 \quad \textit{for all} \quad i = 1,2,3, \quad i \neq j.
$$

Proof. Exercise (follows from Theorem 6.5). □

Remark 6.7. By Theorem 1.1, confocal quadrics intersect each other in curvature lines. Thus, every confocal coordinate system also yields curvature line parametrizations of the quadrics it contains as coordinate surfaces.

Exercise 6.3. Show that (31) satisfies the Euler-Poisson-Darboux system

$$\partial_i \partial_j \boldsymbol{x} = \frac{1}{2(u_i - u_j)} \left(\partial_j \boldsymbol{x} - \partial_i \boldsymbol{x} \right).$$

Start by showing that the partial derivatives of the x (and similarly y and z) satisfy

$$\partial_i x = \frac{x}{2(a + u_i)}.$$

Exercise 6.4. Show that the Lamé coefficients of (31) are given by

$$H_1^2 = \frac{(u_1 - u_2)(u_1 - u_3)}{4(u_1 + a)(u_1 + b)(u_1 + c)}, \quad H_2^2 = \frac{(u_2 - u_3)(u_2 - u_1)}{4(u_2 + a)(u_2 + b)(u_2 + c)},$$
$$H_3^2 = \frac{(u_3 - u_1)(u_3 - u_2)}{4(u_3 + a)(u_3 + b)(u_3 + c)},$$

and thus, in particular,

$$\frac{H_i^2}{H_j^2} = -\frac{\alpha_{jk}(u_j, u_k)}{\alpha_{ik}(u_i, u_k)}, \qquad \alpha_{lm}(u_l, u_m) = \frac{(u_l + a)(u_l + b)(u_l + c)}{(u_l - u_m)}.$$

Conclude that quadrics are isothermic surfaces (see Defintion 3.4 and Exercise 3.1).

Exercise 6.5. Show that the coefficients of the second fundamental forms of the coordinate surfaces (see Section 3.1) of (31), are given by

$$e_{ij} = \frac{1}{H_k} \frac{u_i - u_j}{4(u_i + a)(u_i + b)(u_i + c)}, \qquad g_{ij} = \frac{1}{H_k} \frac{u_j - u_i}{4(u_j + a)(u_j + b)(u_j + c)},$$

and thus, in particular,

$$\frac{e_{ij}}{g_{ij}} = -\frac{\beta_j(u_j)}{\beta_i(u_i)}, \qquad \beta_i(u_i) = (u_i + a)(u_i + b)(u_i + c).$$

The principal curvatures of the coordinate surfaces are given by $\frac{e_{ij}}{E_{ij}}, \frac{g_{ij}}{G_{ij}}$. Show that the principal curvatures coincide if and only if $u_i = u_J$, and conclude that the umbilic points of quadrics are the intersection points with its focal conics.

At least locally we can assume that

$$U = I_1 \times I_2 \times I_3$$

with three intervals $I_1, I_2, I_3 \subset \mathbb{R}$. Then $\boldsymbol{x}$ is a confocal coordinate system if there exist three smooth functions

$$\begin{aligned} u_1 &: I_1 \to \mathcal{I}_1, \quad s_1 \mapsto u_1(s_1), \\ u_2 &: I_2 \to \mathcal{I}_2, \quad s_2 \mapsto u_2(s_2), \\ u_3 &: I_3 \to \mathcal{I}_3, \quad s_3 \mapsto u_3(s_3), \end{aligned}$$

such that (29), or equivalently, (30) is satisfied with $(x, y, z) = \boldsymbol{x}(s_1, s_2, s_3)$ and $u_1 = u_1(s_1)$, $u_2 = u_2(s_2)$, $u_3 = u_3(s_3)$.

Exercise 6.6. How do the coefficients of the second fundamental form change under reparametrization along the coordinate lines? Derive differential equations for the functions $u_i(s_i)$ such that

$$\frac{e_{12}}{g_{12}} = 1, \quad \frac{e_{13}}{g_{13}} = -1, \quad \frac{e_{23}}{g_{23}} = 1.$$

Theorem 6.13. *Let*

$$\boldsymbol{x} : \mathbb{R}^3 \supset I_1 \times I_2 \times I_3 \to \mathbb{R}^3, \quad \boldsymbol{x}(s_1, s_2, s_3) = \begin{pmatrix} x(s_1, s_2, s_3) \\ y(s_1, s_2, s_3) \\ z(s_1, s_2, s_3) \end{pmatrix}$$

be a coordinate system. Then $\boldsymbol{x}$ is a confocal coordinate system if and only if there exist functions

$$f_1, g_1, h_1 : I_1 \to \mathbb{R}, \quad f_2, g_2, h_2 : I_2 \to \mathbb{R}, \quad f_3, g_3, h_3 : I_3 \to \mathbb{R}$$

with

$$\begin{aligned} f_1(s_1)^2 + g_1(s_1)^2 &= a - b, & f_1(s_1)^2 + h_1(s_1)^2 &= a - c, \\ f_2(s_2)^2 - g_2(s_2)^2 &= a - b, & f_2(s_2)^2 + h_2(s_2)^2 &= a - c, \\ f_3(s_3)^2 - g_3(s_3)^2 &= a - b, & f_3(s_3)^2 - h_3(s_3)^2 &= a - c, \end{aligned} \tag{32}$$

such that

$$\begin{cases} x(s_1, s_2, s_3) = \dfrac{f_1(s_1) f_2(s_2) f_3(s_3)}{\sqrt{a-b}\sqrt{a-c}} \\ y(s_1, s_2, s_3) = \dfrac{g_1(s_1) g_2(s_2) g_3(s_3)}{\sqrt{a-b}\sqrt{b-c}} \\ z(s_1, s_2, s_3) = \dfrac{h_1(s_1) h_2(s_2) h_3(s_3)}{\sqrt{a-c}\sqrt{b-c}} \end{cases} \tag{33}$$

Proof. Exercise (similar to Theorem 5.12). Note the fundamental relations

$$\begin{aligned} u_1(s_1) &= f_1(s_1)^2 - a, & u_2(s_2) &= f_2(s_2)^2 - a, & u_3(s_3) &= f_3(s_3)^2 - a, \\ u_1(s_1) &= -g_1(s_1)^2 - b, & u_2(s_2) &= g_2(s_2)^2 - b, & u_3(s_3) &= g_3(s_3)^2 - b, \\ u_1(s_1) &= -h_1(s_1)^2 - c, & u_2(s_2) &= h_2(s_2)^2 - c, & u_3(s_3) &= h_3(s_3)^2 - c. \end{aligned}$$

□

Thus, finding confocal coordinates reduces to solving the three pairs of quadratic equations (32). A distinguished solution is given in terms of Jacboi elliptic functions (see Appendix B.2):

$$\begin{aligned} f_1(s_1) &= \sqrt{a-b}\,\mathrm{sn}(s_1, k_1), & f_2(s_2) &= \sqrt{b-c}\frac{\mathrm{dn}(s_2, k_2)}{k_2}, & f_3(s_3) &= \sqrt{a-c}\frac{1}{\mathrm{sn}(s_3, k_3)} \\ g_1(s_1) &= \sqrt{a-b}\,\mathrm{cn}(s_1, k_1), & g_2(s_2) &= \sqrt{b-c}\,\mathrm{cn}(s_2, k_2), & g_3(s_3) &= \sqrt{a-c}\frac{\mathrm{dn}(s_3, k_3)}{\mathrm{sn}(s_3, k_3)} \\ h_1(s_1) &= \sqrt{a-b}\frac{\mathrm{dn}(s_1, k_1)}{k_1}, & h_2(s_2) &= \sqrt{b-c}\,\mathrm{sn}(s_2, k_2), & h_3(s_3) &= \sqrt{a-c}\frac{\mathrm{cn}(s_3, k_3)}{\mathrm{sn}(s_3, k_3)} \end{aligned} \tag{34}$$

with moduli $0 < k_i < 1$, $i = 1, 2, 3$:

$$k_1^2 = \frac{a-b}{a-c}, \quad k_2^2 = 1 - k_1^2 = \frac{b-c}{a-c}, \quad k_3 = k_1,$$

The associated confocal coordinate system

$$\begin{cases} x(s_1,s_2,s_3) = \sqrt{a-c}\,\mathrm{sn}(s_1,k_1)\,\mathrm{dn}(s_2,k_2)\,\mathrm{ns}(s_3,k_3)\\ y(s_1,s_2,s_3) = \sqrt{a-c}\,\mathrm{cn}(s_1,k_1)\,\mathrm{cn}(s_2,k_2)\,\mathrm{ds}(s_3,k_3)\\ z(s_1,s_2,s_3) = \sqrt{a-c}\,\mathrm{dn}(s_1,k_1)\,\mathrm{sn}(s_2,k_2)\,\mathrm{cs}(s_3,k_3)\end{cases}$$

is naturally periodic in s_1 and s_2, and covers the entire space $\mathbb{R}^3$.

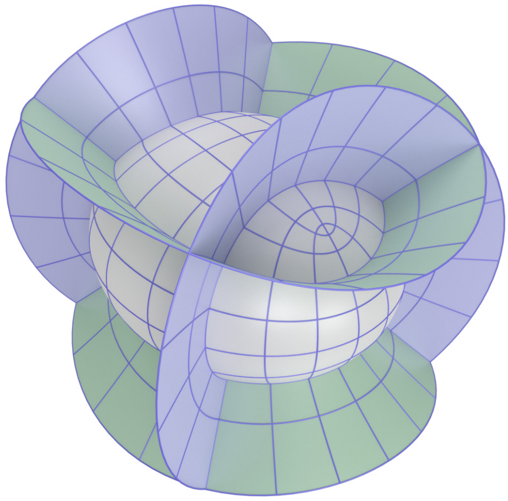

Figure 23. Three quadrics and some parameter lines from a confocal coordinate system using a "Jacobi elliptic functions parametrization".

Theorem 6.14. *Let*

$$\boldsymbol{x} : \mathbb{R}^3 \supset I_1 \times I_2 \times I_3 \to \mathbb{R}^3, \quad \boldsymbol{x}(s_1,s_2,s_3) = \begin{pmatrix} x(s_1,s_2,s_3)\\ y(s_1,s_2,s_3)\\ z(s_1,s_2,s_3)\end{pmatrix}$$

be a coordinate system that satisfies the following two conditions:

(i) $\boldsymbol{x}$ *factorizes:*

$$\begin{cases} x(s_1,s_2,s_3) = f_1(s_1)f_2(s_2)f_3(s_3)\\ y(s_1,s_2,s_3) = g_1(s_1)g_2(s_2)g_3(s_3)\\ z(s_1,s_2,s_3) = h_1(s_1)h_2(s_2)h_3(s_3)\end{cases}$$

with some smooth functions $f_i, g_i, h_i : I_i \to \mathbb{R}, i = 1,2,3$ *that do not vanish:*[5]

$$f_i(s_i) \neq 0, \quad g_i(s_i) \neq 0, \quad h_i(s_i) \neq 0 \quad \textit{for all} \quad s_i \in I_i, \quad i = 1,2,3 \tag{35}$$

and whose derivatives do not constantly vanish:[6]

$$f_i' \neq 0, \quad g_i' \neq 0, \quad h_i' \neq 0, \quad \textit{for all} \quad i = 1,2,3 \tag{36}$$

(ii) $\boldsymbol{x}$ *is orthogonal:*

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x}\rangle = 0 \quad \textit{for all} \quad i,j = 1,2,3 \quad i \neq j \tag{37}$$

[5]By this, we consider the coordinate system away from the coordinate planes.
[6]By this, we exclude the case of coordinate surfaces that are contained in planes, which leads to square grid and cylindrical coordinates.

Then $\boldsymbol{x}$ is a confocal coordinate system (including degenerate cases in which two or three of the semi-axes coincide).

Again for the proof, we introduce the squares

$$F_i := f_i^2, \quad G_i := g_i^2, \quad H_i := h_i^2, \quad \text{for} \quad i = 1, 2, 3.$$

With this we have

$$\begin{cases} x^2 = F_1 F_2 F_3 \\ y^2 = G_1 G_2 G_3 \\ z^2 = H_1 H_2 H_3 \end{cases}$$

The conditions (35) and (36) become

$$F_i(s_i) \neq 0, \quad G_i(s_i) \neq 0, \quad H_i(s_i) \neq 0 \quad \text{for all} \quad s_i \in I_i, \quad i = 1, 2, 3$$

and

$$F_i' \neq 0, \quad G_i' \neq 0, \quad H_i' \neq 0 \quad \text{for all} i = 1, 2, 3.$$

For the orthogonality conditions (37) we obtain

$$\begin{aligned} F_1' F_2' F_3 + G_1' G_2' G_3 + H_1' H_2' H_3 &= 0, \\ F_1 F_2' F_3' + G_1 G_2' G_3' + H_1 H_2' H_3' &= 0, \\ F_1' F_2 F_3' + G_1' G_2 G_3' + H_1' H_2 H_3' &= 0. \end{aligned} \tag{38}$$

To show that $\boldsymbol{x}$ is a confocal coordinate system we show that three equations

$$\frac{F_1 F_2 F_3}{\tilde{a}_i(s_i)} + \frac{G_1 G_2 G_3}{\tilde{b}_i(s_i)} + \frac{H_1 H_2 H_3}{\tilde{c}_i(s_i)} = 1, \quad \text{for} \quad i = 1, 2, 3 \tag{39}$$

hold with some functions $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$, which satisfy

$$\begin{aligned} \tilde{a}_i - \tilde{b}_i = \tilde{a}_j - \tilde{b}_j = \text{const} \neq 0, \\ \tilde{a}_i - \tilde{c}_i = \tilde{a}_j - \tilde{c}_j = \text{const} \neq 0. \end{aligned}$$

for $i, j = 1, 2, 3$, $i \neq j$.

Proof.

- **The orthogonality condition as orthogonality of curves and surfaces**
  Introducing the three curves

  $$\gamma_i(s_i) := \begin{pmatrix} F_i(s_i) \\ G_i(s_i) \\ H_i(s_i) \end{pmatrix}, \quad \text{for} \quad i = 1, 2, 3$$

  and the three surfaces

  $$\Gamma_i(s_j, s_k) := \begin{pmatrix} F_j(s_j) F_k(s_k) \\ G_j(s_j) G_k(s_k) \\ H_j(s_j) H_k(s_k) \end{pmatrix}, \quad \text{for} \quad (ijk) \text{ cyclic permutation of } (123)$$

  the orthogonality conditions (38) can be written as

  $$\langle \gamma_i', \partial_j \Gamma_i \rangle = 0 \quad \text{for} \quad i, j = 1, 2, 3, \quad i \neq j.$$

By Lemma 6.15 the two tangent vectors $\partial_j \Gamma_i$, $\partial_k \Gamma_i$ of the surface Γ_i are linearly independent, and thus, γ_i must lie on a line (while Γ_i must lie in a plane):

$$\gamma_i(s_i) = \alpha_i(s_i) v_i + w_i$$

with some constant vectors $v_i = \begin{pmatrix} v_i^1 \\ v_i^2 \\ v_i^3 \end{pmatrix}, w_i = \begin{pmatrix} w_i^1 \\ w_i^2 \\ w_i^3 \end{pmatrix} \in \mathbb{R}^3$ with

$$v_i^1, v_i^2, v_i^3 \neq 0$$

and non-constant functions $\alpha_i : I_i \to \mathbb{R}$.

By Lemma 6.16 (40), the vectors v_i, w_i satisfy the following additional conditions:[7]

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2, w_3 \rangle = \langle v_1, w_2, v_3 \rangle = \langle w_1, v_2, v_3 \rangle = 0,$$

where $\langle \cdot, \cdot, \cdot \rangle$ denotes the trilinear form

$$\left\langle \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix}, \begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix}, \begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{pmatrix} \right\rangle = v_1^1 v_2^1 v_3^1 + v_1^2 v_2^2 v_3^2 + v_1^3 v_2^3 v_3^3.$$

on $\mathbb{R}^3$.

- **The degenerate cases**

As the reader may verify during the following of the proof, the cases in which one or two of the constants $\langle v_i, w_j, w_k \rangle$ vanish lead to the degenerate cases of confocal coordinates in which two or three of the semi-axis coincide. We exclude these cases from our investigation and only focus on the non-degenerate cases. Thus, we assume

$$\langle v_1, w_2, w_3 \rangle \neq 0, \quad \langle w_1, v_2, w_3 \rangle \neq 0, \quad \langle w_1, w_2, v_3 \rangle \neq 0.$$

- **The quadric equations**

Aiming for (39) we attempt to construct a quadratic equation from

$$\langle v_1, \Gamma_1 \rangle = \langle v_1, \gamma_2, \gamma_3 \rangle = \langle v_1, \alpha_2 v_2 + w_2, \alpha_3 v_3 + w_3 \rangle = \langle v_1, w_2, w_3 \rangle$$

Using $F_1(s_1) \neq 0$, $G_1(s_1) \neq 0$, $H_1(s_1) \neq 0$ we obtain

$$\begin{aligned} \langle v_1, w_2, w_3 \rangle &= v_1^1 F_2 F_3 + v_1^2 G_2 G_3 + v_1^3 H_2 H_3 \\ &= \frac{v_1^1}{F_1} F_1 F_2 F_3 + \frac{v_1^2}{G_1} G_1 G_2 G_3 + \frac{v_1^3}{H_1} H_1 H_2 H_3 \\ &= \frac{v_1^1}{\alpha_1 v_1^1 + w_1^1} F_1 F_2 F_3 + \frac{v_1^2}{\alpha_1 v_1^2 + w_1^2} G_1 G_2 G_3 + \frac{v_1^3}{\alpha_1 v_1^3 + w_1^3} H_1 H_2 H_3. \end{aligned}$$

Treating $\langle v_2, \Gamma_2 \rangle = \langle w_1, v_2, w_3 \rangle$ and $\langle v_3, \Gamma_2 \rangle = \langle w_1, v_2, w_3 \rangle$ in a similar way, we obtain altogether (using $\langle v_i, w_j, w_k \rangle, v_i^1, v_i^2, v_i^3 \neq 0$)

$$\frac{F_1 F_2 F_3}{\tilde{a}_i(s_i)} + \frac{G_1 G_2 G_3}{\tilde{b}_i(s_i)} + \frac{H_1 H_2 H_3}{\tilde{c}_i(s_i)} = 1, \qquad i = 1, 2, 3$$

[7] These generalize the orthogonality condition $\langle v_1, v_2 \rangle = 0$ from the two-dimensional case.

with

$$\tilde{a}_i(s_i) = \langle v_i, w_j, w_k \rangle \left(\alpha_i(s_i) + \frac{w_i^1}{v_i^1} \right),$$

$$\tilde{b}_i(s_i) = \langle v_i, w_j, w_k \rangle \left(\alpha_i(s_i) + \frac{w_i^2}{v_i^2} \right),$$

$$\tilde{c}_i(s_i) = \langle v_i, w_j, w_k \rangle \left(\alpha_i(s_i) + \frac{w_i^3}{v_i^3} \right).$$

where $i, j, k = 1, 2, 3$ distinct.

- **The confocality**
  The differences

$$\tilde{a}_i(s_i) - \tilde{b}_i(s_i) = \langle v_i, w_j, w_k \rangle \left(\frac{w_i^1}{v_i^1} - \frac{w_i^2}{v_i^2} \right)$$

and

$$\tilde{a}_i(s_i) - \tilde{c}_i(s_i) = \langle v_i, w_j, w_k \rangle \left(\frac{w_i^1}{v_i^1} - \frac{w_i^3}{v_i^3} \right)$$

are constant.

Furthermore, by Lemma 6.16 (41), we find that

$$\tilde{a}_i - \tilde{b}_i = \tilde{a}_j - \tilde{b}_j \neq 0, \qquad \tilde{a}_i - \tilde{c}_i = \tilde{a}_j - \tilde{c}_j \neq 0$$

for $i, j = 1, 2, 3$, $i \neq j$,

□

Lemma 6.15. *For $i = 1, 2, 3$ the two tangent vectors $\partial_j \Gamma_i, \partial_k \Gamma_i$ in the proof of Theorem 6.14 are linearly independent.*

Proof. We show this for $i = 1$. The cases $i = 2, 3$ follow analogously. The two tangent vectors under consideration are given by

$$\partial_2 \Gamma_1 = \begin{pmatrix} F_2' F_3 \\ G_2' G_3 \\ H_2' H_3 \end{pmatrix}, \quad \partial_3 \Gamma_1 = \begin{pmatrix} F_2 F_3' \\ G_2 G_3' \\ H_2 H_3' \end{pmatrix}.$$

Multiplying these by the non-singular matrix

$$\begin{pmatrix} 1/F_2F_3 & & \\ & 1/G_2G_3 & \\ & & 1/H_2H_3 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} F_2'/F_2 \\ G_2'/G_2 \\ H_2'/H_2 \end{pmatrix} = 2 \begin{pmatrix} f_2'/f_2 \\ g_2'/g_2 \\ g_2'/g_2 \end{pmatrix}, \quad \begin{pmatrix} F_3'/F_3 \\ G_3'/G_3 \\ H_3'/H_3 \end{pmatrix} = 2 \begin{pmatrix} f_3'/f_3 \\ g_3'/g_3 \\ g_3'/g_3 \end{pmatrix},$$

Now multiplying by the non-singular matrix

$$\frac{1}{2} \begin{pmatrix} f_1 f_2 f_3 & & \\ & g_1 g_2 g_3 & \\ & & h_1 h_2 h_3 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} f_1 f_2' f_3 \\ g_1 g_2' g_3 \\ h_1 h_2' h_3 \end{pmatrix} = \partial_2 \boldsymbol{x}, \quad \begin{pmatrix} f_1 f_2 f_3' \\ g_1 g_2 g_3' \\ h_1 h_2 h_3' \end{pmatrix} = \partial_3 \boldsymbol{x},$$

which are linearly independent since $\boldsymbol{x}$ is a coordinate system. □

Lemma 6.16. *The vectors v_i, w_i, $i = 1, 2, 3$ in the proof of Theorem 6.14 satisfy*

$$\langle v_1, v_2, v_3\rangle = \langle v_1, v_2, w_3\rangle = \langle v_1, w_2, v_3\rangle = \langle w_1, v_2, v_3\rangle = 0, \tag{40}$$

and consequently

$$\langle v_i, w_j, w_k\rangle \left(\frac{w_i^l}{v_i^l} - \frac{w_i^m}{v_i^m}\right) = \langle v_j, w_i, w_k\rangle \left(\frac{w_j^l}{v_j^l} - \frac{w_j^m}{v_j^m}\right) \neq 0 \tag{41}$$

for $i, j, k = 1, 2, 3$ distinct and $l, m = 1, 2, 3$.

Proof. The orthogonality conditions (38) can also be written as

$$\partial_i \partial_j \langle \gamma_1, \gamma_2, \gamma_3\rangle = 0 \quad \text{for} \quad i, j = 1, 2, 3 \quad i \neq j$$

which is equivalent to

$$\langle \gamma_1, \gamma_2, \gamma_3\rangle = A_1(s_1) + A_2(s_2) + A_3(s_3)$$

with three functions A_1, A_2, A_3. Substituting $\gamma_i(s_i) = \alpha_i(s_i)v_i + w_i$ into $\langle \gamma_1, \gamma_2, \gamma_3\rangle$ we obtain

$$\begin{aligned}\langle \alpha_1 v_1 + w_1, \alpha_2 v_2 + w_2, \alpha_3 v_3 + w_3\rangle = \\ \alpha_1\alpha_2\alpha_3 \langle v_1, v_2, v_3\rangle + \alpha_1\alpha_2 \langle v_1, v_2, w_3\rangle + \alpha_2\alpha_3 \langle w_1, v_2, v_3\rangle + \alpha_3\alpha_1 \langle v_1, w_2, v_3\rangle \\ +\alpha_1 \langle v_1, w_2, w_3\rangle + \alpha_2 \langle w_1, v_2, w_3\rangle + \alpha_3 \langle w_1, w_2, v_3\rangle + \langle w_1, w_2, w_3\rangle .\end{aligned}$$

Since α_i are non-constant functions, the right-hand side is a sum of functions each depending only on one variable, if and only if (40).

