Geometry 2
Non-Euclidean Geometries

Jan Techter

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Lecture notes for Geometry 2, Summer 2020.

Contents

Part I

Projective Geometry

> In this chapter we shall learn about geometrical facts that can be formulated and proved without any measurement or comparison of distances or of angles. [...] The elementary figures of projective geometry are points, straight lines, and planes. The elementary results of projective geometry deal with the simplest possible relations between these entities, namely their *incidence*. The word incidence covers all the following relations: A point lying on a straight line, a point lying in a plane, a straight line lying in a plane. Clearly, the three statements that a straight line passes through a point, that a plane passes through a point, that a plane passes through a straight line, are respectively equivalent to the first three. The term incidence was introduced to give these three pairs of statements symmetrical form: a straight line is incident with a point, a plane is incident with a point, a plane is incident with a straight line. (Geometry and the Imagination – Hilbert, Cohn-Vossen)

This is a review and reinforcement of topics from Geometry 1. Yet, on the way, we will tackle some additional ideas which have not been covered there.

1 Projective space

Let V be a vector space of dimension $n+1$ over a field $\mathbb{F}$. Then the *projective space* of V is the set

$$\mathrm{P}(V) := \{\text{1-dimensional subspaces of } V\}$$

Its dimension is given by

$$\dim P(V) := \dim V - 1 = n.$$

For $V = \mathbb{F}^{n+1}$ we write

$$\mathbb{F}\mathrm{P}^n := \mathrm{P}(\mathbb{F}^{n+1}).$$

From now on we assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

1.1 Representative vectors

For $x \in V\backslash\{0\}$ we write $[x] := \operatorname{span}\{x\}$. Then $[x]$ is a point in $\mathrm{P}(V)$, and x is called a *representative vector* for this point.

If $\lambda \in \mathbb{F}\backslash\{0\}$ then $[\lambda x] = [x]$, and λx is another representative vector for the same point. This defines an equivalence relation on $V\backslash\{0\}$

$$x \sim y \quad \Leftrightarrow \quad x = \lambda y, \quad \text{for some } \lambda \in \mathbb{F}\backslash\{0\},$$

and we can identify

$$\mathrm{P}(V) \cong {}^{\textstyle (V\backslash\{0\})}\big/_{\textstyle\sim}.$$

1.2 Projective subspaces

For a $(k+1)$-dimensional linear subspace $U \subset V$ its projective space

$$\mathrm{P}(U) \subset \mathrm{P}(V)$$

is called a k-dimensional *projective subspace* of $\mathrm{P}(V)$.

$\dim \mathrm{P}(U)$	**name**
0	point
1	line
2	plane
k	k-plane
$n-1$	hyperplane

Table 1. Naming conventions for projective (sub)spaces.

1.3 Meet and join

Let $\mathrm{P}(U_1), \mathrm{P}(U_2) \subset \mathrm{P}(V)$ be two projective subspaces. Then their intersection, or *meet*, is given by

$$\mathrm{P}(U_1) \cap \mathrm{P}(U_2) = \mathrm{P}(U_1 \cap U_2),$$

and their span, or *join*, is given by

$$\mathrm{P}(U_1) \wedge \mathrm{P}(U_2) = \mathrm{P}(U_1 + U_2).$$

The dimension formula for linear subspaces carries over to projective subspaces:

$$\dim\left(\mathrm{P}(U_1) \wedge \mathrm{P}(U_2)\right) + \dim\left(\mathrm{P}(U_1) \cap \mathrm{P}(U_2)\right) = \dim \mathrm{P}(U_1) + \dim \mathrm{P}(U_2).$$

In particular, a k_1-plane and a k_2-plane in an n-dimensional projective space with $k_1 + k_2 \geqslant n$ always intersect in an at least $(k_1 + k_2 - n)$-dimensional projective subspace. Thus, certain incidences are always guaranteed in a projective space.

Example 1.1. In a projective plane two (distinct) lines always intersect in a point.

1.4 Sphere model for $\mathbb{R}\mathrm{P}^n$

For a point $P \in \mathbb{R}\mathrm{P}^n$ we can choose a representative vector $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ on the unit sphere. There are exactly two choices: $P = [x] = [-x]$. Thus, $\mathbb{R}\mathrm{P}^n$ is equivalent to the unit sphere with antipodal points identified

$$\mathbb{R}\mathrm{P}^n \cong {\mathbb{S}^n}\big/{\pm 1}.$$

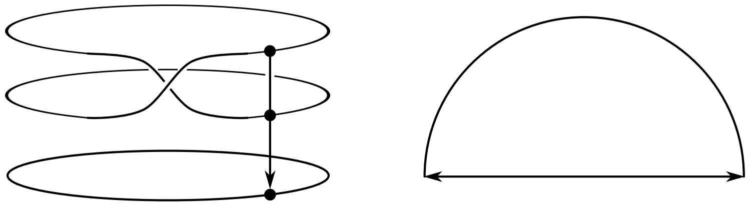


Figure 1. The real projective line $\mathbb{R}\mathrm{P}^1$ can be obtained by identifying opposite points on the unit circle $\mathbb{S}^1$. This yields a double cover of a circle (*left*). Equivalently it can be obtained from identifying the two end points of a semicircle (*right*), which again is homeomorphic to a circle.

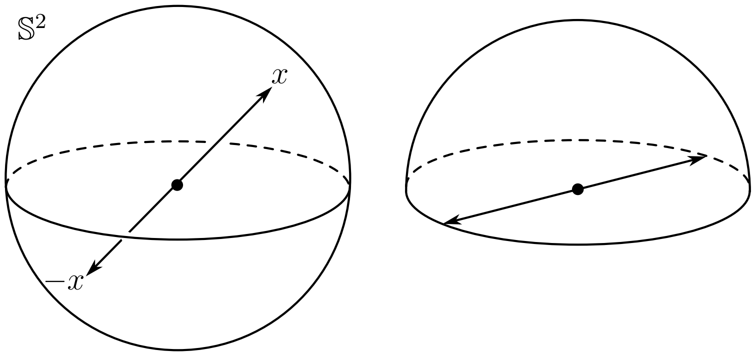


Figure 2. The real projective plane $\mathbb{R}\mathrm{P}^2$ can be obtained by identifying opposite points on the unit 2-sphere $\mathbb{S}^2$ (*left*), or from a hemisphere by identifying opposite points on the equator (*right*).

Remark 1.1. The real projective space $\mathbb{R}\mathrm{P}^n$ is orientable for n odd, and non-orientable for n even.

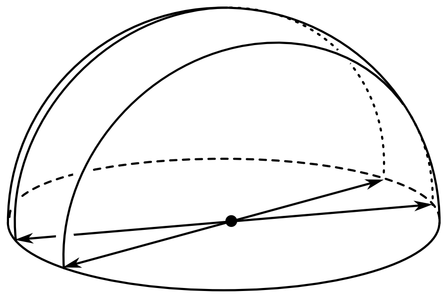


Figure 3. A Möbius strip in the hemisphere model of the real projective plane shows that it is not orientable.

2 Homogeneous coordinates

2.1 Homogeneous coordinates on $\mathbb{F}\mathrm{P}^n$

For a point $[x_1, \ldots, x_{n+1}] \in \mathbb{F}\mathrm{P}^n$ the coordinates of a representative vector $(x_1, \ldots, x_{n+1}) \in \mathbb{F}^{n+1}$ are called *homogeneous coordinates*. They are unique up to a common scalar multiple

$$[x_1, \ldots, x_{n+1}] = [\lambda x_1, \ldots, \lambda x_{n+1}]$$

for $\lambda \in \mathbb{F}\backslash\{0\}$.

If $x_{n+1} \neq 0$ then

$$[x_1, \ldots, x_{n+1}] = \left[\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right] = [y_1, \ldots, y_n, 1],$$

and $(y_1, \ldots, y_n)$ are called *affine coordinates* of the point $[x]$. This yields a decomposition of $\mathbb{F}\mathrm{P}^n$ into an affine part and a *hyperplane at infinity*

$$\mathbb{F}\mathrm{P}^n = \underbrace{\{[x_1, \ldots, x_{n+1}] \mid x_{n+1} \neq 0\}}_{\simeq \mathbb{F}^n} \cup \underbrace{\{[x_1, \ldots, x_{n+1}] \mid x_{n+1} = 0\}}_{\simeq \mathbb{F}\mathrm{P}^{n-1}}.$$

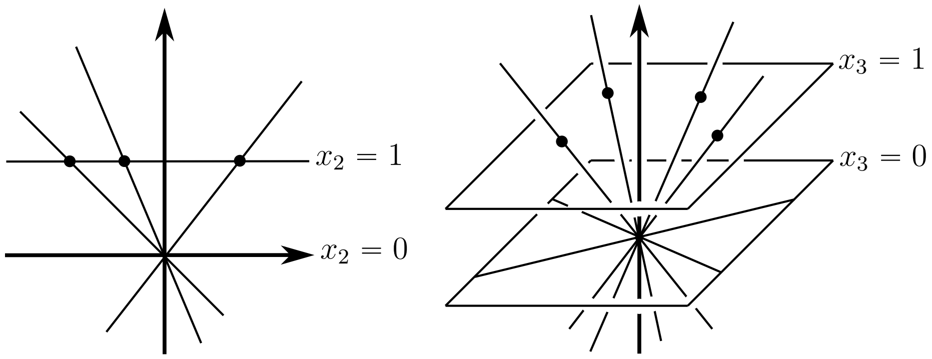


Figure 4. Affine coordinates for $\mathbb{RP}^1$ and $\mathbb{RP}^2$.

2.1.1 The real projective line $\mathbb{RP}^1$

On the real projective line this decomposition is given by

$$\mathbb{RP}^1 \cong \mathbb{R} \cup \mathbb{RP}^0 = \mathbb{R} \cup \{\infty\},$$

where $\mathbb{RP}^0$ consists of only one point $[1, 0]$, which is usually denoted by ∞ in this case and allowed as an “admissible” affine coordinate.

Vice versa, the real projective line $\mathbb{RP}^1$ may be seen as the one-point compactification of the real line $\mathbb{R}$, which again yields a togological circle.