Thus, we have

$$\begin{aligned}&\langle \alpha_1 v_1 + w_1, \alpha_2 v_2 + w_2, \alpha_3 v_3 + w_3\rangle \\ &\qquad = \alpha_1 \langle v_1, w_2, w_3\rangle + \alpha_2 \langle w_1, v_2, w_3\rangle + \alpha_3 \langle w_1, w_2, v_3\rangle + \langle w_1, w_2, w_3\rangle .\end{aligned}$$

Since α_i is a non-constant function, this identity holds for two different values of α_i, and therefore, by linearity, for α_i being replaced by any real number, i.e.,

$$\begin{aligned}&\langle \lambda_1 v_1 + w_1, \lambda_2 v_2 + w_2, \lambda_3 v_3 + w_3\rangle \\ &\qquad = \lambda_1 \langle v_1, w_2, w_3\rangle + \lambda_2 \langle w_1, v_2, w_3\rangle + \lambda_3 \langle w_1, w_2, v_3\rangle + \langle w_1, w_2, w_3\rangle .\end{aligned}$$

for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. In particular, for

$$\lambda_1 = -\frac{w_1^l}{v_1^l}, \quad \lambda_2 = -\frac{w_2^m}{v_2^m}, \quad \lambda_1 = -\frac{w_1^n}{v_1^n}$$

with $l, m, n = 1, 2, 3$ distinct, which leads to

$$0 = \frac{w_1^l}{v_1^l} \langle v_1, w_2, w_3\rangle + \frac{w_2^m}{v_2^m} \langle w_1, v_2, w_3\rangle + \frac{w_3^n}{v_3^n} \langle w_1, w_2, v_3\rangle - \langle w_1, w_2, w_3\rangle .$$

Taking the differences of any pair of these equations leads to the qualities in (41).

Now assume that

$$\langle v_i, w_j, w_k\rangle \left(\frac{w_i^1}{v_i^1} - \frac{w_i^2}{v_i^2}\right) = 0$$

for one and therefore all $i = 1, 2, 3$. Then, by the assumptions $\langle v_i, w_j, w_k \rangle \neq 0$ and $v_i^1, v_i^2 \neq 0$, this is equivalent to

$$\det \begin{pmatrix} v_i^1 & w_i^1 \\ v_i^2 & w_i^2 \end{pmatrix} = 0.$$

Since

$$F_i = \alpha_i v_i^1 + w_i^1, \qquad G_i = \alpha_i v_i^2 + w_i^2,$$

this implies that F_i and G_i are proportional for all $i = 1, 2, 3$, and therefore,

$$\frac{x}{y} = \text{const},$$

which contradicts $\boldsymbol{x}$ being a coordinate system. Similarly, if any of the other terms $\langle v_i, w_j, w_k \rangle \left(\frac{w_i^l}{v_i^l} - \frac{w_i^m}{v_i^m} \right)$ vanishes. $\square$

6.4 Generalization to $\mathbb{R}^N$

A normal form for confocal quadrics in $\mathbb{R}^N$ is given by

$$\mathcal{Q}_\lambda = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^N \frac{x_i^2}{a_i + \lambda} = 1 \right\}, \qquad \lambda \in \mathbb{R}$$

with some $a_1 > a_2 > \cdots > a_N$. With this, all claims about confocal quadrics from this section (including their proofs) easily generalize to arbitrary dimension.

Theorem 6.17. *The family of confocal quadrics is a dual pencil of quadrics.*

Theorem 6.18. *If two confocal quadrics intersect, they intersect orthogonally.*

For a point $(x_1, \ldots, x_N) \in \mathbb{R}^N$ not on the coordinate hyperplanes $(x_1 \cdot \ldots \cdot x_N \neq 0)$, the equation

$$\sum_{i=1}^N \frac{x_i^2}{a_i + \lambda} = 1$$

has N real roots $u_1, \ldots, u_N$ lying in the intervals

$$-a_1 < u_1 < -a_2 < u_2 < \ldots, < -a_N < u_N.$$

These N roots correspond to N confocal quadrics (of different type) of the family $\mathcal{Q}_\lambda$:

$$\begin{aligned} (x_1, \ldots, x_N) \in \bigcap_{i=1}^N \mathcal{Q}_{u_i} \Leftrightarrow \quad & \sum_{k=1}^N \frac{x_k^2}{a_k + u_i} = 1, \qquad i = 1, \ldots, N \\ \Leftrightarrow \quad & x_k^2 = \frac{\prod_{i=1}^N (u_i + a_k)}{\prod_{i \neq k} (a_k - a_i)}, \quad k = 1, \ldots, N. \end{aligned}$$

Theorem 6.19. *Through every point $(x_1, \ldots, x_N) \in \mathbb{R}^N$ not on the coordinate hyperplanes $(x_1 \cdot \ldots \cdot x_N \neq 0)$, there passes exactly one quadric of each type from the confocal family $\mathcal{Q}_\lambda$.*

Theorem 6.20. *N quadrics of different type from a confocal family $\mathcal{Q}_\lambda$ intersect in exactly 2^N points, which lie mirror symmetric with respect to the common principal hyperplanes.*

Definition 6.5. A coordinate system $\boldsymbol{x} : \mathbb{R}^N \supset U \to \mathbb{R}^N$ is called a *confocal coordinate system* if its coordinate hyperplanes $\boldsymbol{x}(s_i = \text{const})$, $i = 1, \ldots, N$, are contained in confocal quadrics.

Theorem 6.21. *Let*

$$\boldsymbol{x} : \mathbb{R}^N \supset I_1 \times \ldots \times I_N \to \mathbb{R}^N, \quad \boldsymbol{x}(s_1, \ldots, s_N) = \begin{pmatrix} x_1(s_1, \ldots, s_N) \\ \vdots \\ x_N(s_1, \ldots, s_N) \end{pmatrix}$$

be a coordinate system. Then $\boldsymbol{x}$ is a confocal coordinate system if and only if there exist functions

$$f_i^k : I_i \to \mathbb{R}, \qquad i, k = 1, \ldots, N,$$

with

$$\begin{cases} f_i^1(s_i)^2 - f_i^k(s_i)^2 = a_1 - a_k, & k \leqslant i, \\ f_i^1(s_i)^2 + f_i^k(s_i)^2 = a_1 - a_k, & k > i, \end{cases}$$

such that

$$x_k(s_1, \cdots, s_N) = \frac{\prod_{i=1}^N f_i^k(s_i)}{\prod_{i \neq k}(a_k - a_i)}.$$

Theorem 6.22. *Let*

$$\boldsymbol{x} : \mathbb{R}^N \supset I_1 \times \ldots \times I_N \to \mathbb{R}^N, \quad \boldsymbol{x}(s_1, \ldots, s_N) = \begin{pmatrix} x_1(s_1, \ldots, s_N) \\ \vdots \\ x_N(s_1, \ldots, s_N) \end{pmatrix}$$

be a coordinate system that satisfies the following two conditions:

(i) $\boldsymbol{x}$ *factorizes:*

$$x_k(s_1, \ldots, s_N) = \prod_{i=1}^N f_i^k(s_i), \qquad k = 1, \ldots, N$$

with some smooth functions $f_i^k : I_i \to \mathbb{R}, i = 1, \ldots, N$ that do not vanish:[8]

$$f_i^k(s_i) \neq 0, \textit{for all} \quad s_i \in I_i, \quad i = 1, \ldots, N$$

and whose derivatives do not constantly vanish:[9]

$$(f_i^k)' \neq 0, \quad \textit{for all} \quad i = 1, \ldots, N$$

(ii) $\boldsymbol{x}$ *is orthogonal:*

$$\langle \partial_i \boldsymbol{x}, \partial_j \boldsymbol{x} \rangle = 0 \quad \textit{for all} \quad i, j = 1, \ldots, N \quad i \neq j$$

Then $\boldsymbol{x}$ is a confocal coordinate system (including degenerate cases in which some of the semi-axes coincide).

[8] By this, we consider the coordinate system away from the coordinate hyperplanes.

[9] By this, we exclude the case of subnets that are contained in affine subspaces, which leads to square grids and generalized cylindrical coordinates.

7 Discrete confocal quadrics

To obtain a discretization of confocal quadrics we apply the characterizing properties from Theorem 6.22 for (smooth) confocal coordinates to Definition 2.3 of discrete orthogonal nets.

Thus, consider applying the factorizability condition

$$x_k(\boldsymbol{n}) = f_1^k(n_1) f_2^k(n_2) \cdots f_N^k(n_N), \qquad k = 1, \dots, N,$$

to an orthogonal pair of dual discrete nets $\boldsymbol{x} : \mathbb{Z}^N \cup \left(\mathbb{Z} + \frac{1}{2}\right)^N \to \mathbb{R}^N$ defined on the dual pair of square lattices $\mathbb{Z}^N$ and $\left(\mathbb{Z} + \frac{1}{2}\right)^N$. Then the functions f_i^k must each be defined on $\frac{1}{2}\mathbb{Z}$, and thus, the net $\boldsymbol{x}$ can be extended to all of $\left(\frac{1}{2}\mathbb{Z}\right)^N$. The two dual lattices $\mathbb{Z}^M$ and $\left(\mathbb{Z} + \frac{1}{2}\right)^M$ are just one pair of dual sublattices of $\left(\frac{1}{2}\mathbb{Z}\right)^M$. More generally we call two lattices

$$\mathbb{Z}^M + \tfrac{1}{2}\boldsymbol{\delta}, \qquad \mathbb{Z}^M + \tfrac{1}{2}\bar{\boldsymbol{\delta}},$$

a *pair of dual sublattices* of $\left(\frac{1}{2}\mathbb{Z}\right)^M$, where

$$\boldsymbol{\delta} = (\delta_1, \dots, \delta_M) \in \{0,1\}^M, \qquad \bar{\boldsymbol{\delta}} = (1-\delta_1, \dots, 1-\delta_M) \in \{0,1\}^M.$$

The stepsize $\frac{1}{2}$ square lattice $\left(\frac{1}{2}\mathbb{Z}\right)^M$ has 2^{M-1} such pairs of dual sublattices.

Definition 7.1.

(i) A map

$$\boldsymbol{x} : \left(\tfrac{1}{2}\mathbb{Z}\right)^M \to \mathbb{R}^N$$

is called a *stepsize $\frac{1}{2}$ discrete net.*

(ii) A stepsize $\frac{1}{2}$ discrete net is called *regular* if all of its 2^M (stepsize 1) discrete subnets are regular.

(iii) A stepsize $\frac{1}{2}$ discrete net is called *orthogonal* if all of its 2^{M-1} pairs of dual discrete subnets are orthogonal.

Remark 7.1.

(i) For a general stepsize $\frac{1}{2}$ discrete net, the discrete orthogonality constraint (2) only correlates the two nets from each pair of dual discrete subnets. The 2^{M-1} different pairs of dual discrete subnets are not mutually correlated by this condition unless an additional constraint, like the factorizability, is introduced.

(ii) Each of the 2^{M-1} different pairs of dual discrete subnets leads to a different definition of discrete Lamé coefficients on the lattice $\left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right)^M$. In general these do not coincide.

Theorem 7.1. *Let*

$$\boldsymbol{x} : \left(\tfrac{1}{2}\mathbb{Z}\right)^N \supset \mathcal{U} = I_1 \times \ldots \times I_N \to \mathbb{R}^N, \quad \boldsymbol{x}(s_1, \dots, s_N) = \begin{pmatrix} x_1(\boldsymbol{n}) \\ \vdots \\ x_N(\boldsymbol{n}) \end{pmatrix}$$

be a discrete coordinate system that satisfies the following two conditions:

(i) $\boldsymbol{x}$ *factorizes:*

$$x_k(\boldsymbol{n}) = \prod_{i=1}^{N} f_i^k(n_i), \qquad k = 1, \dots, N$$

with some smooth functions $f_i^k : I_i \to \mathbb{R}, i = 1, \dots, N$ *that do not vanish:*

$$f_i^k(n_i) \neq 0, \quad \textit{for all} \quad n_i \in I_i, \quad i = 1, \dots, N \tag{42}$$

and whose differences do not constantly vanish:

$$\Delta(f_i^k) \neq 0, \quad \textit{for all} \quad i = 1, \dots, N \tag{43}$$

(ii) $\boldsymbol{x}$ *is orthogonal, in the sense of Definition 7.1.*

Then there exist $a_1, \dots, a_N \in \mathbb{R}$ *and sequences* $u_i : \left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right) \to \mathbb{R}$, *such that*

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n}) x_k(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})}{a_k + u_i} = 1, \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i), \qquad i = 1, \dots, N \tag{44}$$

for any $\boldsymbol{n} \in \mathcal{U}$ *and* $\boldsymbol{\sigma} \in \{\pm 1\}^N$ *(apart from degenerate cases which arise as limits in which two or more of the values* a_k *coincide).*

As the most instructive example, we will proof the Theorem for $N = 3$. Thus, we denote

$$f_i := f_i^1, \quad g_i := f_i^2, \quad h_i := f_i^3, \quad \text{for} \quad i = 1, 2, 3.$$

and introduce the "discrete squares"

$$F_i(n_i + \tfrac{1}{4}) := f_i(n_i) f_i(n_i + \tfrac{1}{2}), \quad G_i(n_i + \tfrac{1}{4}) := g_i(n_i) g_i(n_i + \tfrac{1}{2}), \quad H_i(n_i + \tfrac{1}{4}) := h_i(n_i) h_i(n_i + \tfrac{1}{2}),$$

for $n_i \in I_i$ and $i = 1, 2, 3$. With this we have, e.g.,

$$\begin{cases} x(\boldsymbol{n}) x(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = x_1(\boldsymbol{n}) x_1(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = F_1 F_2 F_3 \\ y(\boldsymbol{n}) y(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = x_2(\boldsymbol{n}) x_2(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = G_1 G_2 G_3 \\ z(\boldsymbol{n}) z(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = x_3(\boldsymbol{n}) x_3(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma}) = H_1 H_2 H_3 \end{cases}$$

with $\boldsymbol{\sigma} = (1, 1, 1)$. The conditions (42) and (43) become

$$F_i(n_i) \neq 0, \quad G_i(n_i) \neq 0, \quad H_i(n_i) \neq 0 \quad \text{for all} \quad n_i \in I_i, \quad i = 1, 2, 3$$

and

$$\Delta^{1/2} F_i \neq 0, \quad \Delta^{1/2} G_i \neq 0, \quad \Delta^{1/2} H_i \neq 0 \quad \text{for all} \quad i = 1, 2, 3,$$

where

$$\Delta^{1/2} F(n) = F(n + \tfrac{1}{2}) - F(n).$$

For the orthogonality conditions (21) we obtain

$$\begin{aligned} (\Delta^{1/2} F_1)(\Delta^{1/2} F_2) F_3 + (\Delta^{1/2} G_1)(\Delta^{1/2} G_2) G_3 + (\Delta^{1/2} H_1)(\Delta^{1/2} H_2) H_3 &= 0, \\ F_1 (\Delta^{1/2} F_2)(\Delta^{1/2} F_3) + G_1 (\Delta^{1/2} G_2)(\Delta^{1/2} G_3) + H_1 (\Delta^{1/2} H_2)(\Delta^{1/2} H_3) &= 0, \\ (\Delta^{1/2} F_1) F_2 (\Delta^{1/2} F_3) + (\Delta^{1/2} G_1) G_2 (\Delta^{1/2} G_3) + (\Delta^{1/2} H_1) H_2 (\Delta^{1/2} H_3) &= 0. \end{aligned} \tag{45}$$

For (44) we show:

$$\frac{F_1F_2F_3}{\tilde{a}_i(n_i)} + \frac{G_1G_2G_3}{\tilde{b}_i(n_i)} + \frac{H_1H_2H_3}{\tilde{c}_i(n_i)} = 1, \quad \text{for} \quad i = 1,2,3$$

hold with some functions $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$, which satisfy

$$\tilde{a}_i - \tilde{b}_i = \tilde{a}_j - \tilde{b}_j = \text{const} \neq 0,$$
$$\tilde{a}_i - \tilde{c}_i = \tilde{a}_j - \tilde{c}_j = \text{const} \neq 0.$$

for $i, j = 1,2,3$, $i \neq j$. This now looks identical to the smooth case

Proof for $N = 3$.

- **The orthogonality condition as orthogonality of discrete curves and surfaces**
  Introducing the three discrete curves

  $$\gamma_i(n_i) := \begin{pmatrix} F_i(n_i) \\ G_i(n_i) \\ H_i(n_i) \end{pmatrix}, \quad \text{for} \quad i = 1,2,3$$

  and the three discrete surfaces

  $$\Gamma_i(n_j, n_k) := \begin{pmatrix} F_j(n_j)F_k(n_k) \\ G_j(n_j)G_k(n_k) \\ H_j(n_j)H_k(n_k) \end{pmatrix}, \quad \text{for} \quad (ijk) \text{ cyclic permutation of } (123)$$

  the orthogonality conditions (45) can be written as

  $$\left\langle \Delta^{1/2}\gamma_i, \Delta_j^{1/2}\Gamma_i \right\rangle = 0 \quad \text{for} \quad i, j = 1,2,3, \quad i \neq j.$$

  By Lemma 7.2 the two discrete tangent vectors $\Delta_j^{1/2}\Gamma_i$, $\Delta_k^{1/2}\Gamma_i$ of the discrete surface Γ_i are linearly independent, and thus, γ_i must lie on a line (while Γ_i must lie in a plane):

  $$\gamma_i(n_i) = \alpha_i(n_i)v_i + w_i$$

  with some non-constant functions $\alpha_i : I_i \to \mathbb{R}$. and constant vectors $v_i w_i \in \mathbb{R}^3$ with $v_i^1, v_i^2, v_i^3 \neq 0$, which satisfy some further conditions according to Lemma 7.3.

- **The degenerate cases, quadric equations, and confocality**
  The remainder of the proof (deriving the quadratic equations and showing that they belong to a confocal family) is identical to the proof of the smooth case (Theorem 6.14).

□

Lemma 7.2. *For $i = 1,2,3$ the two difference vectors $\Delta_j^{1/2}\Gamma_i, \Delta_k^{1/2}\Gamma_i$ in the proof of Theorem 7.1 are linearly independent.*

Proof. Same as Lemma 6.15 upon replacing F_i' by $\Delta^{1/2}F_i$ and f_i' by Δf_i etc. □

Lemma 7.3. *The vectors v_i, w_i, $i = 1,2,3$ in the proof of Theorem 7.1 satisfy*

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2, w_3 \rangle = \langle v_1, w_2, v_3 \rangle = \langle w_1, v_2, v_3 \rangle = 0,$$

and consequently

$$\langle v_i, w_j, w_k \rangle \left(\frac{w_i^l}{v_i^l} - \frac{w_i^m}{v_i^m} \right) = \langle v_j, w_i, w_k \rangle \left(\frac{w_j^l}{v_j^l} - \frac{w_j^m}{v_j^m} \right) \neq 0$$

for $i, j, k = 1,2,3$ distinct and $l, m = 1,2,3$.

Proof. Same as Lemma 6.16 upon replacing $\partial_i\partial_j \langle\gamma_1, \gamma_2, \gamma_3\rangle = 0$ by $\Delta_i\Delta_j \langle\gamma_1, \gamma_2, \gamma_3\rangle = 0$. ☐

Theorem 7.1 motivates the following definition for discrete confocal quadrics:

Definition 7.2. A discrete coordinate system $\boldsymbol{x} : \left(\frac{1}{2}\mathbb{Z}\right)^N \supset \mathcal{U} \to \mathbb{R}^N$ is called a *discrete confocal coordinate system* if there exist $a_1, \ldots, a_N \in \mathbb{R}$, and sequences $u_i : \left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right) \to \mathbb{R}$, $i = 1, \ldots, N$ such that

$$\sum_{k=1}^{N} \frac{x_k(\boldsymbol{n})x_k(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})}{a_k + u_i} = 1, \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i), \qquad i = 1, \ldots, N \tag{46}$$

for any $\boldsymbol{n} \in \mathcal{U}$ and $\boldsymbol{\sigma} \in \{\pm 1\}^N$, or equivalently,

$$x_k(\boldsymbol{n})x_k(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma}) = \frac{\prod_{j=1}^{N}(u_j + a_k)}{\prod_{j\neq k}(a_k - a_j)}, \quad u_j = u_j(n_j + \tfrac{1}{4}\sigma_j), \qquad k = 1, \ldots, N.$$

for any $\boldsymbol{n} \in \mathcal{U}$ and $\boldsymbol{\sigma} \in \{\pm 1\}^N$.

Remark 7.2.

(i) Different choices for the discrete functions u_i lead to different "discrete reparametrizations" of the system of confocal coordinates.

(ii) By relabeling, we can assume $a_1 > \ldots > a_N$. A reasonable additional condition on discrete confocal quadrics is to require the sequences u_i to lie in the intervals

$$-a_1 < u_1 < -a_2 < u_2 < \ldots < -a_N < u_N.$$

Discrete confocal coordinates admit the following geometric interpretation via polarity with respect to sequences of classical confocal quadrics.

Theorem 7.4. *Let $\boldsymbol{n} \in \mathcal{U}$ and $\boldsymbol{\sigma} \in \{\pm 1\}^N$. Then the two adjacent points $\boldsymbol{x}(\boldsymbol{n}), \boldsymbol{x}(\boldsymbol{n} + \frac{1}{2}\boldsymbol{\sigma})$ of a discrete confocal coordinate system are related by polarity with respect to the N confocal quadrics*

$$\sum_{k=1}^{N} \frac{x_k^2}{a_k + u_i} = 1, \quad u_i = u_i(n_i + \tfrac{1}{4}\sigma_i), \qquad i = 1, \ldots, N$$

Proof. Equation (46) describes exactly the stated polarity relation. ☐

This yields a geometric construction for one pair of discrete dual nets from a system of discrete confocal coordinates:

- Choose a family of classical confocal quadrics $\mathcal{Q}_\lambda$ (choice of $a_1, \ldots, a_N \in \mathbb{R}$).
- Sample each subfamily arbitrarily (choice of sequences $u_i(\frac{1}{2}n_i + \frac{1}{4})$).
- Choose one point (per pair of dual subnets) and propagate by polarity with respect to the quadrics $\mathcal{Q}_{u_i}$

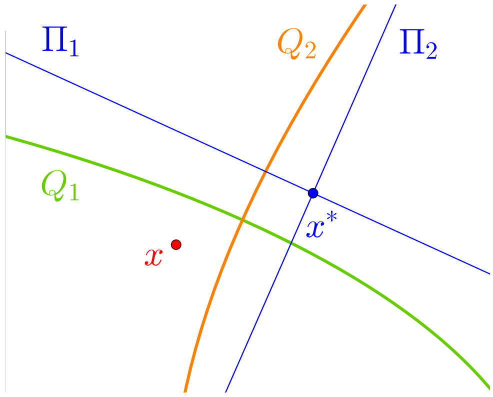

Figure 24. Geometric construction of the point $\boldsymbol{x}^*$ in the case $N = 2$ as the intersection of the polar lines Π_1 and Π_2 of $\boldsymbol{x}$ with respect to $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Exercise 7.1. Show that this geometric construction always closes (is independent of the path).

Exercise 7.2. Show that for a discrete confocal coordinate system the discrete Lamé coefficients defined by the different pairs of dual discrete subnets coincide on $\left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right)^N$, i.e, at each $\boldsymbol{n} \in \left(\frac{1}{2}\mathbb{Z}\right)^N$ and for $i = 1, ..., N$ the 2^{N-1} scalar products

$$\left\langle \Delta_i \boldsymbol{x}(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\delta}), \bar{\Delta}_i \boldsymbol{x}(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{\sigma} + \tfrac{1}{2}\bar{\boldsymbol{\delta}}) \right\rangle$$

are equal for all $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_M) \in \{0,1\}^M$ and $\bar{\boldsymbol{\delta}} = (1-\delta_1, \ldots, 1-\delta_M)$ and $\boldsymbol{\sigma} = (1, \ldots, 1)$. They are given by

$$H_i^2(\boldsymbol{n}) = \left(u_i(n_i + \tfrac{1}{2}) - u_i(n_i)\right)\left(u_i(n_i) - u_i(n_i - \tfrac{1}{2})\right) \frac{\prod_{j \neq i} (u_i(n_i) - u_j(n_j))}{\prod_{k=1}^{N} (u_i(n_i) - a_k)}.$$

This resembles the property derived in Exercise 6.4 which describes that quadrics are isothermic.

If we rescale the functions f_i^k we obtain:

Theorem 7.5. *Let $a_1, \ldots, a_N \in \mathbb{R}$ and $u_i : \left(\frac{1}{2}\mathbb{Z} + \frac{1}{4}\right) \to \mathbb{R}$, $i = 1, \ldots, N$ some sequences Let $f_i^k : \frac{1}{2}\mathbb{Z} \to \mathbb{R}$ be solutions of the difference equations*

$$f_i^k(n_i) f_i^k(n_i + \tfrac{1}{2}) = \begin{cases} u_i(n_i + \tfrac{1}{4}) + a_k, & k \leqslant i, \\ -\left(u_i(n_i + \tfrac{1}{4}) + a_k\right), & k > i. \end{cases} \tag{47}$$

Then, $\boldsymbol{x}$ defined by

$$x_k(\boldsymbol{n}) = \frac{\prod_{i=1}^{N} f_i^k(n_i)}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^{N} \sqrt{a_k - a_i}}$$

is a discrete confocal coordinate system.

Remark 7.3. If u_i are chosen such that

$$-a_1 < u_1 < -a_2 < u_2 < \ldots < -a_N < u_N,$$

all discrete squares of f_i^k and therefore of x_k are positive:

$$f_i^k(n_i) f_i^k(n_i + \tfrac{1}{2}) > 0.$$

By eliminating the sequences u_i from (47), we obtain a characterization of discrete confocal coordinates similar to Theorem 6.13:

Theorem 7.6. *Let*

$$f_i^k : \tfrac{1}{2}\mathbb{Z} \to \mathbb{R}, \qquad i, k = 1, \dots, N$$

be functions satisfying

$$\begin{aligned} f_i^1(n_i) f_i^1(n_i + \tfrac{1}{2}) - f_i^k(n_i) f_i^k(n_i + \tfrac{1}{2}) &= a_1 - a_k, \quad k \leqslant i, \\ f_i^1(n_i) f_i^1(n_i + \tfrac{1}{2}) + f_i^k(n_i) f_i^k(n_i + \tfrac{1}{2}) &= a_1 - a_k, \quad k > i. \end{aligned}$$

Then, $\boldsymbol{x}$ defined by

$$x_k(\boldsymbol{n}) = \frac{\prod_{i=1}^N f_i^k(n_i)}{\prod_{i=1}^{k-1} \sqrt{a_i - a_k} \prod_{i=k+1}^N \sqrt{a_k - a_i}}$$

is a discrete confocal coordinate system.

7.1 Discrete confocal coordinates in terms of Γ-functions

The parametrization of smooth confocal coordinates in terms of square roots (31) was characterized by taking the quantities u_i as coordinates with further reparametrization, i.e., choosing the function $u_i(s_i) = s_i$. To derive a discrete version of this parametrization we set

$$u_i(n_i + \tfrac{1}{4}) = n_i + \varepsilon_i, \quad i = 1, \dots, N,$$

where $\varepsilon_i \in \mathbb{R}$ are some fixed shifts. With this choice, equations (47) turn into

$$f_i^k(n_i) f_i^k(n_i + \tfrac{1}{2}) = \begin{cases} n_i + a_k + \varepsilon_i, & k \leqslant i, \\ -\big(n_i + a_k + \varepsilon_i\big), & k > i. \end{cases} \tag{48}$$

These equations can be solved in terms of Γ-functions.

The Γ-function is given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

It solves the functional equation

$$\Gamma(x+1) = x\Gamma(x)$$

with initial value $\Gamma(1) = 1$, and can be taken as an analytic continuation of the factorial function:

$$\Gamma(n) = (n-1)!, \quad \text{for} \quad n \in \mathbb{N}.$$

We use the gamma function to define a "discrete square root" by[10]

$$(u)_{1/2} = \frac{\Gamma(u + \frac{1}{2})}{\Gamma(u)},$$

which satisfies the identities

$$(u)_{1/2} (u + \tfrac{1}{2})_{1/2} = u, \quad (-u)_{1/2} (-u - \tfrac{1}{2})_{1/2} = -u - \tfrac{1}{2}.$$

[10]The Pochhammer symbol is more generally defined by $(u)_\delta = \frac{\Gamma(u+\delta)}{\Gamma(u)}$.

With this we can write solutions of (48) as

$$f_i^k(n_i) = \begin{cases} (n_i + a_k + \varepsilon_i)_{1/2} & \text{for} \quad i \geqslant k, \\ (-n_i - a_k - \varepsilon_i + \frac{1}{2})_{1/2} & \text{for} \quad i < k. \end{cases}$$

One can impose boundary conditions

$$\begin{aligned} x_k|_{n_k=-\alpha_k} &= 0 \quad \text{for} \quad k = 1, \dots, N, \\ x_k|_{n_{k-1}=-\alpha_k} &= 0 \quad \text{for} \quad k = 2, \dots, N, \end{aligned}$$

on the integer lattice $\mathbb{Z}^N$ for certain integers $\alpha_1 > \cdots > \alpha_N$, which imitate the corresponding property of the continuous confocal coordinates. These boundary conditions are satisfied provided that

$$a_k - \alpha_k + \varepsilon_k = 0, \quad a_k - \alpha_k + \varepsilon_{k-1} = \frac{1}{2},$$

for which the shifts ε_k should satisfy $\varepsilon_{k-1} - \varepsilon_k = \frac{1}{2}$. Choosing $\varepsilon_k = -\frac{k}{2}$ and $a_k = \alpha_k + \frac{k}{2}$, we finally arrive at the solutions

$$f_i^k(n_i) = \begin{cases} (n_i + \alpha_k + \frac{k-i}{2})_{1/2} & \text{for} \quad i \geqslant k, \\ (-n_i - \alpha_k - \frac{k-i}{2} + \frac{1}{2})_{1/2} & \text{for} \quad i < k. \end{cases} \tag{49}$$

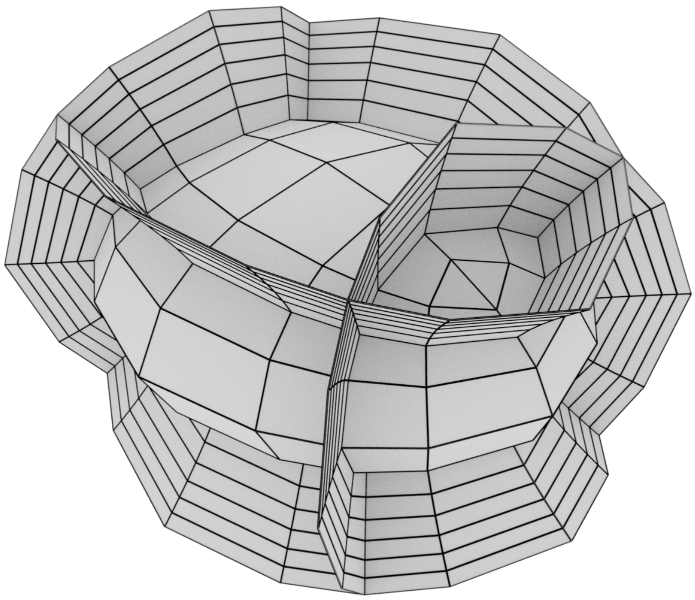

Figure 25. Three discrete confocal quadrics as part of a (stepsize 1) subnet of a discrete confocal coordinate system (49) in $\mathbb{R}^3$ given in terms of Γ-functions.

7.2 Discrete confocal conics in terms of trigonometric functions

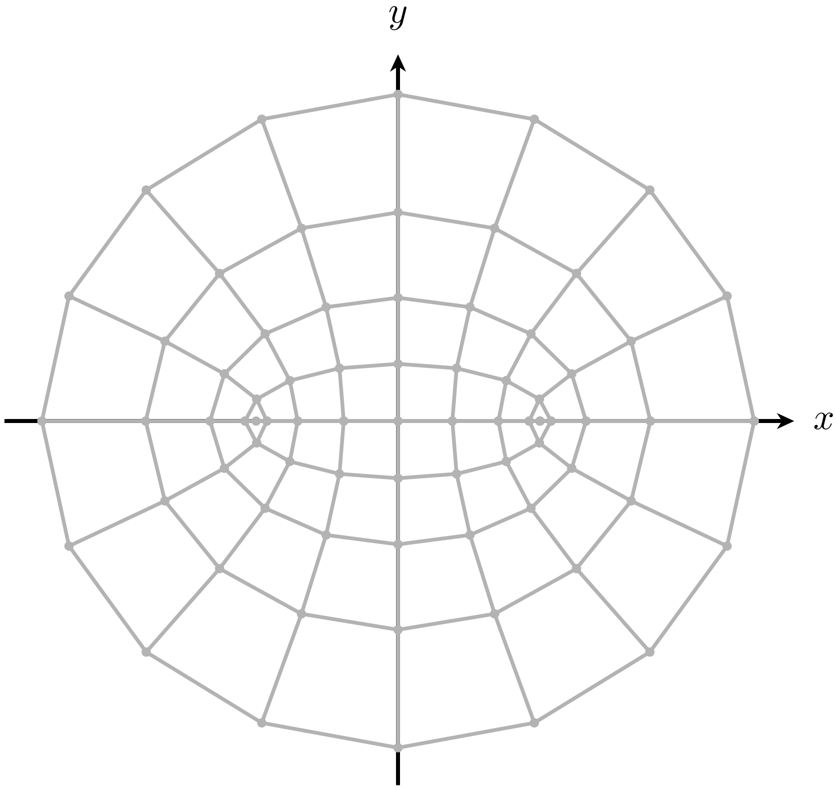

Figure 26. Two-dimensional discrete confocal coordinate system on $\left(\frac{1}{2}\mathbb{Z}\right)^2$ in terms of trigonometric functions.

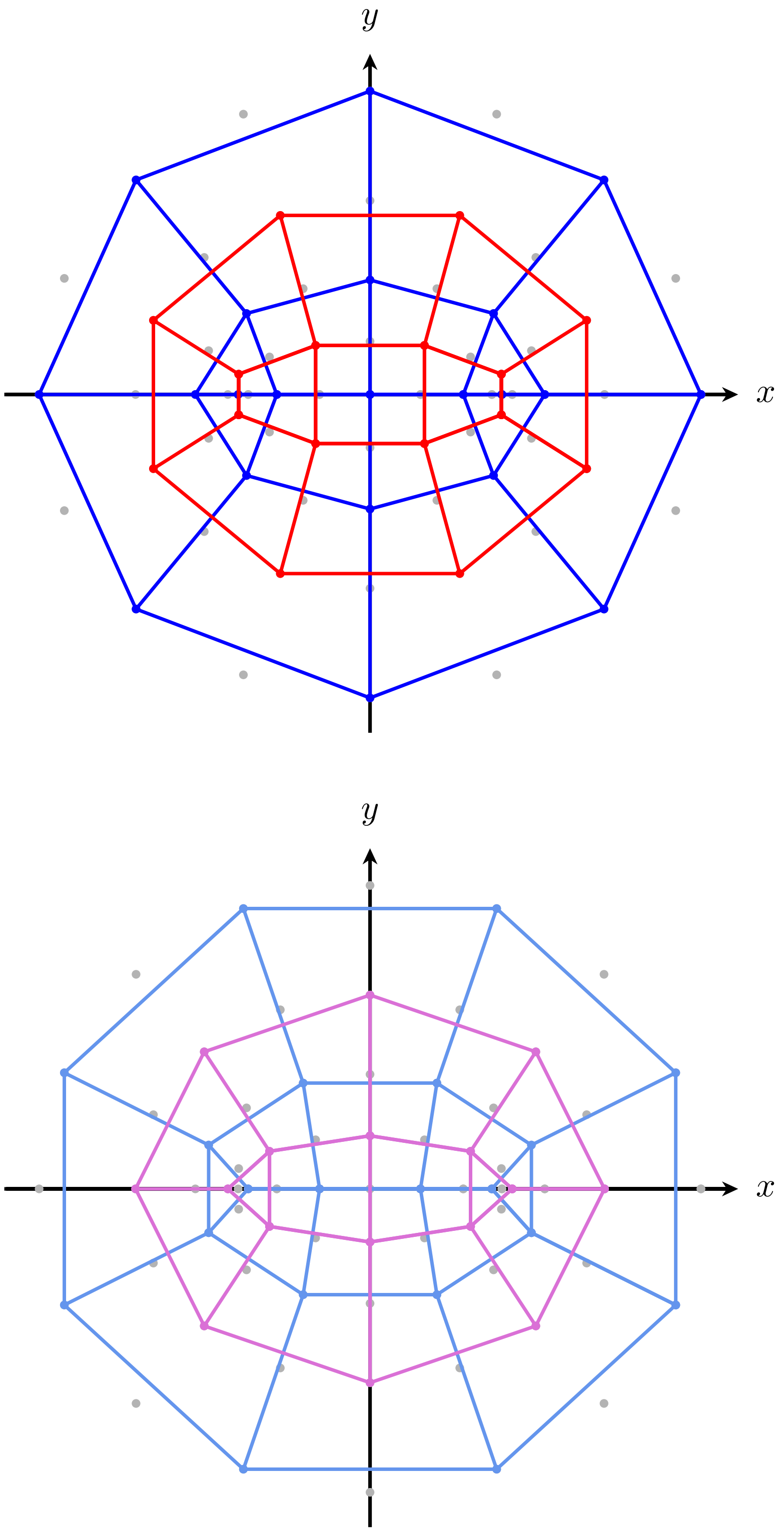

Figure 27. Pairs of dual orthogonal sublattices. (top) Sublattice on $\mathbb{Z}^2$ in blue and on $\left(\mathbb{Z}+\frac{1}{2}\right)^2$ in red. (bottom) Sublattice on $\mathbb{Z}\times(\mathbb{Z}+\frac{1}{2})$ in blue and on $(\mathbb{Z}+\frac{1}{2})\times\mathbb{Z}$ in pink.

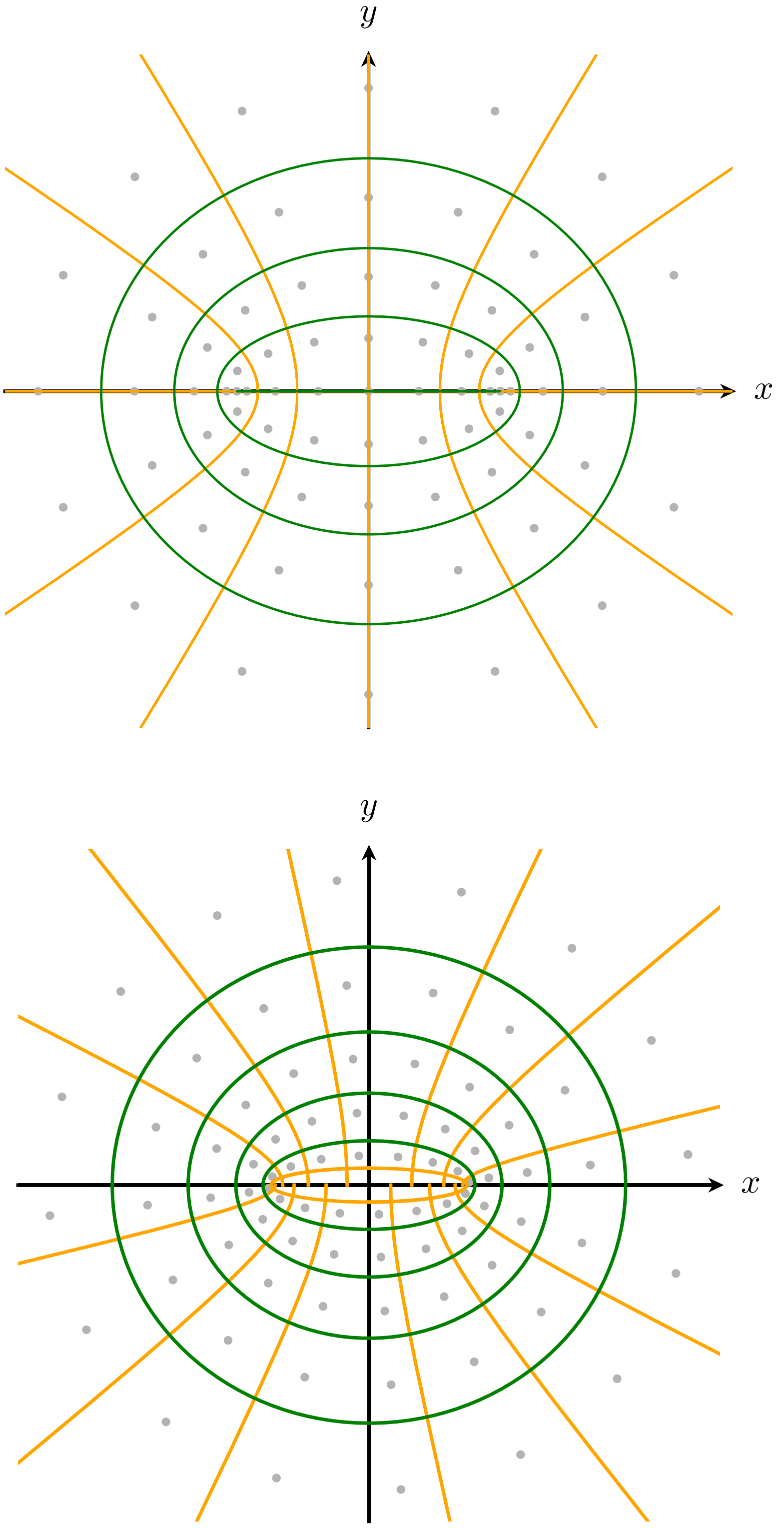

Figure 28. Polarity relation for discrete confocal conics. Gray points show discrete confocal coordinates on $\left(\frac{1}{2}\mathbb{Z}\right)^2$. The corresponding classical confocal conics which give rise to the polarity relation between gray points are shown in orange (for the values $u\left(n_1 + \frac{1}{4}\right)$) and green (for the values $v\left(n_2 + \frac{1}{4}\right)$). (left) Symmetric case with $c_1 = c_2 = 0$. All orange conics are hyperbolas and all green conics are ellipses. Note that near the coordinate axes those conics become degenerate and the polarity relation is not injective anymore. (right) Asymmetric case with $c_1 = 0.1$, $c_2 = 0.3$. Moving along the n_2-direction, the polarity across the y-axis is established by a conic with value $u\left(n_1 + \frac{1}{4}\right) < -a$, which is purely imaginary, while the polarity across the x-axis is established by a conic with value $u\left(n_1 + \frac{1}{4}\right) > -b$, which is an ellipse.

7.3 Discrete confocal coordinates in terms of Jacobi elliptic functions

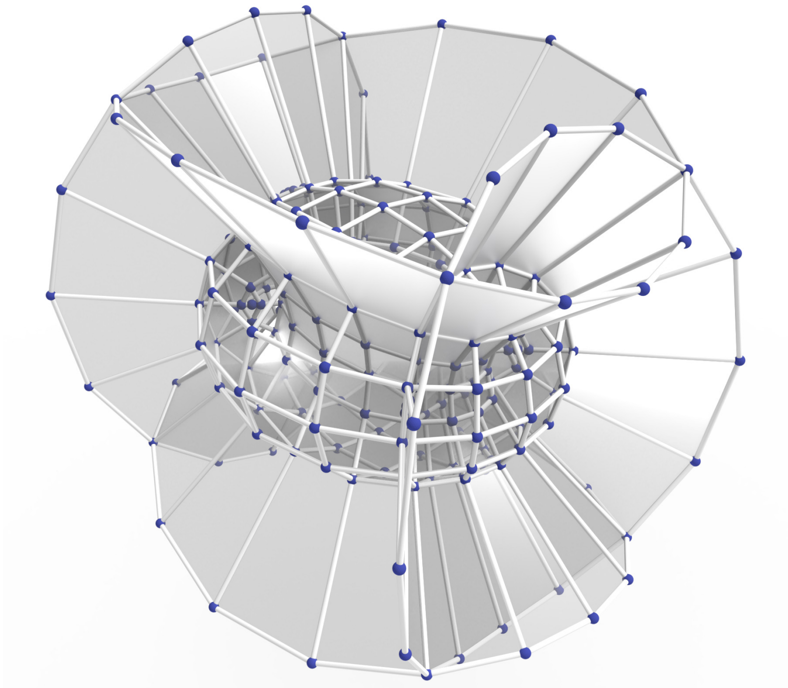

Figure 29. Three discrete confocal quadrics as part of a (stepsize 1) subnet of a discrete confocal coordinate system in $\mathbb{R}^3$ in terms of Jacobi elliptic functions.

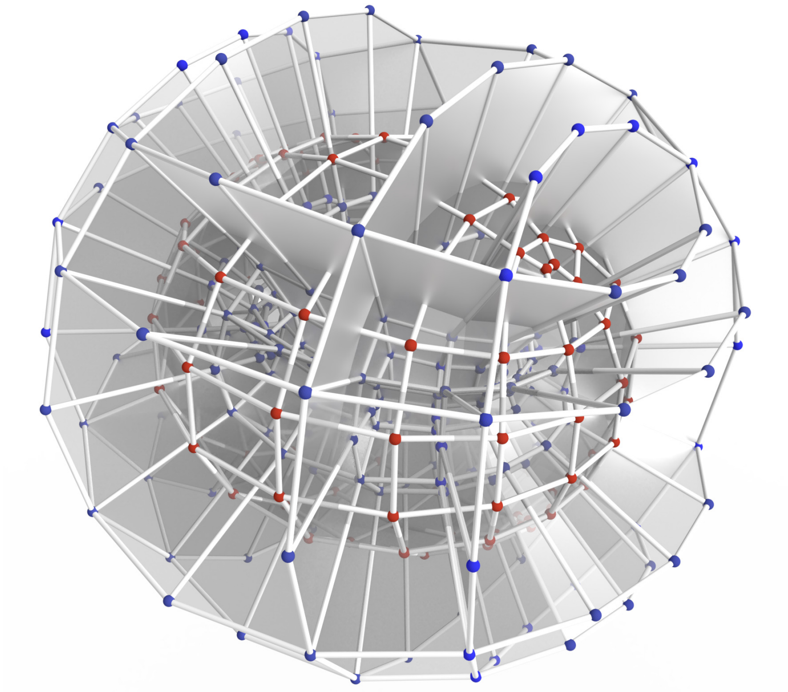

Figure 30. Part of an orthogonal pair of dual (stepsize 1) subnets of a discrete confocal coordinate system in $\mathbb{R}^3$ in terms of Jacobi elliptic functions

8 Diagonally related nets on surfaces

For the purposes of this chapter we adopt a slightly more general notion of nets (on surfaces):

Definition 8.1. A *net* $\mathcal{N}$ on a surface Σ is a collection of two (one-parameter) families of curves on Σ, such that for every point on Σ there exists exactly one curve from each of the two families through that point.

Let

$$\mathcal{N} = \left((\alpha_{s_1})_{s_1 \in I_1}, (\beta_{s_2})_{s_2 \in I_2}\right)$$

be a net on a surface Σ. Then curves from different families intersect in a unique point

$$P_{s_1,s_2} = \alpha_{s_1} \cap \beta_{s_2}.$$

We call (s_1, s_2) the *coordinates* of the point P_{s_1,s_2}. Note that the coordinates of a point are not uniquely defined by the net, but only a net together with a specific parametrization of the two families of curves.

Let $\alpha_{s_1}, \alpha_{\tilde{s}_1}, \beta_{s_2}, \beta_{\tilde{s}_2}$ be two pairs of curves from the two families of $\mathcal{N}$. They form a quadrilateral with vertices $P_{s_1,s_2}, P_{s_1,\tilde{s}_2}, P_{\tilde{s}_1,\tilde{s}_2}, P_{\tilde{s}_1,s_2}$, where $(P_{s_1,s_2}, P_{\tilde{s}_1,\tilde{s}_2})$ and $(P_{\tilde{s}_1,s_2}, P_{s_1,\tilde{s}_2})$ are pairs of opposite vertices.

Definition 8.2. Let $\mathcal{N}_1, \mathcal{N}_2$ be two nets on a surface Σ. Then $\mathcal{N}_2$ is called *diagonal* to $\mathcal{N}_1$ if the following condition is satisfied for any quadrilateral formed by four curves of $\mathcal{N}_1$: If one pair of opposite vertices is connected by a curve from $\mathcal{N}_2$, then the other pair of opposite vertices is connected by a curve from $\mathcal{N}_2$.