2.1.2 The complex projective line $\mathbb{CP}^1$

Similarly, for the complex projective line one obtains

$$\mathbb{CP}^1 \cong \mathbb{C} \cup \mathbb{CP}^0 = \mathbb{C} \cup \{\infty\},$$

which is the one-point compactification of the complex plane $\mathbb{C} \cong \mathbb{R}^2$. This yields a topological 2-sphere, and a corresponding homeomorphism is given by stereographic projection

$$\sigma : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}, \qquad (1,0,0) \mapsto \infty, \quad (x,y,z) \mapsto \frac{x+iy}{1-z},$$

which has the additional properties of

- being conformal (angle preserving), and
- mapping planar sections of the sphere (circles) to circles and lines in the complex plane.

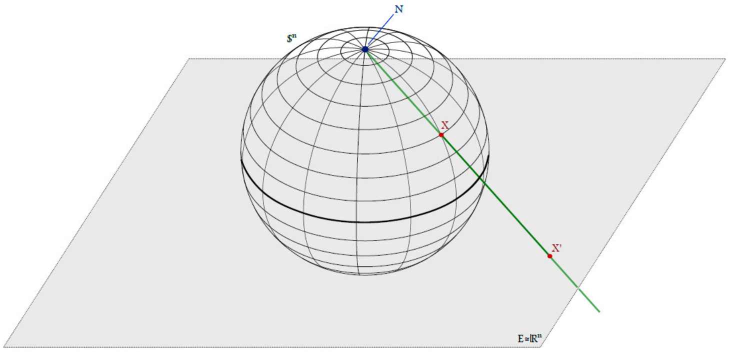


Figure 5. Stereographic projection.

2.2 General homogeneous coordinates

More generally let $b_1, \ldots, b_{n+1}$ be a basis of V. This gives an identification of V with $\mathbb{F}^{n+1}$, and, in turn, of $\mathrm{P}(V)$ with $\mathbb{F}\mathrm{P}^n$. For $x \in V$ let $x_1, \ldots, x_{n+1} \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n+1} x_i b_i.$$

Then $(x_1, \ldots, x_{n+1})$ are called *homogeneous coordinates* of the point $[x] \in \mathrm{P}(V)$. They depend on the chosen basis and are unique up to a common scalar multiple. We then identify

$$[x] \cong [x_1, \ldots, x_{n+1}].$$

A change of basis acts on the homogeneous coordinates as a general linear transformation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \end{bmatrix} \tag{1}$$

with $A \in \mathrm{GL}(\mathbb{F}^{n+1})$.

2.3 General position and projective frames

$n+2$ points in an n-dimensional projective space $\mathrm{P}(V)$ are said to be in *general position* if no $n+1$ of the points are contained in an $(n-1)$-dimensional projective subspace. Equivalently, any $n+1$ of the points have linearly independent representative vectors.

Lemma 2.1. *Let $P_1, \ldots, P_{n+2} \in \mathrm{P}(V)$ be in general position. Then representative vectors $p_1, \ldots, p_{n+2} \in V$ may be chosen such that $P_1 = [p_1], \ldots, P_{n+2} = [p_{n+2}]$ and*

$$p_1 + p_2 + \cdots + p_{n+1} + p_{n+2} = 0.$$

This choice is unique up to a common scalar multiple.

Proof. Geometry 1. □

$n+2$ points in general position are also called a *projective frame*. They can be used to define homogeneous coordinates in a projective space. The homogeneous coordinates with respect to a projective frame $P_1, \dots, P_{n+2}$ are the homogeneous coordinates with respect to the basis $p_1, \dots, p_{n+1}$. They satisfy

$$\begin{aligned} P_1 = [1, 0, \dots, 0], \quad \dots, \quad P_{n+1} = [0, \dots, 0, 1], \\ P_{n+2} = [-1, \dots, -1] = [1, \dots, 1]. \end{aligned}$$

2.4 Projective coordinate grid

In the projective plane consider the complete quadrangle give by the four points

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

Its diagonal points are given by $\left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$, which together span the line at infinity, and $\left[\begin{smallmatrix} 1/2 \\ 1/2 \\ 1 \end{smallmatrix}\right]$. Adding the diagonal lines of the complete quadrilateral leads to a recursive construction of a *projective coordinate grid*.

3 Cross-ratio

Let P_1, P_2, P_3, P_4 be four points on a projective line $\mathrm{P}(U)$. Choose a basis of U and denote the homogeneous coordinates of the four points with respect to this basis by $P_i = [x_i, y_i]$, $i = 1, \dots, 4$. Then the *cross-ratio* of P_1, P_2, P_3, P_4 is defined by

$$\begin{aligned} \mathrm{cr}(P_1, P_2, P_3, P_4) := \frac{\det \left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right) \det \left(\begin{smallmatrix} x_3 & x_4 \\ y_3 & y_4 \end{smallmatrix}\right)}{\det \left(\begin{smallmatrix} x_2 & x_3 \\ y_2 & y_3 \end{smallmatrix}\right) \det \left(\begin{smallmatrix} x_4 & x_1 \\ y_4 & y_1 \end{smallmatrix}\right)} &= \frac{(x_1 y_2 - x_2 y_1)(x_3 y_4 - x_4 y_3)}{(x_2 y_3 - x_3 y_2)(x_4 y_1 - x_1 y_4)} \\ &= \frac{(u_1 - u_2)(u_3 - u_4)}{(u_2 - u_3)(u_4 - u_1)} =: \mathrm{cr}(u_1, u_2, u_3, u_4). \end{aligned}$$

where $u_i = \frac{x_i}{y_i} \in \mathbb{F} \cup \{\infty\}$ are affine coordinates.

Proposition 3.1. *The cross-ratio is independent of the choice of the basis used for the homogeneous coordinates.*

Proof. Exercises. □

3.1 Coordinates from cross-ratios

By Lemma 2.1, we can choose a basis such that

$$P_1 = \begin{bmatrix} x \\ y \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then the cross-ratio is equal to the affine coordinate $u = \frac{x}{y}$ of P_1 with respect to the projective frame P_4, P_2, P_3:

$$\mathrm{cr}(P_1, P_2, P_3, P_4) = \mathrm{cr}\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \mathrm{cr}(u, 0, 1, \infty) = u.$$

Put differently, if we choose a basis such that the affine coordinates of P_2, P_3, P_4 are given by $0, 1, \infty$ respectively, then the affine coordinate of P_1 is equal to the cross-ratio $\mathrm{cr}(P_1, P_2, P_3, P_4)$.

In higher dimensions the affine coordinates with respect to a projective frame can still be expressed in terms of cross-ratios: E.g., let $P_1^\infty, P_2^\infty, P^0, P^1$ be a projective frame in a projective plane.

$$P_1^\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_2^\infty = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad P^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Define the two projections

$$\pi_1 : P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto (P^0 \wedge P_1^\infty) \cap (P \wedge P_2^\infty) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix},$$
$$\pi_2 : P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto (P^0 \wedge P_2^\infty) \cap (P \wedge P_1^\infty) = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}.$$

Now for an arbitrary point

$$P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

we find

$$u = \mathrm{cr}(\pi_1(P), P^0, \pi_1(P^1), P_1^\infty), \qquad v = \mathrm{cr}(\pi_2(P), P^0, \pi_2(P^1), P_2^\infty).$$

In particular, if we assign affine coordinates $0, 1, \infty$ to the points $P^0, \pi_1(P^1), P_1^\infty$ on the line $P^0 \wedge P_1^\infty$, the point $\pi_1(P)$ has affine coordinate u.

3.2 Symmetries of the cross-ratio

We study the dependence of the cross-ratio on the order of the points. The cross-ratio is invariant under the action of the Klein four-group

$$K_4 = \{(),\ (12)(34),\ (13)(24),\ (14)(23)\},$$

i.e., under any simultaneous interchange of two disjoint pairs of points

$$\mathrm{cr}(P_1, P_2, P_3, P_4) = \mathrm{cr}(P_2, P_1, P_4, P_3) = \mathrm{cr}(P_3, P_4, P_1, P_2) = \mathrm{cr}(P_4, P_3, P_2, P_1).$$

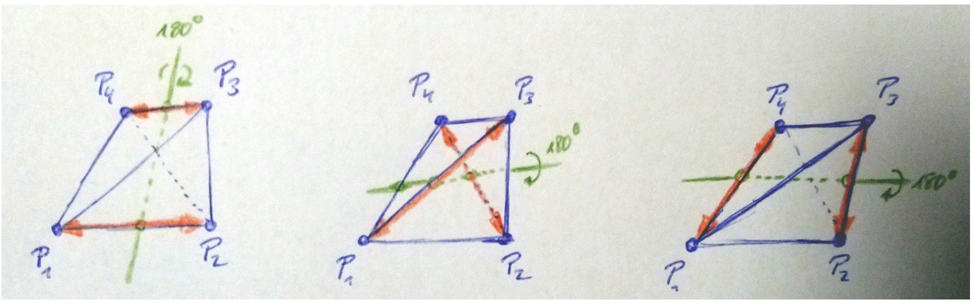


Figure 6. If we associate the points in the cross-ratio with the vertices of a regular tetrahedron, each of the 24 permutations in S_4 corresponds to a symmetry of the tetrahedron. The cross-ratio is invariant under simultaneously interchanging the vertices of two opposite edges of the tetrahedron, or equivalently, under 180°-rotation about an axis connecting the midpoints of two opposite edges.