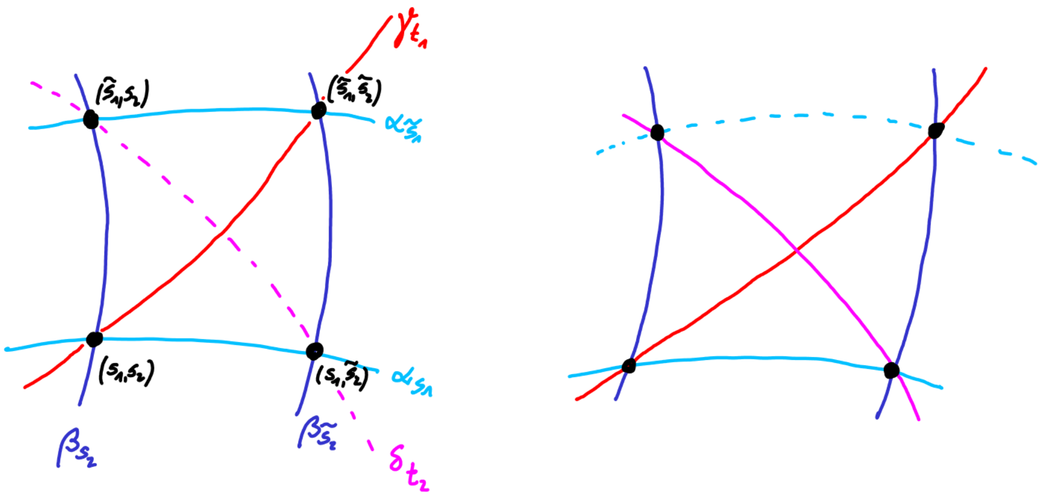

Figure 31. Diagonal relation between two nets, and equivalent characterization by Lemma 8.1

Exercise 8.1. Let

$$\mathcal{N}_1 = \left((\alpha_{s_1})_{s_1 \in I_1}, (\beta_{s_2})_{s_2 \in I_2}\right), \quad \mathcal{N}_2 = \left((\gamma_{t_1})_{t_1 \in J_1}, (\delta_{t_2})_{t_2 \in J_2}\right)$$

be two nets on a surface Σ. Show that the following four conditions are equivalent:

(i) For any quadrilateral formed by four curves of $\mathcal{N}_1$:
If one pair of opposite vertices is connected by a curve γ_{t_1} then the other pair of opposite vertices is connected by a curve δ_{t_2}.

(ii) For any quadrilateral formed by four curves of $\mathcal{N}_1$:
If one pair of opposite vertices is connected by a curve δ_{t_2} then the other pair of opposite vertices is connected by a curve γ_{t_1}.

In terms of using coordinates for the two nets we may reformulate the statement as the equivalence of the following two conditions:

(i) For any two points with coordinates $(s_1, s_2), (\tilde{s}_1, \tilde{s}_2) \in I_1 \times I_2$:
If (s_1, s_2) and $(\tilde{s}_1, \tilde{s}_2)$ have the same t_1-coordinate then $(\tilde{s}_1, s_2)$ and $(s_1, \tilde{s}_2)$ have the same t_2-coordinate.

(ii) For any two points with coordinates $(s_1, s_2), (\tilde{s}_1, \tilde{s}_2) \in I_1 \times I_2$:
If (s_1, s_2) and $(\tilde{s}_1, \tilde{s}_2)$ have the same t_2-coordinate then $(\tilde{s}_1, s_2)$ and $(s_1, \tilde{s}_2)$ have the same t_1-coordinate.

Lemma 8.1. *Let*

$$\mathcal{N}_1 = \left((\alpha_{s_1})_{s_1 \in I_1}, (\beta_{s_2})_{s_2 \in I_2}\right), \quad \mathcal{N}_2 = \left((\gamma_{t_1})_{t_1 \in J_1}, (\delta_{t_2})_{t_2 \in J_2}\right)$$

be two nets on a surface Σ. Then $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$ if and only if the following holds: For any three curves $\alpha_{s_1}, \beta_{s_2}, \beta_{\tilde{s}_2}$ let γ_{t_1} be the curve through P_{s_1,s_2} and δ_{t_2} be the curve through $P_{s_1,\tilde{s}_2}$. Then the two points

$$P_{\tilde{s}_1,s_2} = \beta_{s_2} \cap \delta_{t_2}, \qquad P_{\tilde{s}_1',\tilde{s}_2} = \beta_{\tilde{s}_2} \cap \gamma_{t_1}$$

lie on the common curve $\alpha_{\tilde{s}_1}$, i.e., $\tilde{s}_1 = \tilde{s}_1'$

Proof. Exercise. □

It turns out that the notion of diagonally related nets is symmetric:

Theorem 8.2. *Let $\mathcal{N}_1, \mathcal{N}_2$ be two nets on a surface Σ. Then $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$ if and only if $\mathcal{N}_1$ is diagonal to $\mathcal{N}_2$.*

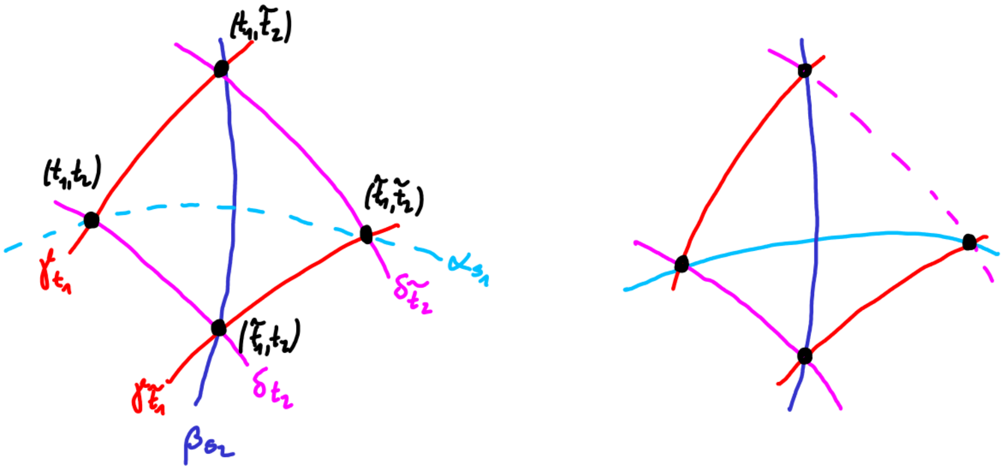

Figure 32. Symmetry of the diagonal relation.

Proof. Let $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$. Consider three curves $\gamma_{t_1}, \gamma_{\tilde{t}_1}, \delta_{t_2}$ from $\mathcal{N}_2$. Let α_{s_1} be the curve through P_{t_1,t_2} and β_{s_2} be the curve through $P_{\tilde{t}_1,t_2}$. By Lemma 8.1, we need to show that the two points

$$P_{t_1,\tilde{t}_2} = \gamma_{t_1} \cap \beta_{s_2}, \qquad P_{\tilde{t}_1,\tilde{t}'_2} = \gamma_{\tilde{t}_1} \cap \alpha_{s_1}$$

lie on the common curve $\delta_{\tilde{t}_2}$, i.e., $\tilde{t}_2 = \tilde{t}'_2$.

Let

$$\begin{aligned}
&\beta_{\hat{s}_2} \text{ be the curve through } P_{t_1,t_2},\\
&\beta_{\tilde{s}_2} \text{ be the curve through } P_{\tilde{t}_1,\tilde{t}'_2},\\
&\alpha_{\hat{s}_1} \text{ be the curve through } P_{\tilde{t}_1,t_2},\\
&\alpha_{\tilde{s}_1} \text{ be the curve through } P_{t_1,\tilde{t}_2}.
\end{aligned}$$

Let $O := \alpha_{s_1} \cap \beta_{s_2}$ and

$$\gamma_{\hat{t}_1}, \delta_{\hat{t}_2} \text{ be the two curves through } O.$$

Since $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$

$$\begin{aligned}
&\gamma_{\hat{t}_1} \text{ is the diagonal of the quadrilateral } \alpha_{s_1}, \alpha_{\hat{s}_1}, \beta_{s_2}, \beta_{\hat{s}_2},\\
&\delta_{\hat{t}_2} \text{ is the diagonal of the quadrilateral } \alpha_{s_1}, \alpha_{\tilde{s}1}, \beta_{s_2}, \beta_{\hat{s}_2},\\
&\delta_{\hat{t}_2} \text{ is the diagonal of the quadrilateral } \alpha_{s_1}, \alpha_{\hat{s}_1}, \beta_{s_2}, \beta_{\tilde{s}_2},
\end{aligned}$$

Thus,

$$\delta_{\hat{t}_2} \text{ is the diagonal of the big quadrilateral } \alpha_{\hat{s}_1}, \alpha_{\tilde{s}_1}, \beta_{\hat{s}_2}, \beta_{\tilde{s}_2}.$$

Since $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$

$$\gamma_{\hat{t}_1} \text{ is the diagonal of the big quadrilateral } \alpha_{\hat{s}_1}, \alpha_{\tilde{s}_1}, \beta_{\hat{s}_2}, \beta_{\tilde{s}_2}.$$

Thus,

$$\gamma_{\hat{t}_1} \text{ is the diagonal of the quadrilateral } \alpha_{s_1}, \alpha_{\tilde{s}_1}, \beta_{s_2}, \beta_{\tilde{s}_2}.$$

Finally, again since $\mathcal{N}_2$ is diagonal to $\mathcal{N}_1$, the two points $P_{t_1,\tilde{t}_2}, P_{\tilde{t}_1,\tilde{t}'_2}$ must lie on the same curve $\delta_{\tilde{t}_2}$. $\square$

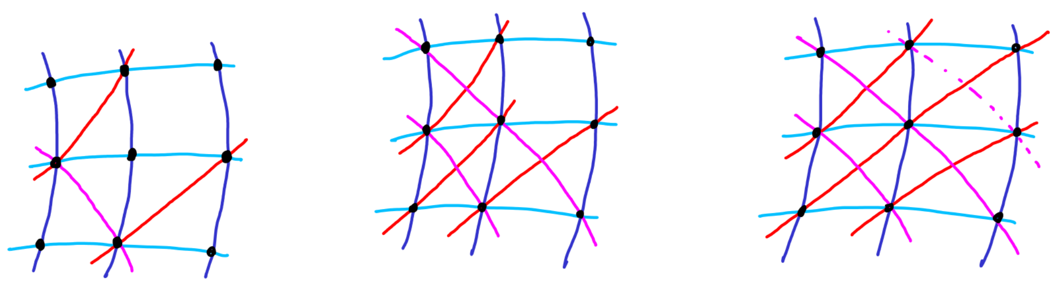

Figure 33. Proof of Theorem 8.2.

8.1 Dual pencils and tangent lines

As an example of diagonally related nets in the plane we will show the following theorem:

Theorem 8.3. *Let $\mathcal{N}_1$ be a net formed formed by conics from a dual pencil $\mathcal{P}$ of conics in $\mathbb{RP}^2$, and let $\mathcal{N}_2$ be a net formed by the tangent lines of one of the conics of $\mathcal{P}$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are diagonally related.*

Proof. Follows from Lemma 8.4. □

Lemma 8.4. *Let a_1, a_2, a_3, a_4 be four tangent lines of a conic $\mathcal{Q}$ in $\mathbb{RP}^2$, and let (L_1, L_2) and (M_1, M_2) be two opposite pairs of vertices of the quadrilateral formed by these lines. Let $\mathcal{Q}_1$ be a conic containing the two points L_1, L_2. Then there exists a conic $\mathcal{Q}_2$ in the dual pencil of conics spanned by $\mathcal{Q}$ and $\mathcal{Q}_1$ that contains the two points M_1, M_2.*

Moreover, in this case, the two tangent lines of $\mathcal{Q}_1$ in L_1, L_2 and the two tangent lines of $\mathcal{Q}_2$ in M_1, M_2 intersect in a common point.

Proof. We prove the (projective) dual statement, which is Lemma 8.5. □

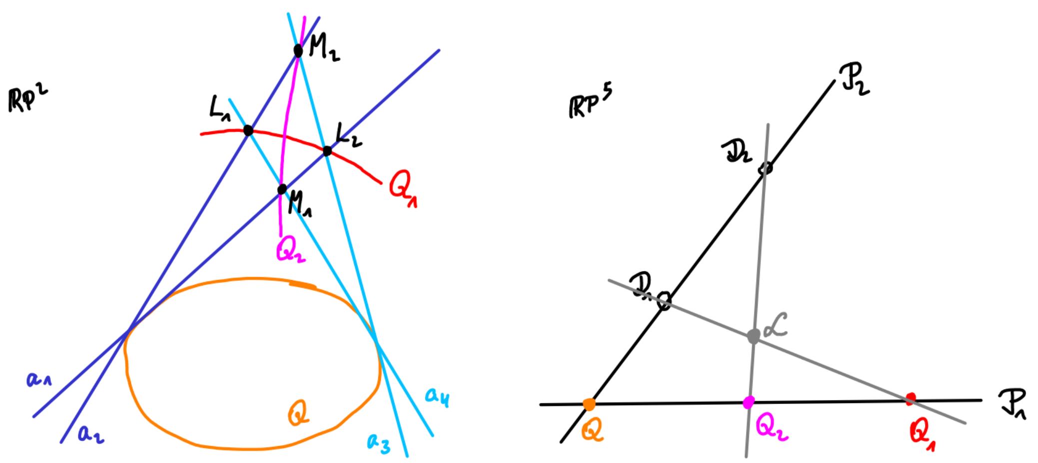

Figure 34. Lemma 8.4 and the proof of the dual statement, Lemma 8.5

Lemma 8.5. *Let A_1, A_2, A_3, A_4 be four points on a conic $\mathcal{Q}$ in $\mathbb{RP}^2$, and let (ℓ_1, ℓ_2) and (m_1, m_2) be two opposite pairs of edges of the quadrangle formed by these four points. Let $\mathcal{Q}_1$ be a conic tangent to the two lines ℓ_1, ℓ_2. Then there exists a conic $\mathcal{Q}_2$ in the pencil of conics spanned by $\mathcal{Q}$ and $\mathcal{Q}_1$ that is tangent to the two lines m_1, m_2.*

Moreover, in this case, the two touching points of $\mathcal{Q}_1$ and L_1, L_2 and the two touching points of $\mathcal{Q}_2$ and M_1, M_2 are collinear.

Proof. Let $\mathcal{P}_1$ be the pencil of conics spanned by $\mathcal{Q}$ and $\mathcal{Q}_1$. Then $\mathcal{P}_1$ defines a line in the space conics, which is a 5-dimensional projective space. Let $\mathcal{P}_2$ be the pencil of all conics through the four points A_1, A_2, A_3, A_4. Then $\mathcal{P}_2$ is another line in the space of conics, that intersects the line $\mathcal{P}_1$. Consider the two pairs of lines $\mathcal{D}_1 = \ell_1 \cup \ell_2$ and $\mathcal{D}_2 = m_1 \cup m_2$ as two degenerate conics in the pencil $\mathcal{P}_2$. The two lines ℓ_1 and ℓ_2 are tagent lines of $\mathcal{Q}_1$. Thus, by Lemma 8.6, the pencil spanned by $\mathcal{D}_1$ and $\mathcal{Q}_1$ contains a double line $\mathcal{L}$. In the space of conics, the point corresponding to $\mathcal{L}$ lies in the plane spanned by the two concurrent lines corresponding to $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus, the pencil spanned by $\mathcal{D}_2$ and $\mathcal{L}$ contains a conic $\mathcal{Q}_2$ which is also contained in $\mathcal{P}_1$. By Lemma 8.6 the conic $\mathcal{Q}_2$ is tangent to the lines m_1 and m_2. □

Lemma 8.6. *Let ℓ_1, ℓ_2 be two lines in $\mathbb{R}\mathrm{P}^3$ and $X_1 \in \ell_1, X_2 \in \ell_2$ a point on each line. Then the family of all conics tangent to ℓ_1 in X_1 and to ℓ_2 in X_2 is a pencil of conics containing the degenerate conic consisting of the two lines ℓ_1, ℓ_2 and the degenerate conic consisting of the double line joining X_1 and X_2*

Proof. Exercise. □

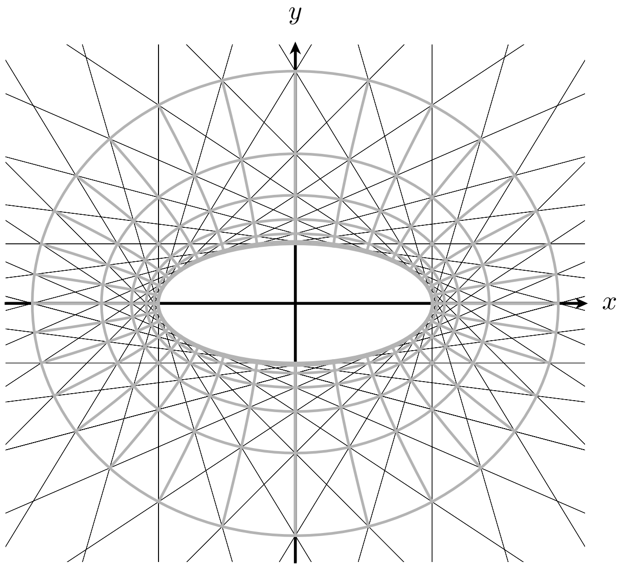

Figure 35. Confocal conics constitute a dual pencil of conics. Thus, by Theorem 8.3, the net of confocal ellipses and hyperbolas is diagonally related to the net formed by the tangent lines of one conic of the confocal family.

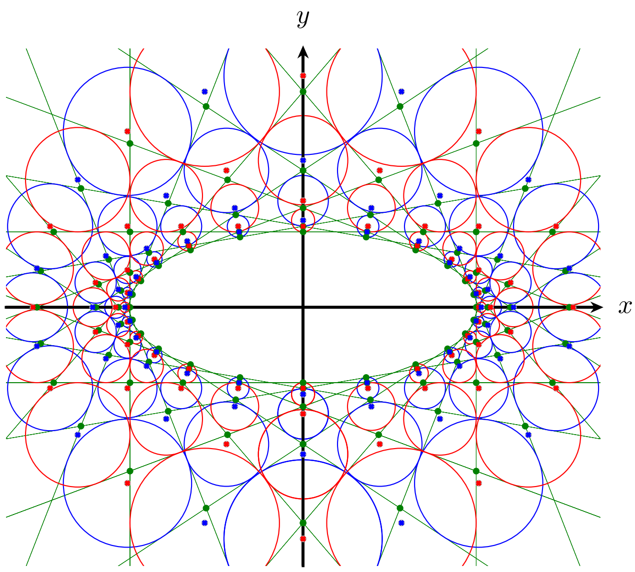

Figure 36. By Graves-Chasles Theorem the quadrilaterals in Figure 35 possess incircles. Furthermore the centers of these incircles constitute a special case of a discrete confocal coordinate system.

8.2 Diagonally related parametrizations

If $\boldsymbol{x} : \mathbb{R}^2 \supset U = I_1 \times I_2 \to \mathbb{R}^N$ is a parametrization of a surface then its coordinate lines

$$\mathcal{N} = \left(\left(\boldsymbol{x}(s_1, s_2 = s_2^0)\right)_{s_1 \in I_1}, \left(\boldsymbol{x}(s_1 = s_1^0, s_2)\right)_{s_2 \in I_2} \right) \qquad \text{for some } s_1^0 \in I_1, s_2^0 \in I_2$$

define a net on the surface $\boldsymbol{x}(U)$. Note that a reparametrization along the coordinate lines $\boldsymbol{x}(\varphi_1(s_1), \varphi_2(s_2))$ does not change the net it defines.

We now introduce the new variables

$$t_1 = s_1 + s_2, \qquad t_2 = s_1 - s_2,$$

which generate the parametrization

$$\boldsymbol{y}(t_1, t_2) = \boldsymbol{x}\left(\frac{t_1 + t_2}{2}, \frac{t_1 - t_2}{2}\right).$$

Lemma 8.7. *The nets corresponding to the parametrizations $\boldsymbol{x}$ and $\boldsymbol{y}$ are diagonally related.*

Proof. Let (s_1, s_2) and $(\tilde{s}_1, \tilde{s}_2)$ be two points with the same t_1-coordinate:

$$t_1 = s_1 + s_2 = \tilde{s}_1 + \tilde{s}_2.$$

Then $(\tilde{s}_1, s_2)$ and $(s_1, \tilde{s}_2)$ have the same t_2-coordinate:

$$t_2 = \tilde{s}_1 - s_2 = s_1 - \tilde{s}_2.$$

□

The following theorem, which we give without proof, states that up to reparametrization along the coordinate lines, this change of variables generates all diagonal nets for a given parametrization

Theorem 8.8. *Let $\boldsymbol{x}(s_1, s_2)$ and $\boldsymbol{y}(t_1, t_2)$ be two parametrizations of the same surface. Then the corresponding nets are diagonally related if and only there exist two smooth functions φ_1 and φ_2 such that*

$$\boldsymbol{y}(t_1, t_2) = \boldsymbol{x}\left(\varphi_1(t_1 + t_2), \varphi_2(t_1 - t_2)\right).$$

8.3 Diagonally related nets on quadrics

We now look at some examples of nets that diagonal to the net of curvature lines on quadrics, which we have derived as side product of our studies of confocal coordinate systems in Section 6.

Theorem 8.9. *The net of curvature lines on a one-sheeted hyperboloid and the net of asymptotic lines (generators of the hyperboloid) are diagonal. Furthermore, the deformation of this hyperboloid along its confocal family:*

- *preserves the curvature lines, the asymptotic lines, and their diagonal relation,*
- *preserves the distance between any two points on an asymptotic line ("isometric along asymptotic lines"),*

- *in the planar limits becomes a net of confocal conics (confocal to one of the focal conics) and a net of tangent lines of the focal conic (see Section 8.1).*

Idea of the proof. This can be derived from the parametrization (34) of confocal quadrics in terms of Jacobi elliptic functions. In particular, for any $s_2 \in (0, \mathsf{K}(k_2))$

$$(s_1, s_3) \mapsto \boldsymbol{x}(s_1, s_2, s_3)$$

is a curvature line parametrization of a one-sheeted hyperboloid. In this parametrization its asymptotic lines are given by $s_1 + s_3 = \text{const}$ and $s_1 - s_3 = \text{const}$. The deformation is described by change of the parameter s_2. □

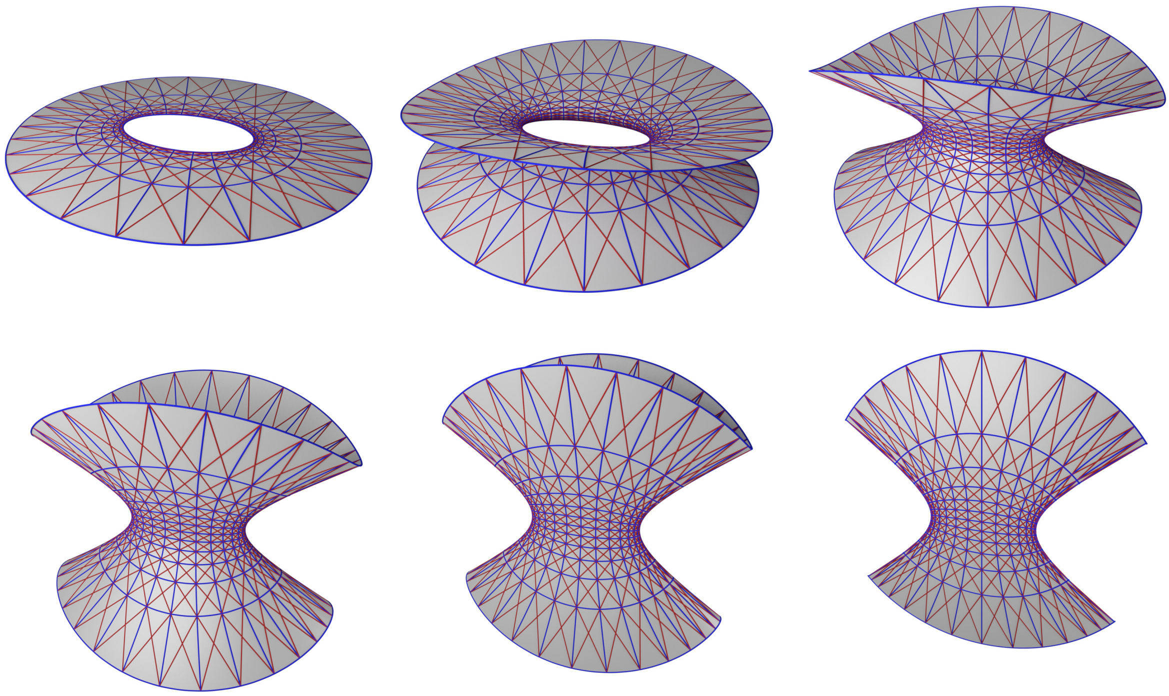

Figure 37. Diagonally related nets of curvature lines and asymptotic lines on a one-sheeted hyperboloid and its deformation along confocal quadrics.

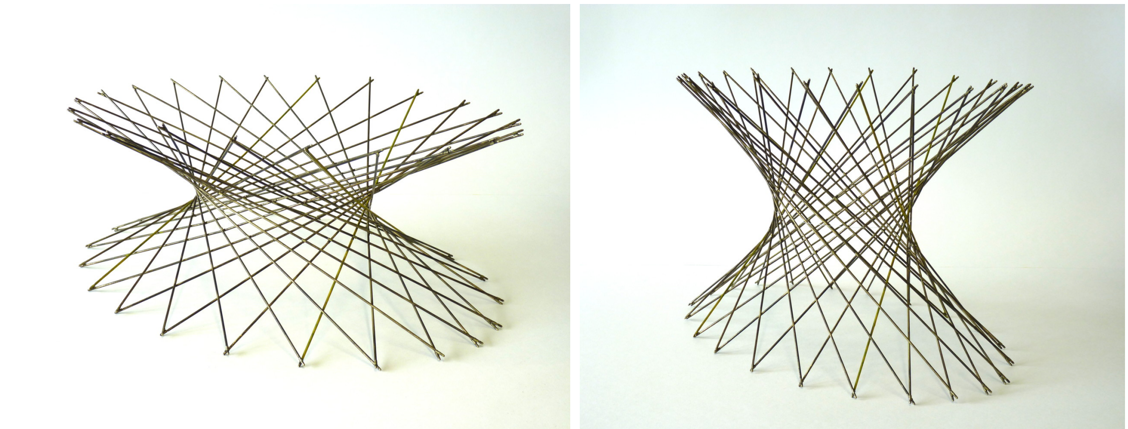

Figure 38. "Isometrically" deformable model of a one-sheeted hyperboloid at TU Wien.

On ellipsoids and two-sheeted hyperboloids have no asymptotic lines (since they have positive Gaussian curvature). Instead *(conjugate) characteristic lines* can be viewed as an analogous net of lines on positively curved surfaces. Characteristic lines are characterized by the two properties of being conjugate and bisected by the curvature lines. Analogous to the diagonal relation of curvature lines and asymptotic lines on one-sheeted hyperboloids, the parametrization (34) of confocal quadrics in terms of Jacobi elliptic functions yields the diagonal relation of curvature lines and conjugate characteristic lines.

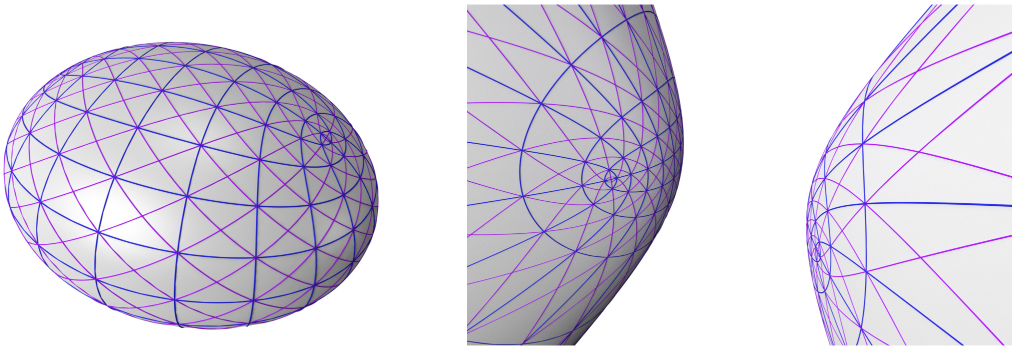

Figure 39. Diagonally related nets of curvature lines and characteristic lines on an ellipsoid and a two-sheeted hyperboloid.

As opposed to the asymptotic lines on a one-sheeted hyperboloid the characteristic lines on ellipsoids and two-sheeted hyperboloid do not give rise to an "isometric" deformation of the quadrics. Yet for ellipsoids a different net of lines, which are also diagonally related to curvature lines does.

8.3.1 Circular cross sections of quadrics

To find circles on quadrics we use the following projective characterization:

Proposition 8.10. *Consider the embedding of Euclidean space into projective space* $\mathbb{R}^3 \subset \mathbb{RP}^3 \subset \mathbb{CP}^3$*, together with the* ***absolute (imaginary) conic*** *at infinity:*

$$\mathcal{Z} : x_1^2 + x_2^2 + x_3^2 = 0, \quad x_4 = 0.$$

Then a quadric $\mathcal{Q} \subset \mathbb{RP}^3$ *is a sphere if and only if it (its complexification) contains* $\mathcal{Z}$*.*

Proof. Exercise. □

Considering circles as the intersection of spheres, this implies that a conic $\mathcal{C} \subset \Pi$ in some plane $\Pi \subset \mathbb{RP}^3$ is a circle if and only if it intersects the absolute conic $\mathcal{Z}$ in two points.

Thus, to find the circular sections of a quadric $\mathcal{Q}$, one should consider the restriction of $\mathcal{Q}$ to the plane at infinity $x_4 = 0$. The resulting conic generically intersects $\mathcal{Z}$ in four points, which are pairs of complex conjugate points. Each pair of these complex conjugate points spans a real line at infinity, and each plane through one of those two lines intersects the quadric $\mathcal{Q}$ in a circle (if the intersection is not empty). Thus, generically a quadric in $\mathbb{R}^3$ has two famalies of circular sections.

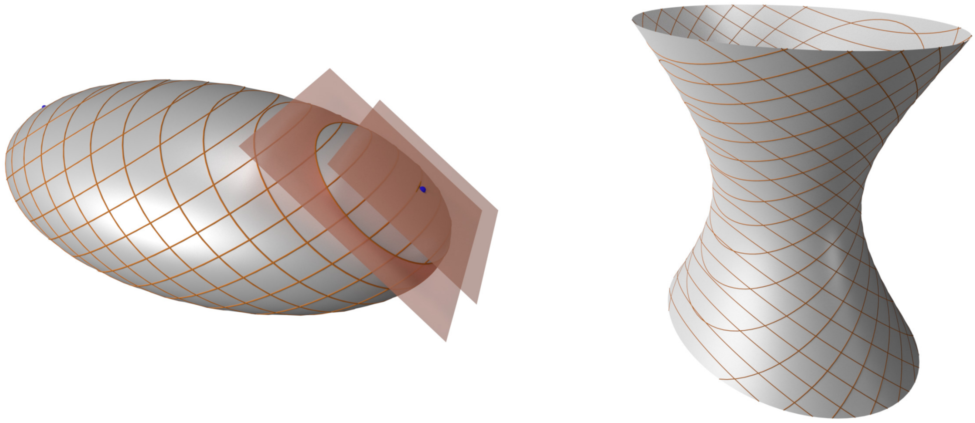

Figure 40. Circular cross sections of an ellipsoid and a one-sheeted hypreboloid.

We specify this claim for the case of ellipsoids:

Theorem 8.11. *Let $a > b > c > 0$ and $\mathcal{Q} \subset \mathbb{R}^3$ the ellipsoid*

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

Then the circular sections of $\mathcal{Q}$ are given by the two families of parallel planes

$$\Pi_\pm(\lambda_\pm): \sqrt{\tfrac{1}{b} - \tfrac{1}{a}}\, x \pm \sqrt{\tfrac{1}{c} - \tfrac{1}{b}}\, z = \lambda_\pm, \qquad \lambda_\pm \in \left[-\sqrt{\tfrac{a-c}{b}}, \sqrt{\tfrac{a-c}{b}}\right]. \tag{50}$$

Proof. We introduce homogeneous coordinates $x = \frac{x_1}{x_4}, y = \frac{x_2}{x_4}, z = \frac{x_3}{x_4}$ and the constants $\alpha := \frac{1}{a}, \beta := \frac{1}{b}, \gamma := \frac{1}{c}$, which satisfy $0 < \alpha < \beta < \gamma$. Then the ellipsoid $\mathcal{Q}$ is given by

$$\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 - x_4^2 = 0.$$

Its four intersection points with $\mathcal{Z}$ are given by

$$P_{\sigma,\tau} = \begin{bmatrix} \sigma\sqrt{\gamma - \beta} \\ \tau i \sqrt{\gamma - \alpha} \\ \sqrt{\beta - \alpha} \\ 0 \end{bmatrix}, \qquad \sigma, \tau \in \{+, -\},$$

and come in two complex conjugate pairs

$$P_{+,+} = \bar{P}_{+,-}, \quad P_{-,+} = \bar{P}_{-,-}.$$

Thus they span two real lines at infinity

$$\ell_\pm = P_{\pm,+} \vee P_{\pm,-} = \operatorname{span}\left\{ \begin{pmatrix} \pm\sqrt{\gamma - \beta} \\ 0 \\ \sqrt{\beta - \alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The two one-parameter families of planes that contain ℓ_σ, respectively, are given by

$$\Pi_\pm(\lambda_\pm): \sqrt{\beta - \alpha}\, x_1 \mp \sqrt{\gamma - \beta}\, x_3 - \lambda_\pm x_4 = 0$$

for $\lambda_\pm \in \mathbb{R} \cup \{\infty\}$. The planes from each family $\Pi_\pm$ intersect in a line at infinity and therefore are parallel planes. Furthermore, by construction, their intersection with $\mathcal{Q}$ (if not empty) gives all circles contained in $\mathcal{Q}$.

It remains to show for which values of $\lambda_\pm$ the intersection $\Pi_\pm(\lambda_\pm) \cap \mathcal{Q}$ is not empty. Let

$$Q = \operatorname{diag}(\alpha, \beta, \gamma, -1)$$

be the Gram matrix of $\mathcal{Q}$, and

$$q(x) = x^\intercal Q x$$

the corresponding quadratic form. With

$$p_\pm(\lambda_\pm) = \begin{pmatrix} \sqrt{\beta - \alpha} \\ 0 \\ \mp\sqrt{\gamma - \beta} \\ -\lambda_\pm \end{pmatrix}$$

the poles of $\Pi_\pm(\lambda_\pm)$ have homogeneous coordinates $Q^{-1}p_\pm(\lambda_\pm)$. Thus, $\mathcal{Q} \cap \Pi_\pm(\lambda_\pm)$ is not empty if and only if

$$0 \leqslant q(Q^{-1}p_\pm(\lambda_\pm)) = p_\pm(\lambda_\pm)^\intercal Q^{-1} p_\pm(\lambda_\pm) = \frac{\beta - \alpha}{\alpha} + \frac{\gamma - \beta}{\gamma} - \lambda_\pm^2 = \beta\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right) - \lambda_\pm^2.$$

□

Exercise 8.2. Show that the poles of the two families of planes $\Pi_\pm$ lie (on the outside segments) of the lines that intersect opposite umbilic points of the ellipsoid $\mathcal{Q}$. Thus, in particular, the planes $\Pi_\pm(\lambda_\pm)$ are parallel to the tangent planes of $\mathcal{Q}$ in its umbilic points.

The two families of circular sections of an ellipsoid constitute a net on the ellipsoid.

Theorem 8.12. *On any ellipsoid, the net of curvature lines and the net of circular sections are diagonally related.*

Proof. Let $a > b > c > 0$. We consider the ellipsoid as part of the confocal family (26) with $\lambda = 0$. Then the corresponding confocal coordinates (33) yield a curvature line parametrization of the ellipsoid for

$$u_3(s_3) = 0,$$

or equivalently,

$$f_3(s_3)^2 = a, \quad g_3(s_3)^2 = b, \quad h_3(s_3)^2 = c.$$

Thus, this parametrization is given by

$$\begin{cases} x(s_1, s_2) = \sqrt{\dfrac{a}{(a-b)(a-c)}}\, f_1(s_1) f_2(s_2) \\[2ex] y(s_1, s_2) = \sqrt{\dfrac{b}{(a-b)(b-c)}}\, g_1(s_1) g_2(s_2) \\[2ex] z(s_1, s_2) = \sqrt{\dfrac{c}{(a-c)(b-c)}}\, h_1(s_1) h_2(s_2) \end{cases} \tag{51}$$

with

$$f_1(s_1)^2 + g_1(s_1)^2 = a - b, \quad f_1(s_1)^2 + h_1(s_1)^2 = a - c,$$
$$f_2(s_2)^2 - g_2(s_2)^2 = a - b, \quad f_2(s_2)^2 + h_2(s_2)^2 = a - c,$$

We show that there exist solutions $f_1, f_2, g_1, g_2, h_1, h_2$ such that the diagonal net given by the curves

$$s_\pm = s_1 \pm s_2 = \text{const}$$

are the circular cross sections of the ellipsoid. Substituting the parametrization (51) into the planes $\Pi_\pm(\lambda_\pm)$ given by (50) we obtain

$$f_1(s_1)f_2(s_2) \pm h_1(s_1)h_2(s_2) = \sqrt{b(a-c)}\,\lambda_\pm. \tag{52}$$

For the diagonal lines $s_\pm = \text{const}$ to lie in these planes, the parameters λ_+ and λ_- must be functions only depending on s_+ and s_-, respectively. Thus, the functions f_1, f_2, h_1, h_2 must be solutions of the equations

$$\begin{aligned} f_1(s_1)^2 + h_1(s_1)^2 &= a - c, \\ f_2(s_2)^2 + h_2(s_2)^2 &= a - c, \\ f_1(s_1)f_2(s_2) \pm h_1(s_1)h_2(s_2) &= \sqrt{b(a-c)}\,\lambda_\pm(s_1 \pm s_2), \end{aligned}$$

which are readily solved by trigonometric functions

$$\begin{aligned} f_1(s_1) &= \sqrt{a-c}\sin(s_1), \quad & f_2(s_2) &= \sqrt{a-c}\cos(s_1), \\ h_1(s_1) &= \sqrt{a-c}\cos(s_1), \quad & h_2(s_2) &= \sqrt{a-c}\sin(s_2). \end{aligned}$$

The last two equations due to the addition law

$$\sin(s_1)\cos(s_2) \pm \cos(s_1)\sin(s_2) = \sin(s_1 \pm s_2),$$

and thus the functions $\lambda_\pm$ are given by

$$\lambda_\pm(s_\pm) = \sqrt{\frac{a-c}{b}}\sin(s_\pm)$$

The functions g_1 and g_2 are then obtained from

$$\begin{aligned} g_1(s_1)^2 &= a - b - (a-c)\sin^2(s_1), \\ g_2(s_2)^2 &= (a-c)\cos^2(s_1) - a + b. \end{aligned} \tag{53}$$

The right-hand sides are positive as long as

$$\sin^2(s_1) < \frac{a-b}{a-c}, \qquad \cos^2(s_2) > \frac{a-b}{a-c},$$

which have open intervals as solutions since

$$0 < \frac{a-b}{a-c} < 1.$$

□

Remark 8.1. Since g_1 and g_2 are determined by the square roots of (53), we obtain two separate parametrizations of the ellipsoid, one for $y > 0$ and one for $y < 0$. Geometrically, this reflects the fact that the net of circular sections becomes degenerate for $y = 0$ in the sense that the two families of circles become tangent in these points and furthermore tangent one of the families o curvature lines.

Similar to the asymptotic lines on one-sheeted hyperboloids, the circular sections of ellipsoids admit an "isometric" deformation:

Theorem 8.13. *The deformation of an ellipsoid given by its confocal family scaled to have the same second semi-axis:*

- *preserve the curvature lines, the circular sections, and their diagonal relation*
- *preserves the distance between any two points on a circular section ("isometric along circular sections"),*
- *in the planar limits becomes a net of confocal conics and a net of circles touching a conic.*

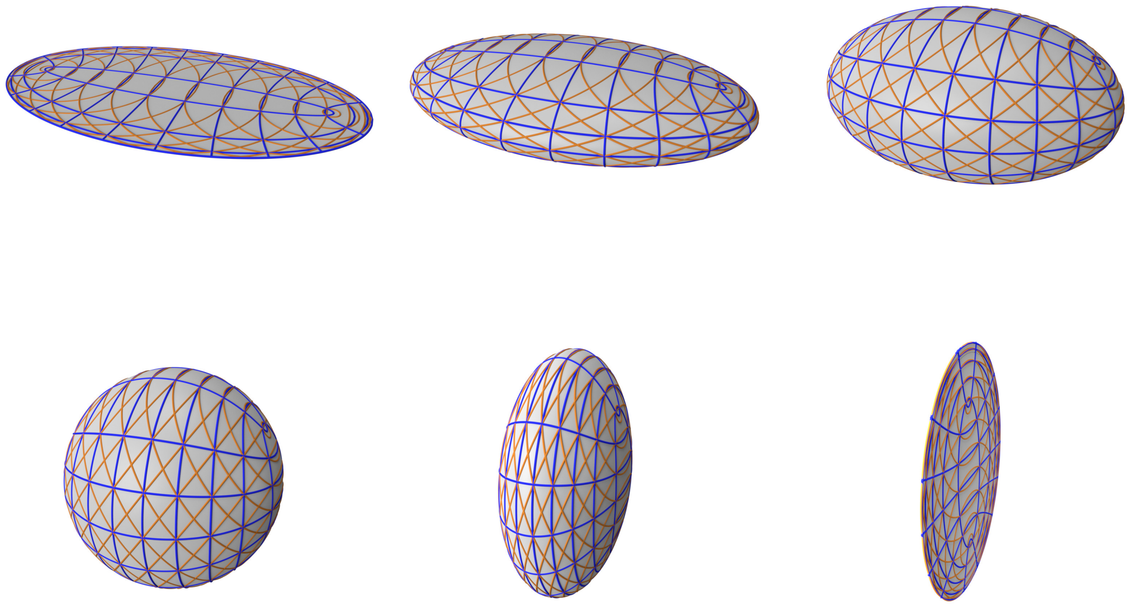

Figure 41. Diagonally related nets of curvature lines and circular sections on an ellipsoid and its "isometric" deformation along the circular sections.

8.3.2 Discrete ellipsoid with circular cross sections

Let $a > b > c > 0$. Similar to (51) we obtain a discrete curvature line parametrized ellipsoid by taking two layers from a 3-dimensional discrete confocal coordinate system:

$$\begin{cases} x(n_1,n_2) = \sqrt{\dfrac{a}{(a-b)(a-c)}}\, f_1(n_1)f_2(n_2) \\ y(n_1,n_2) = \sqrt{\dfrac{b}{(a-b)(b-c)}}\, g_1(n_1)g_2(n_2) \\ z(n_1,n_2) = \sqrt{\dfrac{c}{(a-c)(b-c)}}\, h_1(n_1)h_2(n_2) \end{cases} \tag{54}$$

with

$$\begin{aligned} &f_1(n_1)f_1(n_1+\tfrac{1}{2}) + g_1(n_1)g_1(n_1+\tfrac{1}{2}) = a-b, \quad f_1(n_1)f_1(n_1+\tfrac{1}{2}) + h_1(n_1)h_1(n_1+\tfrac{1}{2}) = a-c, \\ &f_2(n_2)f_2(n_2+\tfrac{1}{2}) - g_2(n_2)g_2(n_2+\tfrac{1}{2}) = a-b, \quad f_2(n_2)f_2(n_2+\tfrac{1}{2}) + h_2(n_2)h_2(n_2+\tfrac{1}{2}) = a-c, \end{aligned}$$

for $(n_1,n_2) \in \mathbb{Z}^2 \cup (\mathbb{Z}+\frac{1}{2})^2$.

Proposition 8.14. *The discrete net $\boldsymbol{x}(n_1,n_2)$ on* (54) *is a discrete curvature line parametrization of an ellipsoid in the following sense:*

(i) Any two points $\boldsymbol{x}(n_1,n_2)$ and $\boldsymbol{x}(n_1 \pm \frac{1}{2}, n_2 \pm \frac{1}{2})$ are polar points with respect to the ellipsoid

$$\mathcal{Q}: \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1. \tag{55}$$

(ii) All quadrilaterals $\boldsymbol{x}(n_1,n_2), \boldsymbol{x}(n_1+1,n_2), \boldsymbol{x}(n_1+1,n_2+1), \boldsymbol{x}(n_1,n_2+1)$ are planer.

(iii) All edges $\Delta_1\boldsymbol{x}(n_1,n_2)$ and $\Delta_2\boldsymbol{x}(n_1+\frac{1}{2}, n_2-\frac{1}{2})$ are orthogonal.

The discrete diagonal nets of $\boldsymbol{x}$ are given by introducing the coordinates $(n_+,n_-) \in \mathbb{Z}^2$:

$$n_\pm = \frac{n_1 \pm n_2}{2}.$$

This yields four diagonal sublattices of stepsize 1

$$\mathbb{Z}\times\mathbb{Z}, \quad (\mathbb{Z}+\tfrac{1}{2})\times(\mathbb{Z}+\tfrac{1}{2}), \quad (\mathbb{Z}+\tfrac{1}{2})\times\mathbb{Z}, \quad \mathbb{Z}\times(\mathbb{Z}+\tfrac{1}{2}),$$

which come as two dual pairs.

The discrete ellipsoid (54) is closely related to the smooth ellipsoid (55). Thus, we may try to find a parametrization such that the coordinate polygons of the discrete diagonal nets lie in the planes $\Pi_\pm(\lambda_\pm)$ given in (50). Substituting (54) into (50) we find, in the same way as (52):

$$f_1(n_1)f_2(n_2) \pm h_1(n_1)h_2(n_2) = b(a-c)\lambda_\pm.$$

To have the diagonal polygons $n_\pm = \text{const}$ to lie in these planes, the functions f_1, f_2, h_1, h_2 must satisfy the equations

$$\begin{aligned} f_1(n_1)f_1(n_1+\tfrac{1}{2}) + h_1(n_1)h_1(n_1+\tfrac{1}{2}) &= a-c, \\ f_2(n_2)f_2(n_2+\tfrac{1}{2}) + h_2(n_2)h_2(n_2+\tfrac{1}{2}) &= a-c, \\ f_1(n_1)f_2(n_2) \pm h_1(n_1)h_2(n_2) &= \sqrt{b(a-c)}\lambda_\pm(n_1 \pm n_2), \end{aligned}$$

which are again solved by trigonometric functions

$$f_1(n_1) = \varepsilon\sqrt{a-c}\sin(\delta n_1), \quad f_2(n_2) = \varepsilon\sqrt{a-c}\cos(\delta n_1),$$
$$h_1(n_1) = \varepsilon\sqrt{a-c}\cos(\delta n_1), \quad h_2(n_2) = \varepsilon\sqrt{a-c}\sin(\delta n_2)$$

with some constant $0 < \delta < \pi$ and

$$\varepsilon = \frac{1}{\sqrt{\cos\frac{\delta}{2}}},$$

and

$$\lambda_\pm(n_\pm) = \varepsilon^2\sqrt{\frac{a-c}{b}}\sin(2\delta n_\pm).$$

The functions g_1 and g_2 are then obtained by the recurrence relations

$$g_1(n_1+\tfrac{1}{2}) = \frac{a-b-f_1(n_1)f_1(n_1+\frac{1}{2})}{g_1(n_1)}, \qquad g_2(n_2+\tfrac{1}{2}) = \frac{f_2(n_2)f_2(n_2+\frac{1}{2})-a+b}{g_2(n_2)}.$$

The following proposition further establishes why the diagonal polygons $n_\pm = \text{const}$ may be thought of as discrete circles:

Proposition 8.15. *Along a diagonal polygon $n_- = n_-^0 = const$ a point $\boldsymbol{x}(n_+, n_- = n_-^0)$ and the line through $\boldsymbol{x}(n_+ - \frac{1}{2}, n_- = n_-^0)$, $\boldsymbol{x}(n_+ + \frac{1}{2}, n_- = n_-^0)$ are polar in the plane $\Pi_-(\lambda_-(n_-^0))$ are polar with respect to the circle $\mathcal{Q} \cap \Pi_i(\lambda_-(n_-^0))$. Similarly, along the diagonals $n_+ = const$.*

Proof. By Proposition 8.14 (i), the line through $\boldsymbol{x}(n_+ - \frac{1}{2}, n_- = n_-^0)$, $\boldsymbol{x}(n_+ + \frac{1}{2}, n_- = n_-^0)$ lies in the polar plane of the point $\boldsymbol{x}(n_+, n_- = n_-^0)$ with respect to the ellipsoid $\mathcal{Q}$ Since the three points lie in the plane $\Pi_-(\lambda_-(n_-))$ the polarity can be restricted to the intersection of $\mathcal{Q}$ with this plane, which is a circle. ☐

Remark 8.2. Note that by the same reasoning we obtain discrete tangent cones to the discrete circles: The planes of the planar quadrilaterals of $\boldsymbol{x}$ along any diagonal $n_\pm = \text{const}$ intersect in a common point.