Thus, from the 24 possible permutations of the four points we only need to consider 6 that may change the cross-ratio.

$$
\begin{aligned}
\mathrm{cr}(P_1, P_2, P_3, P_4) &= \mathrm{cr}(q, 0, 1, \infty) = \frac{(q-0)(1-\infty)}{(0-1)(\infty-q)} = q, \\
\mathrm{cr}(P_1, P_3, P_2, P_4) &= \mathrm{cr}(q, 1, 0, \infty) = \frac{(q-1)(0-\infty)}{(1-0)(\infty-q)} = 1-q, \\
\mathrm{cr}(P_1, P_2, P_4, P_3) &= \mathrm{cr}(q, 0, \infty, 1) = \frac{(q-0)(\infty-1)}{(0-\infty)(1-q)} = \frac{q}{q-1}, \\
\mathrm{cr}(P_1, P_4, P_3, P_2) &= \mathrm{cr}(q, \infty, 1, 0) = \frac{(q-\infty)(1-0)}{(\infty-1)(0-q)} = \frac{1}{q}, \\
\mathrm{cr}(P_1, P_3, P_4, P_2) &= \mathrm{cr}(q, 1, \infty, 0) = \frac{(q-1)(\infty-0)}{(1-\infty)(0-q)} = \frac{q-1}{q} = 1-\frac{1}{q}, \\
\mathrm{cr}(P_1, P_4, P_2, P_3) &= \mathrm{cr}(q, \infty, 0, 1) = \frac{(q-\infty)(0-1)}{(\infty-0)(1-q)} = \frac{1}{1-q}.
\end{aligned}
$$

For certain values of q more of these 6 values coincide:

3.2.1 Harmonic points

For $q = -1, \frac{1}{2}, 2$ they coincide in pairs. If for instance $q = -1$, we have

$$
\begin{aligned}
\mathrm{cr}(P_1, P_2, P_3, P_4) &= \mathrm{cr}(P_1, P_4, P_3, P_2) = -1, \\
\mathrm{cr}(P_1, P_2, P_4, P_3) &= \mathrm{cr}(P_1, P_4, P_2, P_3) = \frac{1}{2}, \\
\mathrm{cr}(P_1, P_3, P_2, P_4) &= \mathrm{cr}(P_1, P_3, P_4, P_2) = 2.
\end{aligned}
$$

Thus, the cross-ratio is additionally invariant under the permutation (24) and therefore also under the permutation (13). In this case its symmetry group may be generated by the permutations

$$(13), \quad (24), \quad (12)(34),$$

and we say that the pair of points $\{P_1, P_3\}$ separates the pair $\{P_2, P_4\}$ *harmonically.*

If we assign affine coordinates $0, 1, \infty$ to the points P_2, P_3, P_4 respectively, then P_1 has affine coordinate -1, i.e. P_2 is the midpoint of P_1 and P_3.

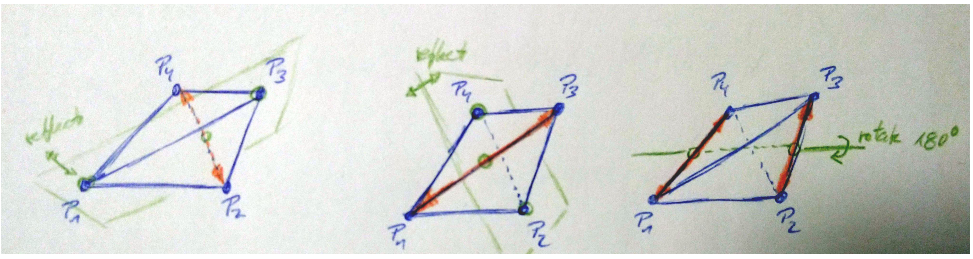


Figure 7. In the case $\mathrm{cr}(P_1, P_2, P_3, P_4) = -1$ the symmetries of the cross-ratio are generated by the three permutations $(13), (24), (12)(34)$. The permutation (13) for instance corresponds to a reflections of the tetrahedron in a plane through the opposite edge (P_2, P_4) and the midpoint of the edge (P_1, P_3).

3.2.2 Tetrahedral points

For $q = e^{\pm i\frac{\pi}{3}}$ they coincide in triples. This case can only be realized on a complex projective line. If for instance $q = e^{-i\frac{\pi}{3}}$, we have

$$\begin{aligned}
\mathrm{cr}(P_1, P_2, P_3, P_4) &= \mathrm{cr}(P_1, P_3, P_4, P_2) = \mathrm{cr}(P_1, P_4, P_2, P_3) = e^{-i\frac{\pi}{3}}, \\
\mathrm{cr}(P_1, P_3, P_2, P_4) &= \mathrm{cr}(P_1, P_2, P_4, P_3) = \mathrm{cr}(P_1, P_4, P_3, P_2) = e^{i\frac{\pi}{3}}.
\end{aligned}$$

Thus, the cross-ratio is additionally invariant under the permutations that fix P_1 and interchange the other points cyclically, and therefore under all cyclic permutions of three points. In this case its symmetry group may be generated by the permutations

$$(ijk) \qquad \text{with } i, j, k \in \{1, 2, 3, 4\}.$$

Interchanging one pair (ij) results in the complex conjugate cross-ratio.

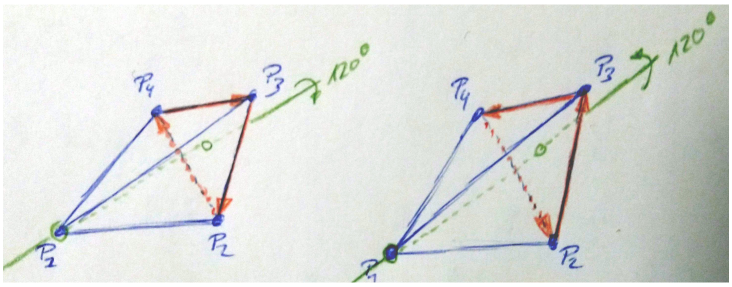


Figure 8. In the case $\mathrm{cr}(P_1, P_2, P_3, P_4) = e^{-i\frac{\pi}{3}}$ the symmetries of the cross-ratio are generated by the permutations (ijk), which correspond to $120°$-rotations about an axis through one of the vertices and the midpoint of its opposite face. These constitute all symmetries of an oriented tetrahedron.

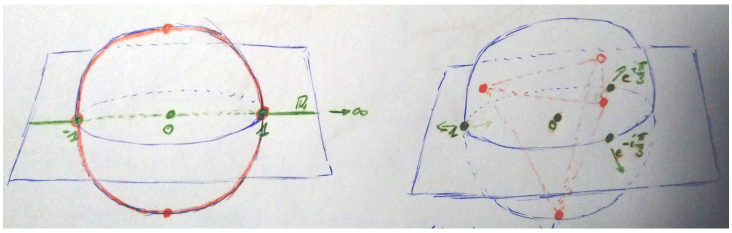


Figure 9. *Left:* In affine coordinates four points with cross-ration $q = -1$ can always be represented by $-1, 0, 1, \infty$, or after stereographic projection, by four equally spaced points on a great circle. *Right:* In affine coordinates four points with cross-ratio $q = e^{-i\frac{\pi}{3}}$ can always be represented by $-1, e^{-i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, 0$, or after stereographic projection, by the four vertices of a regular tetrahedron.

4 Duality

> The interchangeability of points and lines is called the principle of *duality* in the projective plane. According to this principle, there belongs to every theorem a second theorem that corresponds to it dually, and to every figure a second figure that corresponds to it dually. (Geometry and the Imagination – Hilbert, Cohn-Vossen)

In homogeneous coordinates x_1, x_2, x_3, the equation for a line in a projective plane is

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

where not all coefficients a_i are zero. The coefficients a_1, a_2, a_3 can be seen as homogeneous coordinates for the line, because if we replace in the equation a_i by λa_i for some $\lambda \neq 0$ we get an equivalent equation for the same line. Thus, the set of lines in a projective plane is itself a projective plane, the *dual plane*. Points in the dual plane correspond to lines in the original plane. Moreover, if we consider in the above equation the x_i as fixed and the a_i as variables, we get an equation for a line in the dual plane. Points on this line correspond to lines in the original plane that contain $[x]$. Thus, a the points on a line in the dual plane correspond to lines in the original plane through a point.

It makes sense to look at this phenomenon in a basis independent way and for arbitrary dimension. It boils down to the duality of vector spaces.

4.1 Dual space

Let V be a vector space of dimension $n + 1$ over a field $\mathbb{F}$. Then its *dual vector space* is the space of linear functionals $V \to \mathbb{F}$

$$V^* := \{\alpha \mid \alpha : V \to \mathbb{F} \text{ linear}\}.$$

The *dual projective space* of $\mathrm{P}(V)$ is correspondingly defined by

$$\mathrm{P}(V)^* := \mathrm{P}(V^*).$$

The natural identification $V^{**} = V$ carries over to the projective setting $\mathrm{P}(V)^{**} = \mathrm{P}(V)$.

4.2 Dual subspaces

For a projective subspace $\mathrm{P}(U) \subset \mathrm{P}(V)$ its *dual projective subspace* $\mathrm{P}(U)^\star \subset \mathrm{P}(V)^*$ is defined by

$$\mathrm{P}(U)^\star := \{[\alpha] \in \mathrm{P}(V)^* \mid \alpha(x) = 0 \text{ for all } x \in U\}.$$

The dimensions of a projective subspace and its dual projective subspace are related by

$$\dim \mathrm{P}(U) + \dim \mathrm{P}(U)^\star = n - 1.$$

Incidences are reversed by duality

$$\mathrm{P}(U_1) \subset \mathrm{P}(U_2) \quad \Leftrightarrow \quad \mathrm{P}(U_2)^\star \subset \mathrm{P}(U_1)^\star.$$

and meet and join are interchanged

$$\begin{aligned}(\mathrm{P}(U_1) \wedge \mathrm{P}(U_2))^\star &= \mathrm{P}(U_1)^\star \cap \mathrm{P}(U_2)^\star,\\ (\mathrm{P}(U_1) \cap \mathrm{P}(U_2))^\star &= \mathrm{P}(U_1)^\star \wedge \mathrm{P}(U_2)^\star.\end{aligned}$$

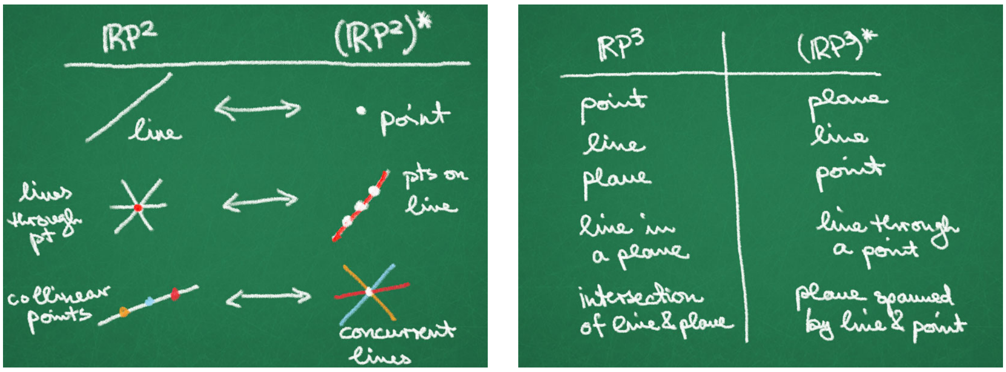


Figure 10. Duality in $\mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{P}^3$.