The discrete circles allow for an “isometric” deformation similar to Theorem 8.13.

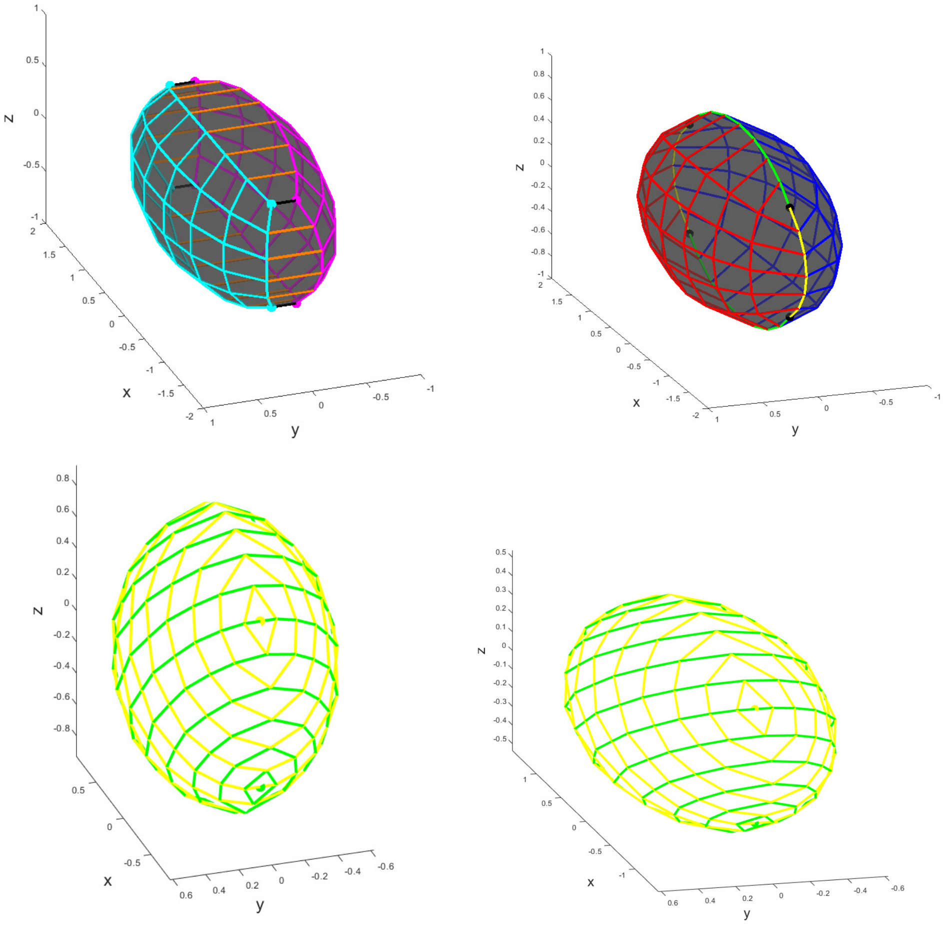

Figure 42. *(top)* Two dual sublattices of a discrete curvature line parametrization of an ellipsoid. *(bottom)* The diagonally related discrete circles in two stages of the “isometric” deformation.

9 Classification of pencils of quadrics

9.1 Polynomial matrices

Definition 9.1. Let $\mathbb{F} = \mathbb{R}$ (or $\mathbb{F} = \mathbb{C}$) be the field of real (or complex) numbers, and let $\mathbb{F}[\lambda]$ be the ring of polynomials over $\mathbb{F}$ in one variable (denoted by λ).

(i) A matrix

$$A \in \mathbb{F}[\lambda]^{m\times n} = \left\{ (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \;\middle|\; a_{ij} \in \mathbb{F}[\lambda] \right\}$$

with polynomial entries is called a *polynomial matrix*.

(ii) The *degree* of a polynomial matrix $A \in \mathbb{F}[\lambda]^{m\times n}$ is given by

$$\deg A = \max \{\deg a_{ij} \mid i = 1,\dots,m,\ j = 1,\dots,n\}.$$

A polynomial matrix $A \in \mathbb{F}[\lambda]^{m\times n}$ is called *constant* if $\deg A = 0$.

(iii) The *rank* of a polynomial matrix $A \in \mathbb{F}[\lambda]^{m\times n}$ is given by

$$\operatorname{rk} A = \max \{k \mid \text{non-zero } k \times k \text{ minor of } A\}.$$ [11]

A square polynomial matrix $A \in \mathbb{F}[\lambda]^{n\times n}$ is called *regular* if $\operatorname{rk} A = n$, or equivalent, if $\det A \neq 0$.

(iv) A square polynomial matrix $A \in \mathbb{F}[\lambda]^{n\times n}$ is called *invertible* if it has an inverse in $\mathbb{F}[\lambda]^{n\times n}$, i.e., if there exists a matrix $A^{-1} \in \mathbb{F}[\lambda]^{n\times n}$ such that

$$A^{-1}A = AA^{-1} = I.$$

For constant square matrices $A \in \mathbb{F}^{n\times n}$ one has

$$A \text{ invertible } \Leftrightarrow A \text{ regular}.$$

The same holds for matrices $A \in \mathbb{F}(\lambda)^{n\times n}$ with entries in the field of rational functions $\mathbb{F}(\lambda)$. For polynomial square matrices $A \in \mathbb{F}[\lambda]^{n\times n}$ one only has

$$A \text{ invertible } \Rightarrow A \text{ regular}.$$

Example 9.1. Consider the polynomial matrix

$$A = \begin{pmatrix} \lambda & 0 \\ \lambda^2 & \lambda+1 \end{pmatrix} \in \mathbb{F}[\lambda]^{2\times 2}.$$

It has $\deg A = 2$ and $\operatorname{rk} A = 2$, and thus is regular. Its determinant is given by

$$\det A = \lambda(\lambda+1) \neq 0,$$

Thus, viewed as a rational matrix, A is invertible in $\mathbb{F}(\lambda)^{2\times 2}$ with

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \lambda+1 & 0 \\ -\lambda^2 & \lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ -\frac{\lambda}{\lambda+1} & \frac{1}{\lambda+1} \end{pmatrix}$$

But since $A^{-1} \notin \mathbb{F}[\lambda]^{2\times 2}$, as a polynomial matrix, A is *not* invertible.

[11] A $k\times k$ minor is the determinant of a $k\times k$ submatrix of A.

Proposition 9.1. *Let $A \in \mathbb{F}[\lambda]^{n\times n}$ be a square polynomial matrix. Then A is invertible if and only if its determinant is a non-zero constant, i.e.,*

$$\det A \in \mathbb{F}\backslash\{0\}.$$

Proof.

($\Leftarrow$) If $\det A \in \mathbb{F}\backslash\{0\}$, then A is invertible in $\mathbb{F}(\lambda)^{n\times n}$. In particular, the entries of the inverse A^{-1} are given by the cofactors[12] of A (which are polynomials) divided by the determinant. Since the determinant is constant, the entries of A^{-1} are polynomials.

($\Rightarrow$) If A has a polynomial inverse $A^{-1} \in \mathbb{F}[\lambda]^{n\times n}$, then $\det A$ and $\det A^{-1}$ are polynomials, and

$$(\det A)(\det A^{-1}) = \det(AA^{-1}) = 1.$$

This implies $\det A \neq 0$ and $\deg \det A = 0$.

□

9.1.1 Multiplication and division of polynomial matrices

For two polynomials $a, b \in \mathbb{F}[\lambda]$ one has

$$\deg(ab) = \deg a + \deg b.$$

For two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ this does not necessarily hold: We can write

$$A = A_s\lambda^s + A_{s-1}\lambda^{s-1} + \cdots + A_0,$$
$$B = B_t\lambda^t + B_{t-1}\lambda^{t-1} + \cdots + B_0,$$

where A_i are the constant matrices containing the coefficients of degree i of A. In particular, $\deg A = s$ and $A_s \neq 0$. Similarly, $\deg B = t$ and $B_t \neq 0$. We call A_s and B_t the *leading coefficient matrices* of A and B, respectively. Then the (possibly) leading coefficient of the product is given by A_sB_t:

$$AB = A_sB_t\lambda^{s+t} + O(s+t-1).$$

In general A_sB_t can be zero.

Exercise 9.1.

(i) Show that if either A_s or B_t are invertible, then

$$\deg(AB) = \deg A + \deg B.$$

(ii) Compute the degree of A^2 from Exercise 9.1.

We now turn to the division with remainder of polynomial matrices:

Proposition 9.2. *Let $A, B \in \mathbb{F}[\lambda]^{n\times n}$ be two square polynomial matrices, where the leading coefficient matrix of B is invertible. Then A can be divided with remainder by B (from the left and from the right) in the following sense:*

*There exist two unique polynomial matrices $Q, R \in \mathbb{F}[\lambda]^{n\times n}$ (**quotient** and **remainder**) such that*

$$A = QB + R \qquad \text{and} \qquad \deg R < \deg B.$$

Similarly, for the division from the right.

[12]The cofactors of A are the signed minors.

Proof.

Existence: We write

$$\begin{aligned} A &= A_s\lambda^s + A_{s-1}\lambda^{s-1} + \cdots + A_0, \\ B &= B_t\lambda^t + B_{t-1}\lambda^{t-1} + \cdots + B_0, \end{aligned}$$

where $s = \deg A$ and $t = \deg B$. Thus, $A_s \neq 0$ and B_t is invertible.

If $t > s$, we can take $Q = 0$, $R = M$. Thus, assume $s \geqslant t$. With

$$\begin{aligned} \tilde{Q} &:= A_s B_t^{-1}\lambda^{s-t}, \\ \tilde{R} &:= A - \tilde{Q}B \end{aligned}$$

we have

$$\tilde{Q}B = A_s B_t^{-1} B\lambda^{s-t} = A_s\lambda^s + A_s B_t^{-1} B_{t-1}\lambda^{s-1} + \cdots + A_s B_t^{-1} B_0 \lambda^{s-t}.$$

and furthermore,

$$A = \tilde{Q}B + \tilde{R}$$

where $\deg \tilde{R} < s$. Applying the same procedure recursively to $\tilde{R}$, we can lower $\deg \tilde{R}$ below $\deg B$ (induction in $s - t$).

Uniqueness: Exercise. ☐

9.1.2 Equivalence of polynomial matrices

Definition 9.2. Let $A \in \mathbb{F}[\lambda]^{n\times n}$ be a square polynomial matrix. Then the following two operations on A are called *elementary operations*:

(i) Multiplying a row (or column) by a number $\alpha \in \mathbb{F}\backslash\{0\}$. Equivalently multiplying A from the left (or right) by the elementary matrix

$$E_i(\alpha) := \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ i \end{array} \end{array}$$

(ii) Adding a row (or column) multiplied by $a \in \mathbb{F}[\lambda]$ to another row (or column). Equivalently multiplying A from the left (or right) by the elementary matrix

$$F_{ij}(a) := \begin{array}{c} \begin{pmatrix} 1 & & a \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{array}{l} \leftarrow i \\ \\ \\ \end{array} \\ \begin{array}{c} \uparrow \\ j \end{array} \end{array} \qquad i \neq j$$

Two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ are called *equivalent* if they are related by a sequence of elementary transformations, or equivalently, if there exist elementary matrices $P_1, \dots, P_r, Q_1, \dots, Q_s$ such that

$$B = P_1 \cdots P_r A Q_1 \cdots Q_s. \tag{56}$$

Lemma 9.3.

(i) The inverse operations of elementary operations are elementary operations. Equivalently, all elementary matrices $E_i(\alpha), F_{ij}(a)$ are intevertible and its inverses given by elementary matrices.

(ii) The equivalence of polynomial matrices is an equivalence relation.

(iii) Interchanging two rows (or columns) is a sequence of elementary operations.

Proof. Exercise. □

We now define invariants of polynomial matrices under elementary operations. Up to a constant the determinant of a polynomial matrix is invariant under elementary operations, and so is the greatest common divisor of all entries of the matrix. This generalizes to the following set of invariants:

Definition 9.3. Let $A \in \mathbb{F}[\lambda]^{n\times n}$ be a square polynomial matrix of rank $\ell = \operatorname{rk} A$. Then for $k = 1, \dots, \ell$ the monic[13] greatest common divisor D_k of all $k \times k$ minors of A is called the *k-th minor divisor* of A. We also define $D_0 := 1$.

Lemma 9.4.

(i) The minor divisors are invariant under elementary operations.

(ii) D_k divides D_{k+1} for $k = 0, \dots, \ell - 1$.

Proof.

(i) Elementary operations turn a minor m into

$$\begin{aligned} & m \mapsto \alpha m, \quad \alpha \in \mathbb{F}\backslash\{0\}, \\ \text{or} \quad & m \mapsto m + a\tilde{m}, \quad a \in \mathbb{F}[\lambda], \end{aligned}$$

where $\tilde{m}$ is another minor of the same size.

(ii) By Laplace expansion of determinants.

□

Since each minor divisor divides the next, we can define further polynomial invariants by their quotients:

Definition 9.4. Let $A \in \mathbb{F}[\lambda]^{n\times n}$ be a square polynomial matrix of rank $\ell = \operatorname{rk} A$. Then for $k = 1, \dots, \ell$

$$I_k = \frac{D_k}{D_{k-1}}$$

is called the *k-th invariant factor* of A.

[13] Monic polynomials are polynomials with leading coefficient equal to 1.

Thus, the invariant factors can be obtained from the minor divisors, and vice versa,

$$D_k = I_1 \cdots I_k, \qquad k = 0, \dots, \ell.$$

By elementary operations a polynomial matrix can be diagonalized in such a way that its minor divisors and the invariant factors can be easily read off.

Theorem 9.5 (Smith normal form)**.**

(i) Any square polynomial matrix $A \in \mathbb{F}[\lambda]^{n\times n}$ is equivalent to a diagonal matrix

$$\begin{pmatrix} I_1 & & & & & \\ & \ddots & & & & \\ & & I_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where $I_1, \dots, I_k \in \mathbb{F}[\lambda]$ are monic polynomials, such that I_i divides I_{i+1}, $i = 1, \dots, k-1$.

(ii) The diagonal matrix in (i) is unique and called the ***Smith normal form*** *of A. Its entries $I_1, \dots, I_k$ are the invariant factors of A.*

Proof.

(i) Denote by

$$\delta(A) := \min\{\deg a_{ij} \mid a_{ij} \neq 0\}$$

the minimal degree of the non-zero entries of A. The decrease of this value is taken as an indicator of progress during the following algorithm, which uses elementary operations to reach the desired form of the matrix.

(a) Choose an entry a_{ij} with $\deg a_{ij} = \delta(A)$.
Make it monic.
Bring it to position a_{11}.

(b) Decrease degree along the 1st column (using polynomial division by a_{11}).
If $\delta(A)$ decreased: Go to (a).
Else: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{pmatrix}$.

(c) Decrease degrees along the 1st row (using polynomial division by a_{11}).
If $\delta(A)$ decreased: Go to (a).
Else: $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{pmatrix}$.

(d) *If* there exists an entry in $\tilde{A}$ not divisible by a_{11}: Decrease degree of this entry (using polynomial division by a_{11}) and go to (a).
Else-if $\tilde{A} = ()$ or $\tilde{A} = 0$: Terminate.
Else: Continue with $\tilde{A}$ at (a).

In section (b) and (c) the value of $\delta(A)$ can only strictly decrease finitely many times. Similarly, in the *if* clause of section (d), the value of $\delta(A)$ is decreased strictly. Thus, in all sections, the *else* clause must be reached after finitely many steps. Once the *else* clause of section (d) is reached, the algorithm continues with a matrix of smaller dimension. Since again, this can only happen finitely many times, the algorithm eventually terminates.

(ii) Because I_k divides I_{k+1} the minor divisors are given by

$$D_k = I_1 \cdots I_k, \qquad k = 0, \ldots, \ell.$$

and thus, $I_1, \ldots, I_\ell$ are the invariant factors. Since the invariant factors are uniquely determined by A, the diagonal matrix in (i) is unique.

□

Corollary 9.6. *The invariant factor I_k divides the invariant factor I_{k+1}.*

Exercise 9.2. Compute the invariant factors of the polynomial matrix

$$\begin{pmatrix} 0 & 0 & \lambda^2 - 1 \\ \lambda^2 + \lambda & \lambda^2 - \lambda & 0 \\ \lambda^2 & 0 & 0 \end{pmatrix}$$

by deriving its Smith normal form.

Theorem 9.7. *Two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ are equivalent, if and only if there exist two invertible polynomial matrices $P, Q \in \mathbb{F}[\lambda]^{n\times n}$ such that*

$$B = PAQ.$$

Proof.

($\Rightarrow$) By (56) and Lemma 9.3 (i).

($\Leftarrow$) We show that every invertible matrix is a product of elementary matrices. Let $P \in \mathbb{F}[\lambda]^{n\times n}$ be an invertible matrix. Then $\det P \in \mathbb{F}\backslash\{0\}$. Thus, the n-th minor divisor is equal to 1, and therefore, all invariant factors are equal to 1. Then its Smith normal form is given by the identity matrix I, and therefore,

$$P = P_1 \cdots P_r I P_1' \cdots P_s = P_1 \cdots P_r P_1' \cdots P_s$$

where $P_1, \ldots, P_r, P_1', \ldots, P_s$ are elementary matrices.

□

We summarize the different characterization of equivalence of polynomial matrices in the following theorem:

Theorem 9.8. *For $A, B \in \mathbb{F}[\lambda]^{n\times n}$ the following statements are equivalent:*

(i) A and B are equivalent (related by elementary operations).

(ii) $B = PAQ$ for some invertible $P, Q \in \mathbb{F}[\lambda]^{n\times n}$.

(iii) A and B have the same minor divisiors

(iv) A and B have the same invariant factors.

(v) A and B have the same Smith normal form.

9.1.3 Elementary divisors and Segre symbols

We introduce yet another way to encode the invariant factors of a polynomial matrix.

Let $A \in \mathbb{F}[\lambda]^{n\times n}$ be a polynomial matrix with invariant factors $I_1, \dots, I_\ell$. Since I_k divides I_{k+1} the irreducible factors of I_k also appear in I_{k+1} with greater or equal multiplicity. Let $e_1, \dots, e_s$ ($s \leqslant \ell$) be the irreducible factors of I_ℓ, then

$$\begin{aligned} I_1 &= e_1^{\mu_{11}} \cdots e_s^{\mu_{s1}}, \\ &\vdots \\ I_\ell &= e_1^{\mu_{1\ell}} \cdots e_s^{\mu_{s\ell}}. \end{aligned}$$

where $\mu_{ij} \in \mathbb{N} \cup \{0\}$ with

$$0 \leqslant \mu_{i1} \leqslant \cdots \leqslant \mu_{i\ell}, \qquad \mu_{i\ell} > 0$$

for $i = 1, \dots, s$.

Definition 9.5. For a polynomial matrix $A \in \mathbb{F}[\lambda]^{n\times n}$, the unordered list

$$e_1^{\mu_{11}}, \dots, e_1^{\mu_{1\ell}}, \dots, e_s^{\mu_{s1}}, \dots, e_s^{\mu_{s\ell}}$$

where entries with $\mu_{ij} = 0$ are dropped, is called the *list of elementary divisors* of A, while each entry $e_i^{\mu_{ij}}$ is called an *elementary divisor* or A.

Theorem 9.9. *Two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ are equivalent, if and only if they have the same rank and the same list of elementary divisors.*

Proof. Exercise. □

Exercise 9.3. Determine the list of elementary divisors for the polynomial matrix from Exercise 9.2.

Let $A \in \mathbb{C}[\lambda]^{n\times n}$ be a complex polynomial matrix. Then all irreducible factors are linear

$$e_i = \lambda - \lambda_i, \qquad i = 1, \dots, s$$

and can be identified with the corresponding roots $\lambda_1, \dots \lambda_s \in \mathbb{C}$.

Remark 9.1. For a real polynomial matrix $A \in \mathbb{R}[\lambda]^{n\times n}$ the irreducible factors can be of degree 1 or 2. While the invariant factors do not depend on whether A is taken as a real or complex matrix, the list of elementary divisors may differ.

If we interpret A as a one-parameter family of matrices, then for $\lambda \in \mathbb{C}$ the constant matrix $A(\lambda)$ has rank ℓ except for the $\lambda = \lambda_i$. Thus, the roots $\lambda_1, \dots, \lambda_s$ correspond to the matrices $A(\lambda_1), \dots, A(\lambda_s)$ in the family of lower rank. More generally:

Theorem 9.10. *Let $A \in \mathbb{C}[\lambda]^{n\times n}$ be a complex polynomial matrix of rank* $\operatorname{rk} A = \ell$, *and*

$$\begin{aligned} I_1 &= (\lambda - \lambda_1)^{\mu_{11}} \cdots (\lambda - \lambda_s)^{\mu_{s1}}, \\ &\vdots \\ I_\ell &= (\lambda - \lambda_1)^{\mu_{1\ell}} \cdots (\lambda - \lambda_s)^{\mu_{s\ell}}. \end{aligned}$$

its invariant factors. Then for $i, k = 1, \dots, \ell$ the value

$$\tilde{\mu}_{ik} := \mu_{i1} + \cdots + \mu_{ik}$$

is the least of the multiplicities of λ_i as a root of the $k \times k$ minors of A (if $\tilde{\mu}_{ik} = 0$, then λ_i is not a common root of the $k \times k$ minors at all). In particular, the rank of the constant matrices $A(\lambda_1), \dots, A(\lambda_s)$ is given by

$$\operatorname{rk} A(\lambda_i) = \max\{k \mid \tilde{\mu}_{ik} = 0\} = \max\{k \mid \mu_{ik} = 0\}.$$

If A is regular ($\ell = n$), then $\lambda_1, \dots, \lambda_s$ are the roots of the determinant

$$\det A \sim I_1 \cdots I_n = (\lambda - \lambda_1)^{\tilde{\mu}_{1n}} \cdots (\lambda - \lambda_s)^{\tilde{\mu}_{sn}}.$$

with multiplicities $\tilde{\mu}_{1n}, \dots, \tilde{\mu}_{sn}$. In particular, the constant matrices $A(\lambda_1), \dots, A(\lambda_s)$ are exactly the non-regular matrices in the family $A(\lambda)$, and furthermore,

$$\sum_{i=1}^{s} \tilde{\mu}_{in} = \sum_{i=1}^{s} \sum_{k=1}^{n} \mu_{ik} = \deg(\det A).$$

Proof. Follows from $D_k = I_1 \cdots I_k$, and in particular, if A is regular, $\det A \sim D_n = I_1 \cdots I_n$. □

Exercise 9.4. Interpret the polynomial matrix from Exercise 9.2 as a one-parameter family of matrices. What are the degenerate matrices in this family and what are their ranks?

If we drop all trivial elementary divisors, we can write the list of elementary divisors as

$$(\lambda - \lambda_1)^{\nu_{11}}, \dots, (\lambda - \lambda_1)^{\nu_{1h_1}}, \dots, (\lambda - \lambda_s)^{\nu_{s1}}, \dots, (\lambda - \lambda_s)^{\nu_{sh_s}}$$

where $\nu_{i1} \geqslant \cdots \geqslant \nu_{ih_i} > 0$, $h_i > 1$.

Definition 9.6. The symbol

$$[(\lambda_1 : \nu_{11}, \dots, \nu_{1h_1}), \dots, (\lambda_s : \nu_{s1}, \dots, \nu_{sh_s})]$$

is called the *characteristic* or *Segre symbol* of A. The numbers $(\nu_{11}, \dots, \nu_{1h_1})$ are called the *characteristic numbers* of λ_i. Given the list of roots $\lambda_1, \dots, \lambda_s$ the Segre symbol is sometimes abbreviated to the characteristic numbers only:

$$[(\nu_{11}, \dots, \nu_{1h_1}), \dots, (\nu_{s1}, \dots, \nu_{sh_s})]$$

Exercise 9.5. Determine the Segre symbol for the polynomial matrix from Exercise 9.2.

We summarize the conditions the Segre symbol of a complex polynomial matrix must satisfy in the following:

Proposition 9.11. *Let $A \in \mathbb{C}[\lambda]^{n \times n}$ a regular complex polynomial matrix. Then its Segre symbol*

$$[(\lambda_1 : \nu_{11}, \dots, \nu_{1h_1}), \dots, (\lambda_s : \nu_{s1}, \dots, \nu_{sh_s})]$$

satisfies

(i) $1 \leqslant s \leqslant \deg(\det A)$.

(ii) $\lambda_1, \dots, \lambda_s \in \mathbb{C}$, $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

(iii) $\nu_{i1} \geqslant \cdots \geqslant \nu_{ih_i} > 0$, $h_i \geqslant 1$.

(iv) $\sum_{i=1}^{s} \sum_{j=1}^{h_i} \nu_{ij} = \deg(\det A)$.

9.1.4 Equivalence of polynomial matrices of degree one

For two polynomial matrices of degree 1, the invertible matrices from Theorem 9.7 can be taken as constant matrices:

Theorem 9.12. *Let $A, B \in \mathbb{F}[\lambda]^{n\times n}$ be two polynomial matrices of degree 1 with invertible leading coefficient matrices. Then A and B are equivalent if and only if there exist two invertible constant matrices $S, T \in \mathbb{F}^{n\times n}$, such that*

$$B = SAT.$$

Proof.

(⇐) Follows directly from Theorem 9.7.

(⇒) Let A and B be equivalent, and $S, T \in \mathbb{F}[\lambda]^{n\times n}$ invertible such that

$$B = SAT,$$

or equivalently

$$S^{-1}B = AT. \tag{57}$$

Divide S^{-1} from the left by A and T from the right by B (the leading coefficient matrix of B is regular):

$$\begin{aligned} S^{-1} &= AQ + R, \\ T &= \tilde{Q}B + \tilde{R}. \end{aligned}$$

with $\deg R < \deg A = 1$ and $\deg \tilde{R} < \deg B = 1$. Thus, R and $\tilde{R}$ are constant matrices and we obtain

$$(AQ + R)B = A(\tilde{Q}B + \tilde{R}),$$

or equivalently,

$$A(Q - \tilde{Q})B = A\tilde{R} - RB.$$

Assume $Q \neq \tilde{Q}$. Then, the left-hand side has degree at least 2 (since the leading coefficient matrices of A and B are regular), and the right-hand side has at most degree 1. Thus, $Q = \tilde{Q}$, and therefore

$$A\tilde{R} = RB.$$

It remains to show that R and $\tilde{R}$ are invertible. Divide S from the left by B (the leading coefficient matrix of B is regular):

$$S = B\hat{Q} + \hat{R}$$

with $\deg \hat{R} < \deg B = 1$, and thus, $\hat{R}$ is a constant matrix. Using (57) we obtain

$$\begin{aligned} I &= S^{-1}S \\ &= S^{-1}(B\hat{Q} + \hat{R}) \\ &= AT\hat{Q} + S^{-1}\hat{R} \\ &= AT\hat{Q} + (AQ + R)\hat{R} \\ &= A(T\hat{Q} + QR) + R\hat{R} \end{aligned}$$

Assume $T\hat{Q}+QR \neq 0$. Then, the right-hand side would have at least degree 1 (since the leading coefficient matrix of A is regular). Thus, $T\hat{Q}+QR = 0$ and therefore

$$I = R\hat{R}.$$

Hence, R is invertible, and so is $\tilde{R}$ by symmetry.

□

Remark 9.2. The equivalence of two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ with constant matrices $S, T \in \mathbb{F}^{n\times n}$ is the same as the simultaneous equivalence of the coefficient matrices of A and B. In particular, in the case of degree 1, let $A_1, A_0, B_1, B_0 \in \mathbb{F}^{n\times n}$ be the coefficient matrices of A and B:

$$A = A_1\lambda + A_0, \qquad B = B_1\lambda + B_0.$$

Then

$$B_1\lambda + B_0 = S(A_1\lambda + A_0)T = SA_1T\lambda + SA_0T$$

is equivalent to

$$B_1 = SA_1T \quad \text{and} \quad B_0 = SA_0T.$$

Thus, the two pairs (A_1, A_0) and (B_1, B_0) of constant coefficient matrices are equivalent.

For complex polynomial matrices of degree 1 we can narrow down the conditions on the Segre symbol from Proposition 9.11:

Proposition 9.13. *Let $A \in \mathbb{C}[\lambda]^{n\times n}$ a complex polynomial matrix of degree 1 with invertible leading coefficient matrix. Then its Segre symbol*

$$[(\lambda_1 : \nu_{11}, \ldots, \nu_{1h_1}), \ldots, (\lambda_s : \nu_{s1}, \ldots, \nu_{sh_s})]$$

satisfies

(i) $1 \leqslant s \leqslant n$.

(ii) $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$, $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

(iii) $\nu_{i1} \geqslant \cdots \geqslant \nu_{ih_i} > 0$, $h_i \geqslant 1$.

(iv) $\sum_{i=1}^{s} \sum_{j=1}^{h_i} \nu_{ij} = n$.

Furthermore, if $A = A_1\lambda + A_0$ with $A_1 \neq \alpha A_0$ for some $\alpha \in \mathbb{C}\backslash\{0\}$, then additionally

(v) $s > 1$ *or* $h_1 < n$.

Proof. $A = A_1\lambda + A_0$ with $\det A_1 \neq 0$ ensures that

$$\det A = \det A_1\lambda^n + O(n-1)$$

has degree n. Thus, conditions (i) - (iv) follow from Proposition 9.11.

Regarding condition (v), assume $s = 1$ and $h_1 \geqslant n$. Then, $h_1 = n$ because of condition (iv). Thus, the list of elementary divisors is

$$(\lambda - \lambda_1), \cdots, (\lambda - \lambda_1).$$

and therefore all invariant factors are equal to $(\lambda - \lambda_1)$. This implies that

$$A_1\lambda + A_0 = (\lambda - \lambda_1)P$$

for some matrix P, and thus

$$A_0 = -\lambda P = -\lambda A_1,$$

which contradicts $A_1 \neq \alpha A_0$. $\square$

The Segre symbol of degree 1 polynomial matrix is invariant under the following change of variable:

$$\lambda = \frac{a\tilde{\lambda} + b}{c\tilde{\lambda} + d}$$

Proposition 9.14. *Let $A \in \mathbb{C}[\lambda]^{n\times n}$ be a polynomial matrix of degree 1 with regular leading coefficient matrix, and let*

$$[(\lambda_1 : \nu_{11}, \dots, \nu_{1h_1}), \dots, (\lambda_s : \nu_{s1}, \dots, \nu_{sh_s})]$$

be the Segre symbol of A.

Let $A = A_1\lambda + A_0$ with $A_1, A_0 \in \mathbb{C}^{n\times n}$, $\det A_1 \neq 0$, and $\tilde{A}_1, \tilde{A}_0 \in \mathbb{C}^{n\times n}$ with

$$\begin{aligned} \tilde{A}_1 &= aA_1 + cA_0 \\ \tilde{A}_0 &= bA_1 + dA_0 \end{aligned}$$

with $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ such that $\det \tilde{A}_1 \neq 0$. Then

$$\left[\left(\tilde{\lambda}_1 : \nu_{11}, \dots, \nu_{1h_1}\right), \dots, \left(\tilde{\lambda}_s : \nu_{s1}, \dots, \nu_{sh_s}\right)\right]$$

is the Segre symbol of $\tilde{A} = \tilde{A}_1\lambda + \tilde{A}_0$, where

$$\lambda_i = \frac{a\tilde{\lambda}_i + b}{c\tilde{\lambda}_i + d}, \qquad i = 1, \dots, s.$$

Proof. We find

$$\begin{aligned} \tilde{A} = \tilde{A}_1\lambda + \tilde{A}_0 &= (aA_1 + cA_0)\lambda + bA_1 + dA_0 \\ &= A_1(a\lambda + b) + A_0(c\lambda + d) \\ &= (c\lambda + d)\left(A_1\frac{a\lambda + b}{c\lambda + d} + A_0\right). \end{aligned}$$

Thus, the polynomial entries a_{ij} of A and $\tilde{a}_{ij}$ of $\tilde{A}$ are related by

$$\tilde{a}_{ij}(\lambda) = (c\lambda + d)a_{ij}\left(\frac{a\lambda + b}{c\lambda + d}\right).$$

Therefore, a k-minor $d(\lambda)$ of A and a k-minor $\tilde{d}(\lambda)$ of $\tilde{A}$ involving the same entries are related by

$$\tilde{d}(\lambda) = (c\lambda + d)^k d\left(\frac{a\lambda + b}{c\lambda + d}\right).$$

For $i = 1, \dots, s$ consider the the matrix $A_1\lambda_i + A_0$, and let $\tilde{\lambda}_i$ such that

$$A_1\lambda_i + A_0 \sim \tilde{A}_1\tilde{\lambda}_i + \tilde{A}_0 = A_1(a\tilde{\lambda}_i + b) + A_0(c\tilde{\lambda}_i + d)$$

This is possible, since $\det \tilde{A}_1 \neq 0$. Furthermore $c\tilde{\lambda}_i + d \neq 0$, and thus,

$$\lambda_i = \frac{a\tilde{\lambda}_i + b}{c\tilde{\lambda}_i + d}.$$

Assume λ_i is a root of $d(\lambda)$ with multiplicity ν, i.e.,

$$d(\lambda) = (\lambda - \lambda_i)^\nu g(\lambda)$$

with $g(\lambda_i) \neq 0$. Then

$$\begin{aligned}
\tilde{d}(\lambda) &= (c\lambda + d)^k \left(\frac{a\lambda + b}{c\lambda + d} - \frac{a\tilde{\lambda}_i + b}{c\tilde{\lambda}_i + d}\right)^\nu g\left(\frac{a\lambda + b}{c\lambda + d}\right) \\
&= \frac{(ad - bc)^\nu}{(c\tilde{\lambda}_i + d)^\nu}(c\lambda + d)^{k-\nu}(\lambda - \tilde{\lambda}_i)^\nu g\left(\frac{a\lambda + b}{c\lambda + d}\right)
\end{aligned}$$

Since $\nu \leqslant k$, $c\tilde{\lambda}_i + d \neq 0$, and $g\left(\frac{a\tilde{\lambda}_i + b}{c\tilde{\lambda}_i + d}\right) \neq 0$, this implies that $\tilde{\lambda}_i$ is a root of $\tilde{d}(\lambda)$ with multiplicity ν. ☐

9.1.5 Similarity of constant matrices

If a constant matrix $A \in \mathbb{F}^{n \times n}$ is interpreted as the representative matrix of a linear endomorphism with respect to a given basis, then the representative matrices of the same linear map with respect to other bases are given by similar matrices.

Definition 9.7. Two constant matrices $A, B \in \mathbb{F}^{n \times n}$ are called *similar* (or *conjugate*) if there exists a constant invertible matrix $S \in \mathbb{F}^{n \times n}$ such that

$$B = SAS^{-1}.$$

Theorem 9.15. *Two constant matrices $A, B \in \mathbb{F}^{n \times n}$ are similar, if and only if their characteristic matrices $\lambda I - A$ and $\lambda I - B$ are equivalent (as polynomial matrices).*

Proof.

($\Rightarrow$) Let A and B be similar, i.e., $B = SAS^{-1}$ with some constant invertible matrix $S \in \mathbb{F}^{n \times n}$. Then the characteristic matrix of B

$$\lambda I - B = \lambda I - SAS^{-1} = S(\lambda I - A)S^{-1}$$

is equivalent to A.

($\Leftarrow$) Let $\lambda I - A$ and $\lambda I - B$ be equivalent. Then, by Theorem 9.12, there exist constant invertible matrixes $S, T \in \mathbb{F}^{n \times n}$, such that

$$\lambda I - B = S(\lambda I - A)T,$$

As in Remark 9.2, this is the simultaneous equivalence of the two pairs of constant coefficient matrices (I, A) and (I, B), i.e.,

$$I = ST, \quad \text{and} \quad B = SAT.$$

In particular, $T = S^{-1}$.

☐

9.1.6 Congruence of pairs of constant matrices

If a constant matrix $A \in \mathbb{F}^{n\times n}$ is interpreted as the Gram matrix of a bilinear form with respect to a given basis, then the Gram matrices of the same bilinear form with respect to other bases are given by congruent matrices.

Definition 9.8. Two constant matrices $A, B \in \mathbb{F}^{n\times n}$ are called *congruent* if there exists a constant invertible matrix $S \in \mathbb{F}^{n\times n}$ such that

$$B = SAS^{\mathsf{T}}.$$

Now, similar to Remark 9.2, the congruence of two polynomial matrices $A, B \in \mathbb{F}[\lambda]^{n\times n}$ with constant matrix $S \in \mathbb{F}^{n\times n}$ is the same as the simultaneous congruence of the coefficient matrices of A and B. In particular, in the case of degree 1, let $A_1, A_0, B_1, B_0 \in \mathbb{F}^{n\times n}$ be the coefficient matrices of A and B:

$$A = A_1\lambda + A_0, \qquad B = B_1\lambda + B_0.$$

Then

$$B_1\lambda + B_0 = S(A_1\lambda + A_0)S^{\mathsf{T}} = SA_1S^{\mathsf{T}}\lambda + SA_0S^{\mathsf{T}}$$

is equivalent to

$$B_1 = SA_1S^{\mathsf{T}} \quad \text{and} \quad B_0 = SA_0S^{\mathsf{T}}.$$

Thus, the two pairs (A_1, A_0) and (B_1, B_0) of constant coefficient matrices are congruent.

The Gram matrix of a symmetric bilinear form is a symmetric matrix. For the congruence of complex symmetric matrices of degree 1 we obtain the following relation to equivalence of polynomial matrices.

Theorem 9.16. *Let $A, B \in \mathbb{C}[\lambda]^{n\times n}$ be two complex symmetric matrices of degree 1 with invertible leading coefficient matrices. Then A and B are equivalent if and only if they are congruent (by a constant congruence matrix), i.e., there exists an invertible constant matrix $S \in \mathbb{C}^{n\times n}$ such that*

$$B = SAS^{\mathsf{T}}.$$

Proof.

($\Leftarrow$) Follows directly from Theorem 9.7.

($\Rightarrow$) Let A and B be equivalent. By Theorem 9.12, there exist two constant matrices $S, T \in \mathbb{C}^{n\times n}$ such that

$$B = SAT.$$

In particular SAT is symmetric. By Lemma 9.17, this implies that

$$UA = AU^{\mathsf{T}}, \quad \text{with} \quad U := T^{-\mathsf{T}}S.$$

By Lemma 9.18, there exists a polynomial square root $R \in \mathbb{C}^{n\times n}$ of U, i.e.,

$$R = \sum_{k=0}^{r} a_k U^k$$

such that

$$U = R^2,$$

Since R is a polynomial expression in U it also satisfies

$$RA = AR^{\intercal},$$

and we obtain

$$B = SAT = T^{\intercal}T^{-\intercal}SAT = T^{\intercal}UAT = T^{\intercal}R^2AT = T^{\intercal}RAR^{\intercal}T$$

Thus, with $\tilde{S} := T^{\intercal}R$, we have

$$B = \tilde{S}^{\intercal}A\tilde{S}.$$

□

Lemma 9.17. *Let $S, T \in \mathbb{F}^{n\times n}$ be two invertible constant matrices, and $A \in \mathbb{F}[\lambda]^{n\times n}$ a symmetric polynomial matrix. Then the following three statements are equivalent:*

(i) SAT is symmetric.

(ii) $T^{-\intercal}SA$ is symmetric.

(iii) $UA = AU^{\intercal}$ with $U := T^{-\intercal}S$.

Proof. Exercise. □

Lemma 9.18. *Let $U \in \mathbb{C}^{n\times n}$ be an complex invertible constant matrix. Then there exist $r \in \mathbb{N}$, and $a_0, \ldots, a_r \in \mathbb{C}$ such that*

$$R = \sum_{k=0}^{r} a_k U^k$$

is a square root of U, i.e.,

$$U = R^2.$$

No proof. □

Corollary 9.19. *Let $A_1, A_0, B_1, B_0 \in \mathbb{C}^{n\times n}$ be symmetric matrices with $\det A_1 \neq 0$, $\det B_1 \neq 0$. Then the following statements are equivalent:*

(i) (A_1, A_0) and (B_1, B_0) are simultaneously congruent (by the same constant congruence).

(ii) $A_1\lambda + A_0$ and $B_1\lambda + B_0$ are equivalent (as polynomial matrices).

(iii) $A_1\lambda + A_0$ and $B_1\lambda + B_0$ have the same Segre symbols.

The conditions from Proposition 9.13 are necessary conditions on the Segre symbol of linear symmetric families, but are they also sufficient for symmetric matrices? Can all such Segre symbols be generated by polynomial matrices of the form

$$A_1\lambda + A_0$$

with $A_1, A_0 \in \mathbb{C}^{n\times n}$ symmetric, $\det A_1 \neq 0$?

Lemma 9.20. *Let $\nu \in \mathbb{N}$, $\alpha \in \mathbb{C}$, and*

$$C_1(\nu) := \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \qquad C_0(\alpha,\nu) := \begin{pmatrix} & & & -\alpha \\ & & \iddots & 1 \\ & -\alpha & \iddots & \\ -\alpha & 1 & & \end{pmatrix} \in \mathbb{C}^{\nu\times\nu}$$

Then the polynomial matrix $C_1(\nu)\lambda + C_0(\alpha,\nu)$ has only one elementary divisor:

$$(\lambda-\alpha)^{\nu}.$$

Proof. The last minor divisor (the determinant up to a constant) of

$$C_1(\nu)\lambda + C_0(\alpha,\nu) = \begin{pmatrix} & & & \lambda-\alpha \\ & & \iddots & 1 \\ & \lambda-\alpha & \iddots & \\ \lambda-\alpha & 1 & & \end{pmatrix},$$

is given by

$$D_\nu = (\lambda-\alpha)^{\nu}.$$

The $(\nu-1)\times(\nu-1)$ minor, obtained by erasing the first row and column is equal to 1. Thus,

$$D_{\nu-1} = \cdots = D_1 = 1.$$

□

Theorem 9.21. *Let $\lambda_1,\dots,\lambda_s \in \mathbb{C}$ and $\nu_{11},\dots,\nu_{1h_1},\dots\nu_{s1},\dots,\nu_{sh_s} \in \mathbb{N}$ satisfying conditions (i) - (iv) from Proposition 9.13. Let*

$$C_1 := \begin{pmatrix} C_1(\nu_{11}) & & & & & \\ & \ddots & & & & \\ & & C_1(\nu_{1h_1}) & & & \\ & & & \ddots & & \\ & & & & C_1(\nu_{s1}) & & \\ & & & & & \ddots & \\ & & & & & & C_1(\nu_{sh_s}) \end{pmatrix}$$

and

$$C_0 := \begin{pmatrix} C_0(\lambda_1,\nu_{11}) & & & & & \\ & \ddots & & & & \\ & & C_0(\lambda_1,\nu_{1h_1}) & & & \\ & & & \ddots & & \\ & & & & C_0(\lambda_s,\nu_{s1}) & & \\ & & & & & \ddots & \\ & & & & & & C_0(\lambda_1,\nu_{sh_s}) \end{pmatrix}$$

Then the polynomial matrix $C_1\lambda + C_0$ is symmetric with $\det C_1 \neq 0$ and has Segre symbol

$$[(\lambda_1 : \nu_{11},\dots,\nu_{1h_1}),\dots,(\lambda_s : \nu_{s1},\dots,\nu_{sh_s})].$$

Proof. By Lemma 9.20, each block $C_1(\nu_{ij})\lambda + C_0(\lambda_i,\nu_{ij})$ has Smith normal form

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (\lambda-\lambda_i)^{\nu_{ij}} \end{pmatrix}.$$

Thus, the matrix $C_1\lambda + C_0$ is equivalent to

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & (\lambda-\lambda_1)^{\nu_{11}} & & & & & \\ & & & & \ddots & & & & \\ & & & & & (\lambda-\lambda_1)^{\nu_{1h_1}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & (\lambda-\lambda_s)^{\nu_{s1}} & \\ & & & & & & & & \ddots \\ & & & & & & & & & (\lambda-\lambda_s)^{\nu_{sh_s}} \end{pmatrix}.$$

From there we read off the minor divisors as

$$\begin{aligned} D_n &= (\lambda - \lambda_1)^{\nu_{11}+\cdots+\nu_{1h_1}} \cdots (\lambda - \lambda_s)^{\nu_{s1}+\cdots+\nu_{sh_s}} \\ D_{n-1} &= (\lambda - \lambda_1)^{\nu_{12}+\cdots+\nu_{1h_1}} \cdots (\lambda - \lambda_s)^{\nu_{s2}+\cdots+\nu_{sh_s}} \\ &\vdots \end{aligned}$$

And thus, the elementary divisors of $C_1\lambda + C_0$ are given by

$$(\lambda - \lambda_1)^{\nu_{11}}, \ldots, (\lambda - \lambda_1)^{\nu_{1h_1}}, \ldots, (\lambda - \lambda_s)^{\nu_{s1}}, \ldots, (\lambda - \lambda_s)^{\nu_{sh_s}}.$$

□

9.2 Classification of pencils of quadrics in $\mathbb{C}\mathrm{P}^n$

Definition 9.9. Let $\mathcal{P} \subset \mathrm{P}\,\mathrm{Sym}(\mathbb{C}^{n+1})$ be a pencil of quadrics. Let $\mathcal{Q}_1, \mathcal{Q}_0$ be two quadrics in $\mathcal{P}$ with representative matrices $Q_1, Q_0 \in \mathrm{Sym}(\mathbb{C}^{n+1})$. Then we call $Q_1\lambda + Q_0 \in \mathbb{C}[\lambda]^{(n+1)\times(n+1)}$ a *characteristic matrix* of $\mathcal{P}$.

A characteristic matrix uniquely determines its pencil together with the two quadrics spanning it. Vice versa, two characteristic matrices

$$Q_1\lambda + Q_0, \quad \text{and} \quad \tilde{Q}_1\lambda + \tilde{Q}_0,$$

describe the same pencil if and only if

$$\begin{aligned} \tilde{Q}_1 &= aQ_1 + cQ_0, \\ \tilde{Q}_0 &= bQ_1 + dQ_0, \end{aligned}$$

with $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. And thus, the corresponding values of λ are related by

$$\lambda = \frac{a\tilde{\lambda} + b}{c\tilde{\lambda} + d}.$$

Furthermore, a change of basis, or a projective transformation acts on the characteristic matrix $Q_1\lambda + Q_0$ as

$$F^\intercal(Q_1\lambda + Q_0)F = F^\intercal Q_1 F\lambda + F^\intercal Q_0 F.$$

Theorem 9.22. *Let $\mathcal{P}, \tilde{\mathcal{P}} \subset \mathrm{P}\,\mathrm{Sym}(\mathbb{C}^{n+1})$ be two regular pencils of quadrics in $\mathbb{C}\mathrm{P}^n$. Then the following statements are equivalent:*

(i) $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are projectively equivalent.

(ii) There exist characteristic matrices of that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are equivalent.

(iii) There exist characteristic matrices of that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ that have the same Segre symbol.

Proof. Follows from Corollary 9.19. □

If two characteristic matrices are equivalent, how are the Segre symbols of all the other characteristic matrices of the same pencils related? This question is answered by Proposition 9.14. We first use this statement to associate the entries from the Segre symbol to the degenerate quadrics of the pencil, before we come back to a refinement of the classification result.

If $\det Q_1 \neq 0$, then the determinant of a characteristic matrix $\det(Q_1\lambda+Q_0)$ has degree $n+1$ and finitely many roots $\lambda_1, \dots, \lambda_s \in \mathbb{C}$, which correspond to the degenerate quadrics $\mathcal{P}(\lambda_1), \dots, \mathcal{P}(\lambda_s)$ of the pencil. Each root λ_i has an entry of characteristic numbers $(\lambda_i : \nu_{i1}, \dots, \nu_{ih_i})$ in the Segre symbol of the characteristic matrix $Q_1\lambda + Q_0$.

Remark 9.3. One can drop the condition $\det Q_1 \neq 0$ and allow ∞ as a root of $\det(Q_1\lambda + Q_0)$. The associated characteristic numbers in the Segre symbol are the ones of the 0 as a root of $\det(Q_1 + Q_0\lambda)$.

By Proposition 9.14, the characteristic numbers in the Segre symbol, are independent of the choice of quadrics to span the pencil. Furthermore, by Corollary 9.19 they are invariant under a change of basis, or a projective transformation. Thus, the characteristic numbers are well-defined attributes of the degenerate quadrics of the pencil

Definition 9.10. Let $\mathcal{P} \subset \mathrm{P}\,\mathrm{Sym}(\mathbb{C}^{n+1})$ be a regular pencil of quadrics with characteristic matrix $Q_1\lambda + Q_0$. Then the *characteristic numbers* of each entry of the Segre symbol

$$(\lambda_i : \nu_{i1}, \dots, \nu_{1h_i}),$$

is called the characteristic numbers of the associated degenerate quadric with representative matrix $Q_1\lambda_i + Q_0$.