4.3 Duality in coordinates

Let $b_1, \dots, b_{n+1}$ be a basis of V and $b_1^*, \dots, b_{n+1}^*$ the corresponding dual basis of V^*, i.e.,

$$b_i^*(b_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

In the homogeneous coordinates with respect to those bases the duality of two points

$$[x_1, \dots, x_{n+1}] \cong [x] \in \mathrm{P}(V), \qquad [\alpha_1, \dots, \alpha_{n+1}] \cong [\alpha] \in \mathrm{P}(V)^*$$

is expressed by

$$\alpha(x) = (\alpha_1 \dots \alpha_{n+1}) \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{pmatrix}^\mathsf{T} \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = 0.$$

Thus, duality in linear algebra as well as in projective geometry expresses in a formal way that a subspace can either be expressed as the span of points or the solutions to a set of linear equations.

If a change of basis acts on the homogeneous coordinates of $\mathrm{P}(V)$ as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \mapsto \left[A \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \right]$$

with $A \in \mathrm{GL}(\mathbb{F}^{n+1})$, it acts on the homogeneous coordinates of the dual space $\mathrm{P}(V)^*$ as

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \mapsto \left[A^{-\intercal} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{pmatrix} \right].$$

5 Embedding $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{C}\mathrm{P}^n$

> Die Einfiihrung komplexer Zahlen in der Algebra hat ihren Grund in der Übersichtlichkeit, welche die Theorie der Gleichungen durch sie gewinnt. Bei Beschrankung auf reelle GroBen konnen wir nämlich nur aussagen, daß eine Gleichung n-ten Grades höchstens n Wurzeln besitzt; wir sind daher gezwungen, bei allen Untersuchungen die Gleichungen mit $0, 1, 2, \ldots$ bis schließlich n Wurzeln voneinander zu unterscheiden. Demgegenüber können wir bei Zulassung komplexer Zahlen behaupten, daß bei richtiger Zählung der vielfachen Wurzeln eine Gleichung n-ten Grades immer n Wurzeln besitzen muß. Derselbe Grund spricht auch für die Zulassung komplexer Koordinaten in der Geometrie. Denn dann wird eine Kurve n-ter Ordnung bei richtiger Zählung der vielfachen Punkte von einer geraden Linie stets in n Punkten getroffen. So hat z.B. ein Kreis mit einer Tangente zwei zusammenfallende Punkte und mit einer im Reellen nicht schneidenden Geraden zwei Punkte mit komplexen Koordinaten gemeinsam. Ferner können wir den Satz, daß eine Kollineation in einer n-dimensionalen Mannigfaltigkeit im allgemeinen $(n + 1)$ Fixpunkte besitzt, nur bei Zulassung von Punkten mit komplexen Koordinaten aussprechen. Der Grund zur Einfiihrung imaginärer Elemente in der Geometrie ist also derselbe, der uns zur Einführung der unendlich fernen oder uneigentlichen Elemente bewogen hat, nämlich die größere Einheitlichkeit der geometrischen Sätze und Beweise. (Vorlesungen über Nicht-Euklidische Geometrie – Felix Klein)

Once we have introduced homogeneous coordinates on $\mathbb{C}\mathrm{P}^n$ we obtain a natural embedding of $\mathbb{R}\mathrm{P}^n$ into $\mathbb{C}\mathrm{P}^n$ by the map

$$\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{C}\mathrm{P}^n, \qquad [x_1, \ldots, x_{n+1}] \mapsto [x_1, \ldots, x_{n+1}] \quad \text{for } x_i \in \mathbb{R}.$$

We will omit the embedding map and just write $\mathbb{R}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^n$.

A point $[z] \in \mathbb{C}\mathrm{P}^n$ is called *real* if one of the following equivalent conditions is satisfied:

(i) It lies in the image of the embedding map: $[z] \in \mathbb{R}\mathrm{P}^n$.

(ii) It possesses real homogeneous coordinates: $[z] = [x]$ for some $x \in \mathbb{R}^{n+1}$.

(iii) It is equal to its complex conjugate point: $[z] = [\bar{z}]$.

Otherwise it is called *imaginary*.

5.1 Real section and complexification

For a projective subspace $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ we denote its *real section* by

$$\mathrm{P}(W)_{\mathbb{R}} := \mathrm{P}(W \cap \mathbb{R}^{n+1}) = \mathrm{P}(W) \cap \mathbb{R}\mathrm{P}^n.$$

This is the set of all real points contained in $\mathrm{P}(W)$.

Proposition 5.1. *For a projective subspace $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ its real section is a projective subspace $\mathrm{P}(W)_{\mathbb{R}} \subset \mathbb{R}\mathrm{P}^n$, which satisfies*

$$\dim_{\mathbb{R}} \mathrm{P}(W)_{\mathbb{R}} \leqslant \dim_{\mathbb{C}} \mathrm{P}(W).$$

Proof.

- For $x, y \in W \cap \mathbb{R}^{n+1}$ and $\lambda, \mu \in \mathbb{R}$ we have $\lambda x + \mu y \in W \cap \mathbb{R}^{n+1}$. Thus, $W \cap \mathbb{R}^{n+1} \subset \mathbb{R}^{n+1}$ is a (real) linear subspace.
- Any (real) basis of $W \cap \mathbb{R}^{n+1}$ is (complex) linear independent in W, and can be completed to a (complex) basis of W.

□

The complex span of real points contains imaginary points. For a projective subspace $\mathrm{P}(U) \subset \mathbb{R}\mathrm{P}^n$ we define its *complexification* by

$$\mathrm{P}(U)_{\mathbb{C}} := \mathrm{P}(U_{\mathbb{C}}) = \mathrm{P}\left(\left\{\lambda x + \mu y \in \mathbb{C}^{n+1} \;\middle|\; x, y \in U,\ \lambda, \mu \in \mathbb{C}\right\}\right).$$

Proposition 5.2. *For a projective subspace $\mathrm{P}(U) \subset \mathbb{R}\mathrm{P}^n$ its complexification $\mathrm{P}(U)_{\mathbb{C}} \subset \mathbb{C}\mathrm{P}^n$ is a projective subspace with*

$$\dim_{\mathbb{R}} \mathrm{P}(U) = \dim_{\mathbb{C}} \mathrm{P}(U)_{\mathbb{C}}.$$

Proof.

- The complexification is closed under (complex) linear combinations by definition.
- Any (real) basis of U serves as a (complex) basis of the complexification.

□

Proposition 5.3. *For a projective subspace $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ the complexification of its real section is given by*

$$(\mathrm{P}(W)_{\mathbb{R}})_{\mathbb{C}} = \mathrm{P}(W) \cap \mathrm{P}(\overline{W}).$$

Proof. First note that

$$(\mathrm{P}(W)_{\mathbb{R}})_{\mathbb{C}} = \left\{[\lambda x + \mu y] \;\middle|\; x, y \in W \cap \mathbb{R}^{n+1},\ \lambda, \mu \in \mathbb{C}\right\},$$

and

$$[z] \in \mathrm{P}(W) \cap \mathrm{P}(\overline{W}) \quad \Leftrightarrow \quad [z], [\bar{z}] \in \mathrm{P}(W).$$

"⊂" Let $[z] \in (\mathrm{P}(W)_{\mathbb{R}})_{\mathbb{C}}$. Thus, we can assume $z = \lambda x + \mu y$ with $x, y \in W \cap \mathbb{R}^{n+1}$, $\lambda, \mu \in \mathbb{C}$. Then $[z] \in \mathrm{P}(W)$ and $[\bar{z}] = [\bar{\lambda} x + \bar{\mu} y] \in \mathrm{P}(W)$.

"$\supset$" Let $[z], [\bar{z}] \in \mathrm{P}(W)$. Then $[z] = \Big[\underbrace{\tfrac{1}{2}(z+\bar{z})}_{\in W \cap \mathbb{R}^{n+1}} + i\,\underbrace{\tfrac{1}{2i}(z-\bar{z})}_{\in W \cap \mathbb{R}^{n+1}}\Big] \in (\mathrm{P}(W)_{\mathbb{R}})_{\mathbb{C}}$.

□

A projective subspace $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ is called *real* if it is the complexification of a projective subspace $\mathrm{P}(U) \subset \mathbb{R}\mathrm{P}^n$:

$$\mathrm{P}(W) = \mathrm{P}(U)_{\mathbb{C}}.$$

Otherwise it is called *imaginary*.

Proposition 5.4. *A projective subspace* $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ *is real if and only if one of the following equivalent conditions is satisfied:*

(i) It is the complexification of its real section: $\mathrm{P}(W) = (\mathrm{P}(W)_{\mathbb{R}})_{\mathbb{C}}$.

(ii) It is invariant under complex conjugation: $\mathrm{P}(\overline{W}) = \mathrm{P}(W)$.

(iii) Its complex dimension is equal to the real dimension of its real section:

$$\dim_{\mathbb{R}} \mathrm{P}(W)_{\mathbb{R}} = \dim_{\mathbb{C}} \mathrm{P}(W).$$

(iv) It can be expressed as the span of real points only.

Proof. Exercises. □

This yields a one-to-one correspondence between projective subspaces of $\mathbb{R}\mathrm{P}^n$ and real projective subspaces of $\mathbb{C}\mathrm{P}^n$.

Proposition 5.5. *Every imaginary point* $[z] \in \mathbb{C}\mathrm{P}^n$ *is contained in exactly one real line, which is given by* $[z] \wedge [\bar{z}]$.

Proof. By Proposition 5.4 (ii) a real line through $[z]$ must contain its complex conjugate point $[\bar{z}]$. $[z] \neq [\bar{z}]$ since $[z]$ is imaginary. Thus, the two (distinct) points $[z]$ and $[\bar{z}]$ already determine the real line ℓ uniquely. □

5.2 Reality and duality

The reality considerations of projective subspaces behave nicely under duality. While we defined the complexification of a real subspace by its complex span, it may dually be characterized by the complex solutions of a system of linear equations with real coefficients. Under the corresponding embedding $(\mathbb{R}\mathrm{P}^n)^* \hookrightarrow (\mathbb{C}\mathrm{P}^n)^*$, one obtains for a projective subspace $\mathrm{P}(U) \subset \mathbb{R}\mathrm{P}^n$:

$$(\mathrm{P}(U)_{\mathbb{C}})^\star = (\mathrm{P}(U)^\star)_{\mathbb{C}}\,.$$

This, in particular, implies that for a projective subspace $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$:

$$\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n \text{ is real} \quad \Leftrightarrow \quad \mathrm{P}(W)^\star \subset (\mathbb{C}\mathrm{P}^n)^* \text{ is real.}$$

For the inclusion of real projective subspaces we obtain the following proposition.

Proposition 5.6. *Let* $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ *a projective subspace and* $\mathrm{P}(U) \subset \mathbb{C}\mathrm{P}^n$ *a real projective subspace. Then*

$$\mathrm{P}(U) \subset \mathrm{P}(W) \quad \Rightarrow \quad \mathrm{P}(W)^\star \subset \mathrm{P}(U)^\star.$$

In particular, since $(\mathrm{P}(W)_\mathbb{R})_\mathbb{C} \subset \mathrm{P}(W)$*,*

$$\mathrm{P}(W)^\star \subset (\mathrm{P}(W)_\mathbb{R})^\star_\mathbb{C}.$$

Proof. Exercise. □

With this we can dualize Proposition 5.4 to obtain:

Proposition 5.7. *For every imaginary hyperplane* $\mathrm{P}(W) \subset \mathbb{C}\mathrm{P}^n$ *(the complexification) of its real section, which is given by* $\mathrm{P}(W) \cap \mathrm{P}(\overline{W})$*, has dimension* $n-2$*.*

Proof. Under duality, and by Proposition 5.6, an imaginary point becomes a imaginary hyperplane, while the real line joining the point and its complex conjugate point becomes a real projective subspace of codimension 2 which is the intersection of the hyperplane and its complex conjugate hyperplane. □

5.3 Examples

We now determine the number of real sub- and superspaces of different projective subspaces of $\mathbb{C}\mathrm{P}^n$ in the cases $n = 1, 2, 3$.

5.3.1 $\mathbb{R}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^1$

- Through every point (real or imaginary) there passes exactly one real line $\mathbb{R}\mathrm{P}^1_\mathbb{C} = \mathbb{C}\mathrm{P}^1$.

Remark 5.1. The real projective line separates the complex projective line into two regions of imaginary points. For all dimensions $n > 1$ the set of imaginary points $\mathbb{C}\mathrm{P}^n \backslash \mathbb{R}\mathrm{P}^n$ is connected.

5.3.2 $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{C}\mathrm{P}^2$

- Through every real point passes a one-parameter family of real lines.
- Through every imaginary point passes exactly one real line.

Dually:

- Every real line contains a one-parameter family of real points.
- Every imaginary line contains exactly one real point.

5.3.3 $\mathbb{RP}^3 \hookrightarrow \mathbb{CP}^3$

- Through every real point passes a two-parameter family of real lines and a two-parameter family of real planes.
- Through every imaginary point passes exactly one real line and a one-parameter family of real planes (all real planes through that line).

Dually:

- Every real plane contains a two-parameter family of real lines and a two-parameter family of real points.
- Every imaginary plane contains exactly one real line and a one-parameter family of real points (all real points on that line).

What about lines in $\mathbb{CP}^3$?

- Every real line contains a one-parameter family of real points and is contained in a one-parameter family of real planes.
- For an imaginary line ℓ there are two cases:
  - Either the line and its complex conjugate line intersect (they are contained in an imaginary plane), in which case ℓ contains exactly one real point $\ell \cap \bar{\ell}$,
  - or the line and its complex conjugate line do not intersect, in which case ℓ contains no real points.

    Example 5.1. To see that imaginary lines of this type do indeed exist, consider the line spanned by the two imaginary points

    $$[z_1] = [1, i, 0, 0], \quad [z_2] = [0, 0, 1, i].$$

    Then $[z_1] \wedge [z_2]$ and $[\bar{z}_1] \wedge [\bar{z}_2]$ do not intersect since

    $$\det(z_1, z_2, \bar{z_1}, \bar{z_2}) = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix}^2 = -4 \neq 0.$$

6 Configurations in the projective plane

> The theorems relating to incidence are by far the most important theorems of projective geometry. [...] We shall study a particularly instructive part of projective geometry – the configurations. This will also reveal certain aspects of various other geometrical problems. It might be mentioned here that there was a time when the study of configurations was considered the most important branch of all geometry. (Geometry and the Imagination – Hilbert, Cohn-Vossen)

Let $p, l, \lambda, \pi \in \mathbb{N}$. A *(plane projective) configuration of type* $(p_\lambda\, l_\pi)$ is an arrangement of p distinct points and l distinct lines in the (real or complex) projective plane, such that every point in the arrangement is incident with λ of the lines, and every line in the arrangement is incident with π of the points.

Proposition 6.1. *The type* $(p_\lambda\, l_\pi)$ *of a configuration satisfies*

(i) $p\,\lambda = l\,\pi$*,*

(ii) $p \geqslant \lambda(\pi - 1) + 1$*, and* $l \geqslant \pi(\lambda - 1) + 1$*.*

Proof.

(i) Each of the l lines contains exactly π points. This gives $l\,\pi$ points, where we have over-counted the number of points λ times, since every point is contained in λ of the lines simultaneously. Thus, the total number of points is given by $p = \frac{l\,\pi}{\lambda}$.

(ii) Every point is contained in λ lines, each of which contains $\pi - 1$ further points.

□

The number $p\,\lambda = l\,\pi$ is the total number of incidences in a configuration. If $p = l$, and consequently $\lambda = \pi$, the type is abbreviated to $(p_\lambda) := (p_\lambda\, p_\lambda)$.

Example 6.1.

(i) A *triangle* consists of 3 non-collinear points and its 3 connecting lines. This constitutes a configuration of type $(3_2) = (3_2\, 3_2)$.

(ii) A *complete quadrangle*, or *Ceva configuration*, consists of 4 points in general position and its 6 connecting lines, which are called its sides, or diagonals. This constitutes a configuration of type $(4_3\, 6_2)$.

The three points of intersection of opposite sides are called its *diagonal points*. They are not points of the configuration.

Proposition 6.2. *The three diagonal points of complete quadrangle in the (real or complex) projective plane are not collinear.*

Proof. Recall that joining two of the diagonal points by a line and intersecting this line with either of the two remaining sides yields four harmonic points on that line. Yet four harmonic points are always distinct. □

Thus, the three diagonal points make a triangle, which is called *diagonal triangle.*

6.1 Incidence structures

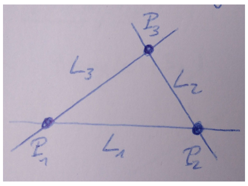
Let $P_1, \ldots, P_p$ be the points and $\ell_1, \ldots, \ell_l$ be the lines of a configuration of type $(p_\lambda\, l_\pi)$. The matrix $(a_{ij}) \in \{0,1\}^{p \times l}$ with

$$a_{ij} := \begin{cases} 1 & \text{if the point } P_i \text{ lies on the line } \ell_j \\ 0 & \text{else} \end{cases}$$

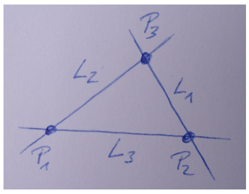
is called an *incidence matrix* of the configuration. Generally, any matrix $A \in \{0,1\}^{p \times l}$ is called an incidence matrix.

The labeling of the points and lines in a configuration is arbitrary. Relabeling of the points and lines corresponds to permutation of the rows and columns of the incidence matrix respectively. This defines an equivalence relation on the set of incidence matrices $\{0,1\}^{p \times l}$. We call the corresponding equivalence class the *incidence structure* of the configuration.

Example 6.2. Two equivalent incidence matrices for a triangle are given by



$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$



Proposition 6.3. *An incidence matrix $(a_{ij}) \in \{0,1\}^{p\times l}$ of a configuration of type $(p_\lambda\, l_\pi)$ has*

(i) constant row sum equal to λ: $\displaystyle\sum_{j=1}^{l} a_{ij} = \lambda$ *for* $i = 1, \ldots, p$,

(ii) constant column sum equal to π: $\displaystyle\sum_{i=1}^{p} a_{ij} = \pi$ *for* $j = 1, \ldots, l$,

(iii) no $2{\times}2$-submatrix containing all ones:

$$\begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \textit{for} \quad 1 \leqslant i_1 < i_2 \leqslant p \quad \textit{and} \quad 1 \leqslant j_1 < j_2 \leqslant l.$$

Proof. In a configuration of type $(p_\lambda\, l_\pi)$ every point is incident with λ of the lines, which implies (i), and every line is incident with π of the points, which implies (ii).

The existence of a 2×2-submatrix containing all ones implies that either $P_{i_1} = P_{i_2}$ or $\ell_{j_1} = \ell_{j_2}$, which contradicts the assumption that all points and lines in a configuration are distinct. □

We call the equivalence class of an incidence matrix $A \in \{0,1\}^{p\times l}$ satisfying conditions (i), (ii), (iii) from Proposition 6.3 an *incidence structure of type* $(p_\lambda\, l_\pi)$.

An incidence structure of type $(p_\lambda\, l_\pi)$ is called *geometrically realizable in the (real or complex) projective plane*, or just *realizable*, if it is the incidence structure of a configuration in the (real or complex) projective plane.

Example 6.3. The incidence structures of type (3_2) and $(4_3\, 6_2)$ of the triangle and the complete quadrilateral are unique and geometrically realizable.

6.2 Representations of incidence structures

An *incidence figure* is a graphical representation of an incidence structure, in which the points are represented by dots and the lines are represented by curves. The crossing of two curves not on a dot as well as the the order of points on a curve has no meaning in terms of the incidence structure.

Alternatively, an incidence structure may be represented by its *incidence graph*, or *Levi graph*. For an incidence structure of type $(p_\lambda\, l_\pi)$ this is a bipartite graph with

- p black vertices of valence λ corresponding points,
- l white vertices of valence π corresponding lines,
- $p\,\lambda = l\,\pi$ edges corresponding to incidences

considered up to graph isomorphisms that preserve the partitioning.