Proposition 9.23. *Let $\mathcal{P} \subset \mathrm{P}\,\mathrm{Sym}(\mathbb{C}^{n+1})$ be a regular pencil of quadrics with characteristic matrix $Q_1\lambda + Q_0$. Let $\mathcal{D}$ be a degenerate quadric in $\mathcal{P}$ with representative matrix $Q_1\lambda_i + Q_0$ and characteristic numbers*

$$(\nu_{i1}, \dots, \nu_{1h_i}).$$

Then the dimension of the projective subspace of singular points of $\mathcal{D}$ is given by

$$n - \mathrm{rk}(Q_1\lambda_i + Q_0) = h_i - 1.$$

Proof. Follows from Theorem 9.10. □

We now come back to the classification of pencils of quadrics in $\mathbb{C}\mathrm{P}^n$.

Theorem 9.24. *Let $\mathcal{P}, \tilde{\mathcal{P}} \subset \mathrm{P}\,\mathrm{Sym}(\mathbb{C}^{n+1})$ be two regular pencils of quadrics in $\mathbb{C}\mathrm{P}^n$. Then the following statements are equivalent:*

(i) $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are projectively equivalent.

(ii) They have same number of degenerate quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_s \in \mathcal{P}$ and $\tilde{\mathcal{Q}}_1, \dots, \tilde{\mathcal{Q}}_s \in \tilde{\mathcal{P}}$, which as points on the line $\mathcal{P}$ and the line $\tilde{\mathcal{P}}$ are related by a (one-dimensional) projective transformation $\mathcal{P} \to \tilde{\mathcal{P}}$, and the characteristic numbers of corresponding degenerate quadrics are equal.

(iii) *Any two characteristic matrices of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ have the same Segre symbols up to a projective transformation of the roots.*

Proof.

($\Rightarrow$) Follows from Proposition 9.14 and Corollary 9.19.

($\Leftarrow$) If the points of the degenerate quadrics are related by a projective transformation, we can choose quadrics $\mathcal{Q}_1, \mathcal{Q}_{\prime} \in \mathcal{P}$ and $\tilde{\mathcal{Q}}_1, \tilde{\mathcal{Q}}_2 \in \tilde{\mathcal{P}}$ to span the two pencils such that the corresponding degenerate quadrics in $\mathcal{P}$ and $\tilde{\mathcal{P}}$ correspond to the same roots λ_i. The two correspoding characteristic matrices $Q_1\lambda + Q_0$ and $\tilde{Q}_1\lambda + Q_0$ therefore have the same Segre symbol, and thus are equivelent. By Theorem 9.19 they are also congruent, which yields the projective equivalence.

$\square$

Thus, the equivalence classes of pencils of quadrics in $\mathbb{C}\mathrm{P}^n$ can be parametrized by the complex roots $\lambda_1, \ldots, \lambda_s \in \mathbb{C} \cup \{\infty\}$ up to projective transformation ($s-3$ parameters) and, by Theorem 9.21, characteristic numbers satisfying the conditions in Proposition 9.13.

9.2.1 Pencils of conics in $\mathbb{C}\mathrm{P}^2$

A pencil in $\mathbb{C}\mathrm{P}^2$ has at most 3 degenerate quadrics. The corresponding roots can always be mapped to, say $0, 1, \infty$. Thus, we obtain 5 different equivalence classes:

(i) $[1, 1, 1]$

(ii) $[(1, 1), 1]$

(iii) $[2, 1]$

(iv) $[3]$

(v) $[(2, 1)]$

9.2.2 Pencils of quadrics in $\mathbb{C}\mathrm{P}^3$

A pencil in $\mathbb{C}\mathrm{P}^3$ has at most 4 degenerate quadrics. In case (i) where the pencil has 4 distinct degenerate quadrics, we can map only 3 of the corresponding roots to, say $0, 1, \infty$, and are left with one further complex parameter, which describes a continues spectrum of equivalence classes. The other cases, where the pencil has at most 3 degenerate quadrics, describe only one single equivalence class:

(i) $[(0:1), (1:1), (\infty:1), (\lambda:1)]$ with $\lambda \in \mathbb{C} \backslash \{0, 1\}$.

(ii) $[(1, 1), 1, 1]$

(iii) $[2, 1, 1]$

(iv) $[(1, 1), (1, 1)]$

(v) $[2, (1, 1)]$

(vi) $[2, 2]$

(vii) $[(1,1,1),1]$

(viii) $[(2,1),1]$

(ix) $[3,1]$

(x) $[(2,1,1)]$

(xi) $[(2,2)]$

(xii) $[(3,1)]$

(xiii) $[4]$

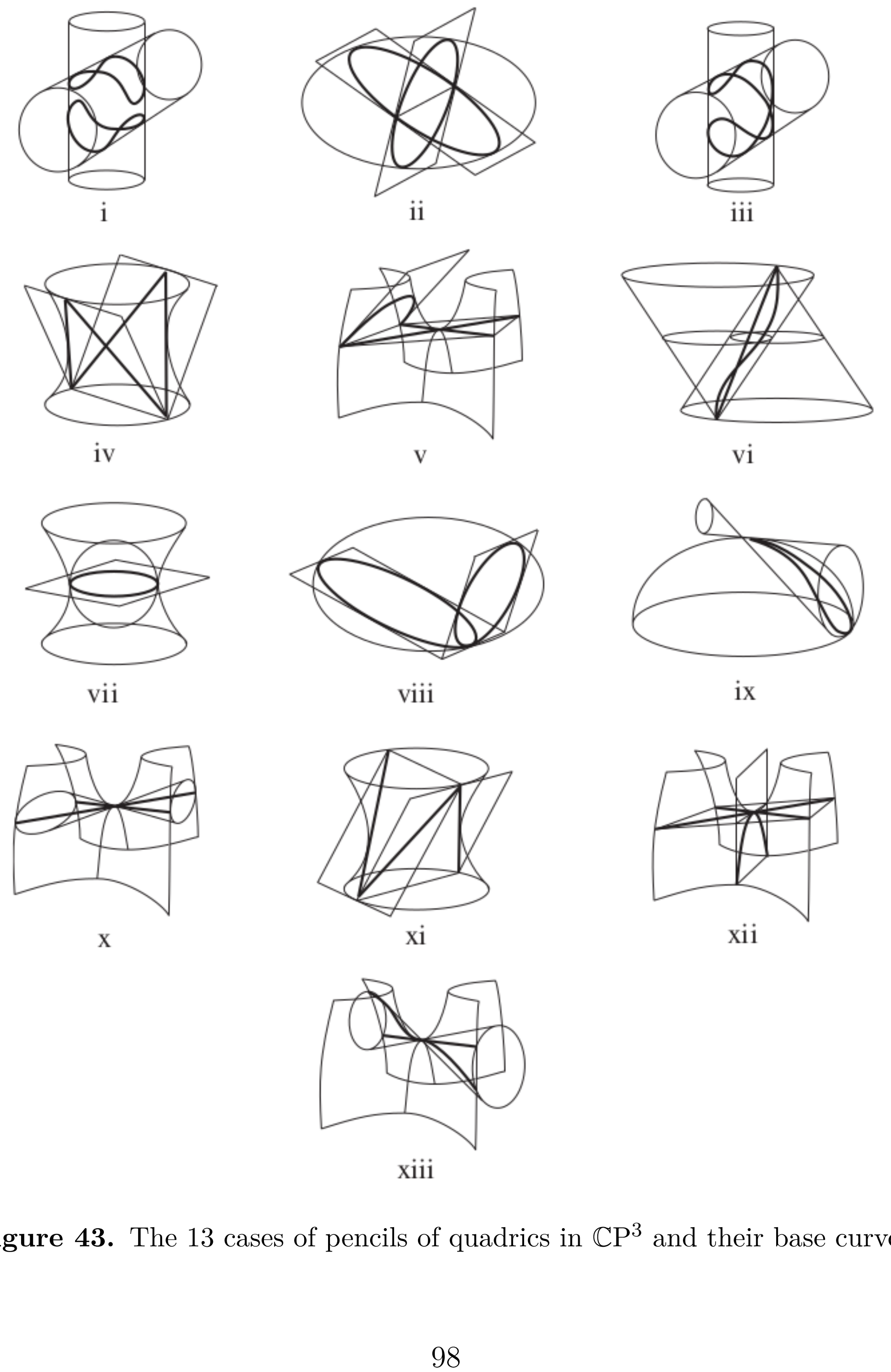

Figure 43. The 13 cases of pencils of quadrics in $\mathbb{C}\mathrm{P}^3$ and their base curves.

A Pencils of conics

- A *conic* in the real projective plane is a set
$$\left\{[x] \subset \mathbb{R}\mathrm{P}^2 \;\middle|\; q(x,x) = 0\right\}$$
where q is a symmetric bilinear form on $\mathbb{R}^3$.

  In homogeneous coordinates w.r.t. a basis $e_1, e_2, e_3 \in \mathbb{R}^3$ the conic can be represented by its *Gram matrix*
$$Q_{ij} = q(e_i, e_j), \qquad i, j = 1, 2, 3.$$
It is a symmetric matrix $Q^\intercal = Q$ and
$$q(x, y) = x^\intercal Q y.$$

- The *dual conic* $\mathcal{C}^\star \subset (\mathbb{R}\mathrm{P}^2)^*$ of a conic $\mathcal{C} \subset \mathbb{R}\mathrm{P}^2$ is the set of all points in the dual space $(\mathbb{R}\mathrm{P}^2)^*$ that correspond to the tangent lines $\mathcal{C}$. Its Gram matrix w.r.t. to the dual basis is given by
$$Q^\star = Q^{-1}$$

- The *space of conics* is a 5-dimensional real projective space for which the 6 non-redundant entries of the Gram-matrix can be taken as homogeneous coordinates.

- A *pencil of conics* is a family of conics corresponding to a line in the space of conics:
$$\mathcal{C}_\lambda = \left\{[x] \in \mathbb{R}\mathrm{P}^2 \;\middle|\; q_1(x,x) + \lambda q_2(x,x) = 0\right\}, \qquad \lambda \in \mathbb{R} \cup \{\infty\}.$$
The *base points* of a pencil is the set of points that is contained in all conics of the pencil. A pencil of conics has up to 4 base points.

  A pencil of conics contains up to 3 degenerate conics given by
$$\det(Q_1 + \lambda Q_2) = 0.$$

- A *dual pencil of conics* is a family of dual conics, which are dual to the conics of a pencil of conics.

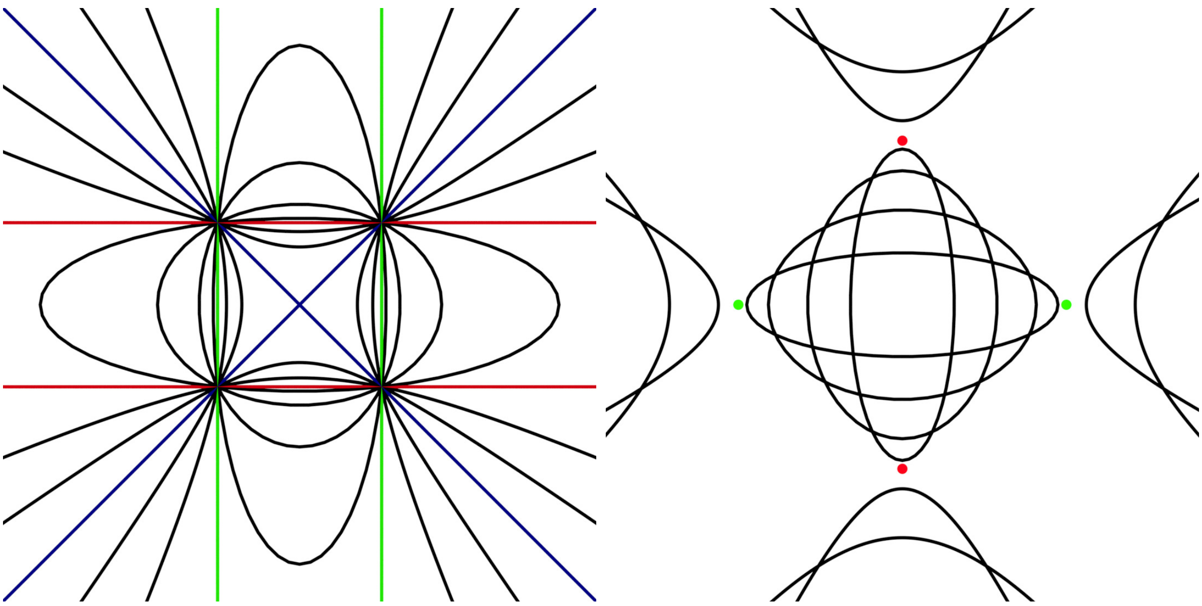

Figure 44. A pencil of conics with 4 real base points and the corresponding dual pencil.

B Elliptic functions

Definition B.1. A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is *periodic* with period $\omega \in \mathbb{C}$ if

$$f(z+\omega) = f(z) \quad \text{for all} \quad z \in \mathbb{C}.$$

Example B.1. The holomorphic sine and cosine functions have periods 2π:

$$\cos(z+2\pi) = \cos(z), \qquad \sin(z+2\pi) = \sin(z).$$

- If $\omega \in \mathbb{C}$ is a period of f then so is $n\omega$ for any $n \in \mathbb{Z}$.
- By the uniqueness theorem for holomorphic fuctions the set of periods cannot have a (finite) accumulation point in $\mathbb{C}$.
- Thus, along any line $\omega\mathbb{R}$ gererated by a period ω we can choose the period ω_1 closest to 0, such that $\mathbb{Z}\omega_1$ is the set of all periods along that line.
- A non-constant meromorphic function cannot have more than two $\mathbb{R}$-linearly independent periods, i.e., two periods $\omega_1, \omega_2 \in \mathbb{C}$ with

  $$\operatorname{Im} \frac{\omega_2}{\omega_1} \neq 0.$$

- If ω_1 and ω_2 are chosen to be the periods closest to 0 along their lines, respectively, then the set of periods of f is given by

  $$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}.$$

  The lattice Λ is the free abelian group generated by ω_1 and ω_2.

Definition B.2. Let

$$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

be a lattice. Then a meromorphic function with

$$f(z+\omega) = f(z) \quad \text{for all} \quad z \in \mathbb{C} \quad \text{and} \quad \omega \in \Lambda$$

is called an *elliptic function* with *period lattice* Λ. For $z_0 \in \mathbb{C}$ the set

$$\Pi = \{z_0 + \lambda_1\omega_1 + \lambda_2\omega_2 \mid 0 \leqslant \lambda_1 < 1,\, 0 \leqslant \lambda_2 < 1\}$$

is called a *fundamental parallelogram* of f.

- The choice of generators for Λ is not unique. Two complex numbers $\tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{C}$, $\operatorname{Im} \frac{\tilde{\omega}_2}{\tilde{\omega}_1} > 0$ generate the same lattice Λ, if and only if

  $$\begin{aligned} \tilde{\omega}_1 &= a\omega_1 + b\omega_2 \\ \tilde{\omega}_2 &= c\omega_1 + d\omega_2 \end{aligned}, \quad \text{with} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = \pm 1,$$

- The quotient group ${}^{\mathbb{C}}\!/_{\Lambda}$, can be thought as the domain of the elliptic function f. It is topologically a torus, obtained by identifying opposite edges of fundamental parallelogram.

B.1 General properties of elliptic functions

The following theorems summarize the main general properties of elliptic functions.

Theorem B.1. *A holomorphic elliptic function is constant.*

Proof. Liouville's theorem for bounded entire functions. □

Theorem B.2. *An elliptic function has finitely many poles in* Π *and the sum of its residues is zero.*

Proof. A meromorphic function can only have finitely many poles in the bounded fundamental parallelogram Π. We can always choose Π such that its boundary $\partial\Pi$ does not contain any of the poles. Then due to the periodicity of f we obtain

$$\frac{1}{2\pi i}\int_{\partial\Pi} f(z)dz = 0.$$

□

Theorem B.3. *A non-constant elliptic function takes on every value in* Π *the same number of times (counting multiplicities). This number is equal to the number of poles in* Π *(counting multiplicities).*

Proof. For any $c \in \mathbb{C}$ the number of points with $f(z) = c$ in minus the number of poles in Π is given by

$$\frac{1}{2\pi i}\int_{\partial\Pi} \frac{f'(z)}{f(z)-c}dz = 0.$$

This integral is equal to zero since $\frac{f'(z)}{f(z)-c}$ is an elliptic function. □

Definition B.3. The number of times a non-constant elliptic function takes on every value in Π is called its *order*.

Corollary B.4. *The order of an elliptic function is at least 2.*

Proof. An non-constant elliptic function must have at least one pole, but it cannot have a single pole of order 1. □

- Non-constant elliptic functions may be viewed as holomorphic maps

  $$f : \mathbb{C}/\Lambda \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

  that take on every value as many times as their order.

Theorem B.5. *Let* f *be an elliptic function of order* m*. Let* $c \in \mathbb{C}$ *and* $z_1, \ldots, z_m \in \Pi$ *be all points in* Π *with* $f(z) = c$ *(appearing multiple times according to their multiplicity). Let* $w_1, \ldots, w_m \in \Pi$ *be all poles in* Π *(appearing multiple times according to their order). Then*

$$z_1 + \cdots + z_m = w_1 + \cdots + w_m \mod \Lambda$$

Proof. Consider the integral

$$\frac{1}{2\pi i}\int_{\partial\Pi} z\frac{f'(z)}{f(z)-c}dz$$

□

Example B.2. The Weierstraß $\wp$-function for a given period lattice Λ is an elliptic function given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

It is constructed to have exactly one double pole per fundamental parallelogram, and thus it has order 2. Any two points $z_1, z_2 \in \Pi$ with $\wp(z_1) = \wp(z_2)$ satisfies

$$z_1 + z_2 = 0 \mod \Lambda.$$

Theorem B.6. *Let f, g be two elliptic functions with the same period lattice Λ. If f and g have the same poles and zeros (with same multiplicities) then*

$$f = \lambda g$$

with some constante $\lambda \in \mathbb{C}$.

Proof. The function $\frac{f}{g}$ is an elliptic function without poles and thus constant. □

B.2 Jacobi elliptic functions

- Denote by

$$\mathsf{K}(k^2) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \mathsf{K}'(k^2) = \mathsf{K}(1-k^2).$$

  the elliptic integral of the first kind, and consider the parallelogram with corners

$$0, \quad \mathsf{K}, \quad \mathsf{K} + i\mathsf{K}', \quad i\mathsf{K}',$$

  which is called the the *auxiliary rectangle.* Its corners are labeled by (in the same order)

$$\mathrm{s}, \quad \mathrm{c}, \quad \mathrm{d}, \quad \mathrm{n}\,.$$

  If the *elliptic modulus* k satisfies $0 < k^2 < 1$ the two values $\mathsf{K}(k^2), \mathsf{K}'(k^2) \in \mathbb{R}$ are real, and the auxiliary rectangle is indeed a rectangle.

- *Jacobi elliptic functions* are certain elliptic functions of order two (with two simple poles and two simple zeros) constructed in the following way:

  - The function

$$\mathrm{pq}(z,k) \qquad \text{with} \quad \mathrm{p},\mathrm{q} \in \{\mathrm{s},\mathrm{c},\mathrm{n},\mathrm{d}\}$$

    has a simple zero at p and a simple pole at q.
  - The pattern of zeros and poles is extended to $\mathbb{C}$ by reflection in the sides of the auxiliary rectangle.[14]
  - The periods are chosen as multiples of K and $i\mathsf{K}'$ such that one obtains an elliptic function of order 2. This leads to periods $2\mathsf{K}$ or $4\mathsf{K}$ in one direction and $2i\mathsf{K}'$ or $4i\mathsf{K}'$ in the other direction.

[14]Indeed the whole Jacobi elliptic function itself is continued across the auxiliary rectangle by the Schwarz reflection principle.

- The leading coefficient in the series expansion of the zero (or equivalently of the pole) is chosen to be 1.

- As the three basic Jacobi elliptic functions one usually takes $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$. For these one obtains
  - cn is an even function with periods $4\mathsf{K}$, $2i\mathsf{K}'$, zeros at K, $3\mathsf{K}$, poles at $i\mathsf{K}'$, $2\mathsf{K}+i\mathsf{K}'$, and $\mathrm{cn}(0) = 1$.
  - sn is an odd function with periods $4\mathsf{K}$, $2i\mathsf{K}'$, zeros at 0, $2\mathsf{K}$, poles at $i\mathsf{K}'$, $2\mathsf{K}+i\mathsf{K}'$, and $\mathrm{sn}(0) = 0$, $\mathrm{sn}'(0) = 0$.
  - dn is an even function with periods $2\mathsf{K}$, $4i\mathsf{K}'$, zeros at $\mathsf{K}+i\mathsf{K}'$, $\mathsf{K}+3i\mathsf{K}'$, poles at $i\mathsf{K}'$, $3i\mathsf{K}'$, and $\mathrm{dn}(0) = 1$.

- All other Jacobi elliptic functions can be constructed as products and quotients of these three functions, e.g.,
$$\mathrm{cs} = \frac{\mathrm{cn}}{\mathrm{sn}}.$$

- For $0 < k^2 < 1$ the values of all Jacobi elliptic functions on the real axis are real

- The Jacobi elliptic functions can be considered as generalizations of trigonometric functions. In the limit $k^2 \to 0$, in which $\mathsf{K}(k^2) \to \frac{\pi}{2}$, one obtains
$$\mathrm{cn} \to \cos, \quad \mathrm{sn} \to \sin, \quad \mathrm{dn} \to 1.$$

- The main Jacobi elliptic functions satisfy the following two quadratic equations:
$$\mathrm{cn}^2 + \mathrm{sn}^2 = 1, \quad \mathrm{dn}^2 + k^2\,\mathrm{sn}^2 = 1$$
Geometrically this means that the curves
$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \mathrm{cn}(t,k) \\ \mathrm{sn}(t,k) \\ \pm\,\mathrm{dn}(t,k) \end{pmatrix}$$
lie on the (and indeed parametrize the entire) intersection curve of the two cylinders
$$x^2 + y^2 = 1, \quad z^2 + k^2 y^2 = 1.$$

- The derivatives of the main Jacobi elliptic functions are given by
$$\mathrm{sn}' = \mathrm{cn}\,\mathrm{dn}, \quad \mathrm{cn}' = -\,\mathrm{sn}\,\mathrm{dn}, \quad \mathrm{dn}' = -k^2\,\mathrm{sn}\,\mathrm{cn}\,.$$

- They satisfy the following differential equations
  - sn solves $z'' + (1+k^2)z - 2k^2z^3 = 0$ and $(z')^2 = (1-z^2)(1-k^2z^2)$
  - cn solves $z'' + (1-2k^2)z + 2k^2z^3 = 0$ and $(z')^2 = (1-z^2)(1-k^2+k^2z^2)$
  - dn solves $z'' - (2-k^2)z + 2z^3 = 0$ and $(z')^2 = (z^2-1)(1-k^2-z^2)$

- They satisfy the following addition laws
$$\begin{aligned}
\mathrm{sn}(x+y) &= \frac{\mathrm{sn}(x)\,\mathrm{cn}(y)\,\mathrm{dn}(y) + \mathrm{sn}(y)\,\mathrm{cn}(x)\,\mathrm{dn}(x)}{1 - k^2\,\mathrm{sn}^2(x)\,\mathrm{sn}^2(y)}, \\
\mathrm{cn}(x+y) &= \frac{\mathrm{cn}(x)\,\mathrm{cn}(y) + \mathrm{sn}(x)\,\mathrm{sn}(y)\,\mathrm{dn}(x)\,\mathrm{dn}(y)}{1 - k^2\,\mathrm{sn}^2(x)\,\mathrm{sn}^2(y)}, \\
\mathrm{dn}(x+y) &= \frac{\mathrm{dn}(x)\,\mathrm{dn}(y) - k^2\,\mathrm{sn}(x)\,\mathrm{sn}(y)\,\mathrm{cn}(x)\,\mathrm{cn}(y)}{1 - k^2\,\mathrm{sn}^2(x)\,\mathrm{sn}^2(y)}.
\end{aligned}$$