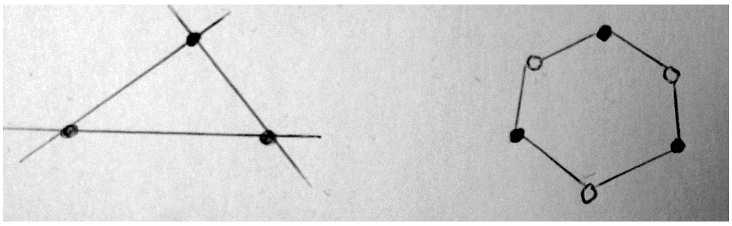


Figure 11. Incidence figure of a triangle (or a geometric realization) and its incidence graph.

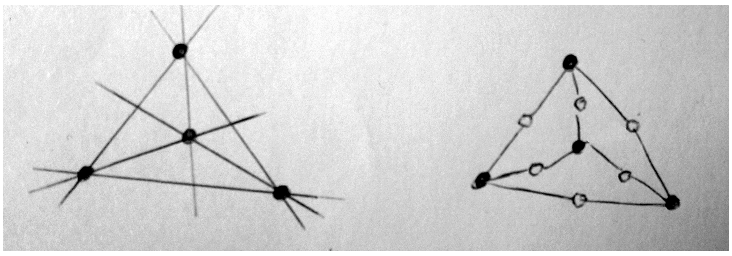


Figure 12. Incidence figure of a complete quadrangle (or a geometric realization) and its incidence graph.

6.3 Symmetries of a configuration

Let $P_1, \dots, P_p$ be the points and $\ell_1, \dots, \ell_l$ be the lines of a configuration of type $(p_\lambda\, l_\pi)$. A pair of permutations $\sigma_P \in S_p$, $\sigma_\ell \in S_l$ is called a *symmetry of the configuration* if

$$P_i \text{ is incident with } \ell_j \quad \Leftrightarrow \quad P_{\sigma_P(i)} \text{ is incident with } \ell_{\sigma_\ell(j)}$$

for $i = 1, \dots, p$ and $j = 1, \dots, l$. The symmetries of a configuration define a group under composition

$$(\sigma_P, \sigma_\ell) \circ (\tilde{\sigma}_P, \tilde{\sigma}_\ell) := (\sigma_P \circ \tilde{\sigma}_P, \sigma_\ell \circ \tilde{\sigma}_\ell),$$

which is called its *symmetry group*, or its *automorphism group*.

Remark 6.1. The symmetries of a configuration correspond to the graph automorphisms of its incidence graph that preserve the partitioning.

A configuration is called *regular* if its automorphism group is transitive, i.e., every point/line of a configuration can be mapped to every other point/line by a symmetry. This means, that all points/lines in the configuration are alike when considered in their relation to the rest of the configuration.

Example 6.4. The triangle and the complete quadrangle are regular.

6.4 Duality of configurations

In a projective plane duality interchanges points and lines while preserving their incidences. Thus, for every configuration $\mathcal{C}$ in $\mathbb{F}\mathrm{P}^2$ there is a *dual configuration* $\mathcal{C}^\star$ in $(\mathbb{F}\mathrm{P}^2)^*$.

Proposition 6.4. *Let $\mathcal{C}$ be a configuration of type $(p_\lambda \, l_\pi)$ with an incidence structure represented by an incidence matrix A, or an incidence graph G.*

Then its dual configuration $\mathcal{C}^\star$ is of type $(l_\pi \, p_\lambda)$ with an incidence structure represented by the transposed incidence matrix $A^\intercal$, or the incidence graph given by G with black and white vertices reversed.

Proof. Exercise. □

A configuration is called *self-dual* if its dual configuration has the same incidence structure. A self-dual configuration satisfies $p = l$, and consequently $\lambda = \pi$.

Proposition 6.5. *A configuration $\mathcal{C}$ is self-dual if and only if one of the following two equivalent conditions is satisfied*

(i) The incidence structure of $\mathcal{C}$ (and therefore of $\mathcal{C}^\star$) can be represented by a symmetric incidence matrix $A^\intercal = A$.

(ii) Its incidence graph of $\mathcal{C}$ (and therefore of $\mathcal{C}^\star$) possesses a graph automorphism which exchanges black and white vertices.

Proof. Exercise. □

Example 6.5.

(i) The triangle (3_2) is a self-dual configuration since its incidence structure can be represented by a symmetric matrix (see Example 6.2).

(ii) The dual configuration of a complete quadrangle $(4_3 \, 6_2)$ is a *complete quadrilateral*, or *Menelaus configuration*. It consists of 4 lines and its 6 pairwise points of intersection, which constitutes a configuration of type $(6_2 \, 4_3)$.

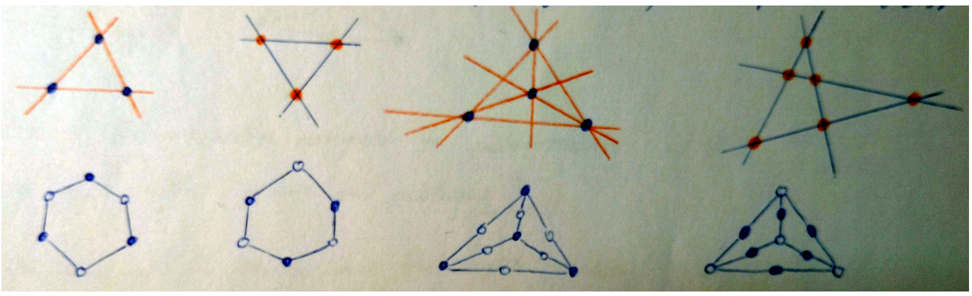


Figure 13. Triangle and complete quadrilateral and its dual configurations.

6.5 Configurations of type (p_3)

We now turn to configurations with $p = l$ and $\lambda = \pi = 3$. The minimum number of points (lines) for such a configuration is 7.

6.5.1 The Fano configuration (7_3)

Proposition 6.6. *There is exactly one incidence structure of type* (7_3)*.*

Proof. Show that up to permutation of there is only one incidence matrix with row and column sums equal to 3 and without any 2×2-submatrices containing all ones (compare video notes):

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

□

Remark 6.2. The incidence matrix given above is symmetric. Thus, the incidence structure of type (7_3) is self-dual. It is also regular.

Proposition 6.7. *The incidence structure of type* (7_3) *is not realizable in the (real or complex) projective plane.*

Proof. Assume the incidence structure of type (7_3) was realizable. Then it would contain a complete quadrangle (a configuration of type $(4_3\, 6_2)$) with diagonal points on a line (see Figure 14). This contradicts Proposition 6.2. □

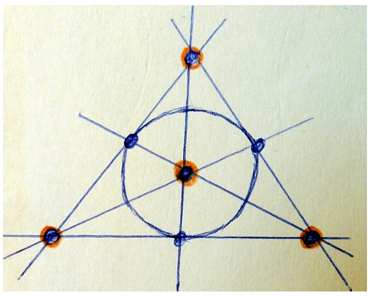


Figure 14. Incidence figure for the Fano configuration. The red points together with its joining lines constitute a complete quadrangle with diagonal points on a line.

Remark 6.3. The incidence structure (7_3) is realizable in the projective plane over the finite field $\mathbb{Z}_2$ of two elements. Indeed, the *Fano plane* $\mathbb{Z}_2\mathrm{P}^2$ consists of all points and lines of such a configuration, which is called the *Fano configuration*. It contains all lines through any two of its points.

6.5.2 The Möbius-Kantor configuration (8_3)

Proposition 6.8. *There is exactly one incidence structure of type* (8_3)*.*

Proof. Exercises. □

Remark 6.4. The incidence structure of type (8_3) is self-dual and regular.

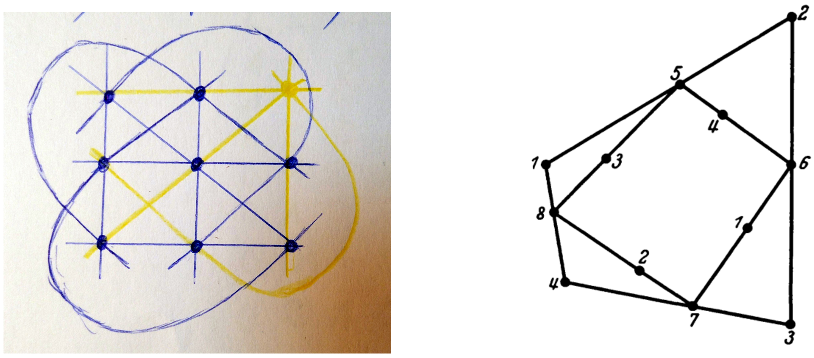


Figure 15. *Left:* Incidence figure for the Möbius-Kantor configuration (8_3) as part of the Hesse configuration $(9_4\,12_3)$. *Right:* (8_3) as two quadrilaterals, "circumscribed" about and "inscribed" into each other.

Proposition 6.9. *The incidence structure of type* (8_3) *is realizable in the complex projective plane, but not in the real projective plane.*

Proof. We choose homogeneous coordinates for P_1, P_2, P_3, P_4 in the following way:

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then the point $P_5 \in P_1 \wedge P_2$, $P_6 \in P_2 \wedge P_3$, $P_7 \in P_3 \wedge P_4$, and $P_8 \in P_4 \wedge P_1$ can be represented by

$$P_5 = \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} \sigma \\ 1 \\ 1 \end{bmatrix}, \quad P_7 = \begin{bmatrix} 1 \\ \rho \\ 1 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 0 \\ \mu \\ 1 \end{bmatrix}.$$

with some $\lambda, \mu, \sigma, \rho \in \mathbb{C}$. Now the remaining four incidences are expressed by the following equations

$$\begin{vmatrix} \lambda & \sigma & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \sigma - \lambda = 0, \qquad \begin{vmatrix} \sigma & 1 & 0 \\ 1 & \rho & 0 \\ 1 & 1 & 1 \end{vmatrix} = \rho\sigma - 1 = 0$$

$$\begin{vmatrix} 1 & 0 & 1 \\ \rho & \mu & 0 \\ 1 & 1 & 0 \end{vmatrix} = \rho - \mu = 0, \qquad \begin{vmatrix} 0 & \lambda & 1 \\ \mu & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \lambda + \mu - \lambda\mu = 0$$

which has two complex conjugate solutions

$$\lambda = \sigma = \frac{1}{\rho} = \frac{1}{\mu} = e^{\pm i\frac{\pi}{3}}.$$

Thus, this configuration can only be realized in the complex projective plane. $\square$

The corresponding configuration is called the *Möbius-Kantor configuration.*

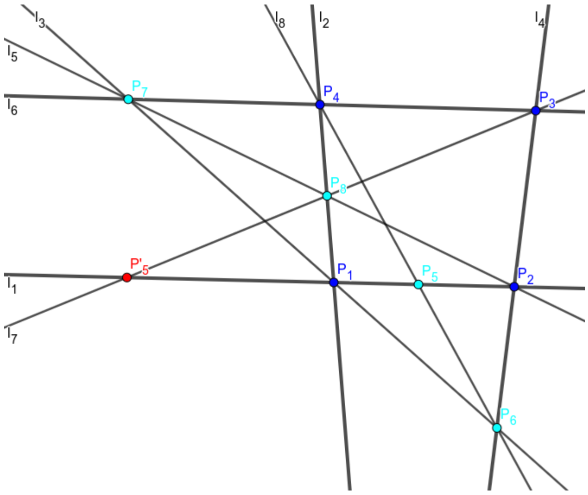


Figure 16. Attempt to construct the configuration of type (8_3) in the real projective plane. The two points P_5 and P_5' cannot be made to coincide.

Remark 6.5. A cubic curve in the projective plane has 9 inflection points, which satisfy that the line through any two of them contains a third one. This corresponds to the unique incidence structure of type $(9_4 12_3)$, whose realization is called the *Hesse configuration*. Note that similar to the Fano configuration it contains all lines through any two of its points. The Möbius-Kantor configuration is obtained from the Hesse configuration by removing one point and all four lines through this point (see Figure 15 (left)). Thus, Proposition 6.9 implies that not all inflection points of a cubic curve can be real.

6.5.3 The Pappus configuration and other configurations of type (9_3)

Proposition 6.10. *There are exactly three incidence structures of type* (9_3).

Remark 6.6. All three incidence structures of type (9_3) are self-dual, while two of them are regular.

Incidence figures for the three incidence structures of type (9_3) are shown in Figure 17. The left most incidence structure corresponds to the Pappus configuration. Its geometric realizability is ensured by the Pappus incidence theorem.

Theorem 6.11 (Pappus)**.** *Let* A, B, C *and* A', B', C' *be two collinear triples of distinct points in the projective plane. Then the points*

$$A'' = (B \wedge C') \cap (B' \wedge C), \quad B'' = (A \wedge C') \cap (A' \wedge C), \quad C'' = (A' \wedge B) \cap (A \wedge B')$$

are collinear.

Proof. Geometry 1. □

Remark 6.7. The other two incidence structures of type (9_3) are also realizable. Yet their construction does not correspond to a fundamental incidence theorem as in the case of the Pappus configuration.

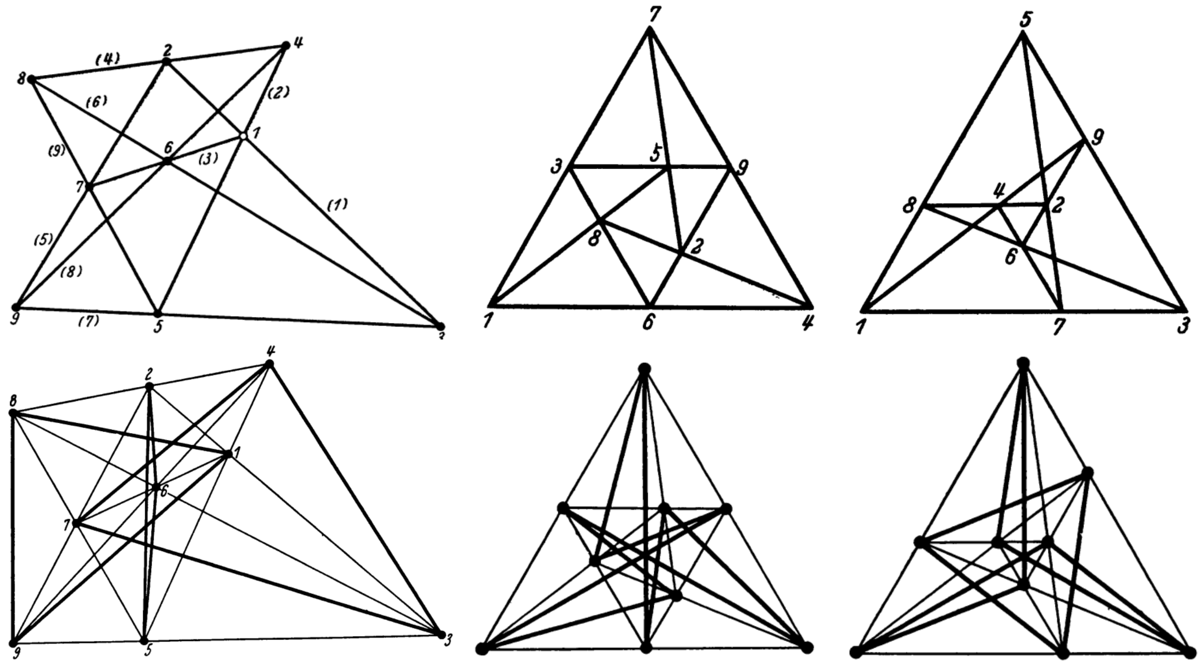


Figure 17. *Top:* Incidence figures (or geometric realizations) for the three incidence structures of type (9_3). *Bottom:* Every point in a configuration of type (9_3) is connected to six other points, and therefore, not connected to two other points. To see that these incidence figures belong to different incidence structure connect the points to its “unconnected points” to obtain “unconnected polygons”. These polygons (which are from left to right three triangles, a nonagon, and a triangle and a hexagon) differ in all three cases. The two different kind of polygons in the right hand case also imply that this incidence structure can not be regular

6.5.4 The Desargues configuration as an example of type (10_3)

Proposition 6.12. *There are exactly ten incidence structures of type* (10_3).

Remark 6.8. Nine of the ten incindence structures of type (10_3) can be realized in the (real or complex) projective plane.

The most important configuration of type (10_3) is the *Desargues configuration.*

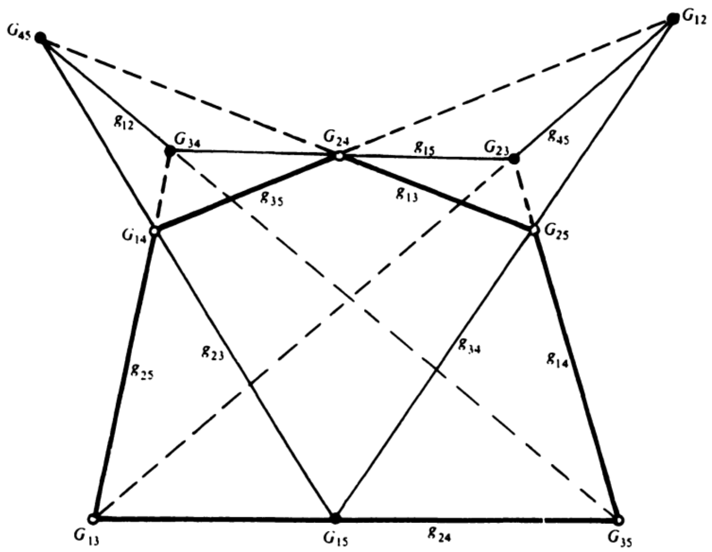


Figure 18. The Desargues configuration.

Remark 6.9. The Desargues configuration is self-dual and regular.

Its realizability is ensured by the Desargues incidence theorem.

Theorem 6.13 (Desargues)**.** *Two triangles are in perspective with respect to a point if and only if they are in perspective with respect to a line.*

Proof. Geometry 1. □

Remark 6.10. The number of incidence structures of type (p_3) starting with $p = 7$ goes:

$$1, 1, 3, 10, 31, 229, 2036, 21399, 245342, \ldots$$

7 Projective transformations

Let V, W be two vector spaces of dimension $n+1$ and $F : V \to W$ an invertible linear map. Then the map

$$[F] : \mathrm{P}(V) \to \mathrm{P}(W), \quad [v] \mapsto [F(v)]$$

is called a *projective transformation.*[1]

Proposition 7.1.

(i) Projective transformations are well-defined maps (do not depend on the representative vectors of points).

(ii) Projective transformations map projective subspaces to projective subspaces, while preserving their dimension and incidences.

(iii) Two invertible linear maps $F, G : V \to W$ give rise to the same projective transformation $\mathrm{P}(V) \to \mathrm{P}(W)$ if and only if $G = \lambda F$ for some scalar $\lambda \neq 0$.

[1] Further common names include *collineation*, *homography*, and *projectivity*.

(iv) Suppose

$$A_1, \dots, A_{n+2} \in \mathrm{P}(V) \quad \textit{and} \quad B_1, \dots, B_{n+2} \in \mathrm{P}(W)$$

are points in general position. Then there exists a unique projective transformation

$$f : \mathrm{P}(V) \to \mathrm{P}(W) \quad \textit{with} \quad f(A_i) = B_i \textit{ for } i = 1, \dots, n+2.$$

(v) Projective transformations preserve the cross-ratio of four points on a line.

(vi) Let $\ell, \tilde{\ell}$ be two projective lines, $A_1, A_2, A_3, A_4 \in \ell$ and $B_1, B_2, B_3, B_4 \in \tilde{\ell}$. Then there exists a projective transformation $f : \ell \to \tilde{\ell}$ with $f(A_i) = B_i$, $i = 1, 2, 3, 4$ if and only if

$$\mathrm{cr}(A_1, A_2, A_3, A_4) = \mathrm{cr}(B_1, B_2, B_3, B_4).$$

Proof. Geometry 1. □

Remark 7.1. From

$$\det(\lambda F) = \lambda^{n+1} \det F$$

we find that over the field of complex numbers we can always choose a linear transformation with $\det F = 1$ to represent a given projective transformation.

Over the field of real numbers we obtain two cases:

(i) If n is even, we can again normalize the linear transformation to have $\det F$, and thus the sign of $\det F$ has no projectively invariant meaning. Recall that $\mathbb{R}\mathrm{P}^n$ is non-orientable for even n.

(ii) If n is odd, it is not possible to change the sign of $\det F$ by multiplication with a scalar. Thus we can only normalize to $\det F = \pm 1$. Recall that $\mathbb{R}\mathrm{P}^n$ is orientable for odd n. The projective transformations with $\det F > 0$ are orientation preserving, while the projective transformations with $\det F < 0$ are orientation reversing.

For real projective spaces, projective transformations are exactly the transformations that map k-planes to k-planes.

Theorem 7.2 (fundamental theorem of real projective geometry)**.** *Let $\mathrm{P}(V)$, $\mathrm{P}(W)$ be two real projective spaces of dimension $n \geqslant 2$. Let $f : \mathrm{P}(V) \to \mathrm{P}(W)$ be a bijective map that maps k-planes to k-planes for some $k = 1, \dots, n-1$. Then f is a projective transformation.*

Proof. Geometry 1. □

Remark 7.2.

(i) Real projective transformations of a line ($n = 1$) are characterized by preserving the cross-ratio, or even stronger, by preserving harmonicity of four points.

(ii) The local version of the fundamental theorem states that an injective map on a open subset of $\mathrm{P}(V)$ satisfying the same conditions can always be extended to a projective transformation.

(iii) For a general field $\mathbb{F}$ the statement is only true up to transformations generated by field automorphisms: A bijective transformation that maps k-planes to k-planes is induced by an *almost linear map*, that is a map $\varphi : V \to W$ with

$$\varphi(u+v) = \varphi(u) + \varphi(v) \qquad \text{and} \qquad \varphi(\lambda v) = \alpha(\lambda)\varphi(v),$$

for all $u, v \in V$, $\lambda \in \mathbb{F}$, where $\alpha : \mathbb{F} \to \mathbb{F}$ is a field automorphism. E.g., complex conjugation is a field automorphism of $\mathbb{C}$, while the field $\mathbb{R}$ of real numbers has no field automorphism except the identity.

7.1 Central projections

As an important example consider the following projective transformation:

Let $\mathrm{P}(V)$ be a projective space of dimension n and let $L_1, L_2, C \subset \mathrm{P}(V)$ be projective subspaces with

- $\dim L_1 = \dim L_2$,
- $C \cap L_1 = C \cap L_2 = \varnothing$,
- $\dim C + \dim L_1 = \dim C + \dim L_2 = n - 1$.

Then the map

$$L_1 \to L_2, \quad X \mapsto (C \wedge X) \cap L_2$$

is a projective transformation, called *(generalized) central projection* from L_1 onto L_2 with center C.

Example 7.1.

(i) If L_1, L_2 are two hyperplanes, i.e. $\dim L_1 = \dim L_2 = n - 1$, the center C is a point.

(ii) If $n = 3$ and L_1, L_2 are two lines, then the center C is another line.

7.2 Projective automorphisms

The projective transformations $\mathrm{P}(V) \to \mathrm{P}(V)$ form a group called the *projective linear group*, or *projective automorphism group*. It is the quotient of the general linear group $\mathrm{GL}(V)$ by the normal subgroup of non-zero multiples of the identity:

$$\mathrm{PGL}(V) = {}^{\mathrm{GL}(V)}\!\big/_{\{\lambda I\}_{\lambda \neq 0}}.$$

In particular for $V = \mathbb{F}^{n+1}$ one writes $\mathrm{PGL}(n+1, \mathbb{F})$.

Proposition 7.3. *Let $[F] : \mathrm{P}(V) \to \mathrm{P}(V)$ be a projective automorphism. Then $[x]$ is a fixed point of $[F]$ if and only if x is an eigenvector of F.*

Proof. Geometry 1. $\square$

7.3 Projective transformations in homogeneous coordinates

A projective transformation $[F] : \mathbb{F}\mathrm{P}^n \to \mathbb{F}\mathrm{P}^n$ is represented by an invertible matrix $F \in \mathbb{F}^{(n+1)\times(n+1)}$ (up to non-zero scalar multiples).

Up to a choice of basis / choice of homogeneous coordinates we can always identify $\mathrm{P}(V) \cong \mathbb{F}\mathrm{P}^n$, $\mathrm{P}(W) \cong \mathbb{F}\mathrm{P}^n$. Thus, in homogeneous coordinates a projective transformation $[F] : \mathrm{P}(V) \to \mathrm{P}(W)$ can be represented by an invertible matricx $F \in \mathbb{F}^{(n+1)\times(n+1)}$ (with respect to the given bases and up to non-zero scalar multiples).

7.3.1 Change of basis

A change of basis induces a change of homogeneous coordinates that is given by an invertible linear transformation (1). Thus, a change of basis / change of homogeneous coordinates corresponds to a projective map $\mathbb{F}\mathrm{P}^n \to \mathbb{F}\mathrm{P}^n$ (projective automporphism).

Let $\mathcal{B}_1, \tilde{\mathcal{B}}_1$ be two bases of V and $\mathcal{B}_2, \tilde{\mathcal{B}}_2$ be two bases of W, where the corresponding change of homogeneous coordinates is given by $[\tilde{x}] = [Ax]$ and $\tilde{y} = [Bx]$, respectively. Let $f : \mathrm{P}(V) \to \mathrm{P}(W)$ be a projective transformation which is represented by the matrix F with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Then a representative matrix of f with respect to the bases $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ is given by

$$\tilde{F} = BFA^{-1}.$$

In particular, for a projective automorphism ($V = W$, $\mathcal{B}_1 = \mathcal{B}_2$, $\tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}}_2$, $A = B$) we obtain

$$\tilde{F} = AFA^{-1}.$$

7.3.2 Affine coordinates

For representative vectors $x = (u_1, \ldots, u_n, 1)$ and with

$$F = \left(\begin{array}{c|c} A & b \\ \hline c^\intercal & d \end{array}\right) \qquad \text{where } A \in \mathbb{F}^{n\times n}, b, c \in \mathbb{F}^n, d \in \mathbb{F}$$

we obtain

$$F(x) = \left(\begin{array}{c|c} A & b \\ \hline c^\intercal & d \end{array}\right) \begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} Au + b \\ c^\intercal u + d \end{pmatrix} \sim \begin{pmatrix} \frac{Au+b}{c^\intercal u+d} \\ 1 \end{pmatrix}$$

if $c^\intercal u + d \neq 0$. Thus, in affine coordinates, projective transformations are *fractional linear transformations.*

7.3.3 Affine transformations

Recall that an *affine transformation* $\mathbb{R}^n \to \mathbb{R}^n$ is a transformation of the form $u \mapsto Au + b$ with $A \in \mathrm{GL}(n, \mathbb{R})$, $b \in \mathbb{R}^n$.

Proposition 7.4. *A projective transformation $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ is an affine transformation if one of the following equivalent conditions is satisfied:*

(i) f maps the hyperplane at infinity $\{[x] \in \mathbb{R}\mathrm{P}^n \mid x_{n+1} = 0\}$ to itself.

(ii) f is represented by the matrix

$$F = \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right).$$

Proof. Geometry 1. □

Remark 7.3. Furthermore, f is a Euclidean transformation if $A \in \mathrm{O}(n)$.

7.4 Dual transformations

Let $f : \mathrm{P}(V) \to \mathrm{P}(W)$ be a projective transformation. Since f maps in particular hyperplanes to hyperplanes this induces a corresponding *dual projective transformation* on the dual spaces

$$f^* : \mathrm{P}(V)^* \to \mathrm{P}(W)^*, \quad X \mapsto f(X^\star)^\star.$$

Remark 7.4. The dual map is the inverse of the adjoint map.

Proposition 7.5. *The dual projective transformation satisfies*

$$f^*(K^\star) = f(K)^\star$$

for every projective subspace $K \subset \mathrm{P}(V)$.

Proof. Exercise. □

If the projective transformation $f : \mathrm{P}(V) \to \mathrm{P}(W)$ is represented by a matrix F with respect to some chosen bases of V and W, then the dual projective transformation f^* is represented by the matrix

$$F^* = F^{-\intercal}.$$

with respect to the corresponding dual bases of V^* and W^*.

7.5 Complexification of projective transformations

Consider a projective transformation $f = [F] : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$. Its representative linear transformation $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ can be uniquely extended to a (complex) linear transformation $F_\mathbb{C} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by

$$F_\mathbb{C}(ix) = iF(x)$$

for all $x \in \mathbb{R}^{n+1}$. Then the projective transformation $f_\mathbb{C} := [F_\mathbb{C}] : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ is called the *complexification* of f.

We call a projective transformation $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ *real* if it maps real points to real points, i.e.,

$$f(\mathbb{R}\mathrm{P}^n) = \mathbb{R}\mathrm{P}^n.$$

Proposition 7.6. *A projective transformation $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ is real if one of the following equivalent conditions is satisfied*

(i) f is invariant under complex conjugation: $\overline{f([x])} = f([\bar{x}])$ for all $[x] \in \mathbb{C}\mathrm{P}^n$.

(ii) f has a real representative matrix: $f = [F]$ with $F \in \mathbb{R}^{(n+1)\times(n+1)}$.

(iii) f is a complexification: $f = g_\mathbb{C}$ for some projective transformation $g : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$.

Proof. Exercise. □

7.6 Involutions

A projective automorphism $f : \mathbb{F}\mathrm{P}^n \to \mathbb{F}\mathrm{P}^n$, $f \neq \mathrm{id}$ is called a *projective involution* if

$$f \circ f = \mathrm{id}\,.$$

7.6.1 Ivolutions on the projective line

Proposition 7.7.

(i) *A projective automorphism $f : \mathbb{F}\mathrm{P}^1 \to \mathbb{F}\mathrm{P}^1$ that exchanges one pair of points, i.e.,*

$$f(A) = \tilde{A}, \quad f(\tilde{A}) = A \quad \text{for some } A, \tilde{A} \in \mathbb{F}\mathrm{P}^1,\ A \neq \tilde{A},$$

is a projective involution.

(ii) *for two pairs of distinct points $A, \tilde{A}, B, \tilde{B} \in \mathbb{F}\mathrm{P}^1$ there exists a unique projective involution $f : \mathbb{F}\mathrm{P}^1 \to \mathbb{F}\mathrm{P}^1$ that exchanges the points from each pair, i.e.,*

$$f(A) = \tilde{A}, \quad f(B) = \tilde{B}.$$

Proof. Geometry 1. ☐

Proposition 7.8.

(i) *A projective involution $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ has exactly two fixed points $A_1, A_2 \in \mathbb{C}\mathrm{P}^1$.*

(ii) *f maps a point $X \in \mathbb{C}\mathrm{P}^1$ to the harmonic conjugate with respect to the pair of fixed points $\{A_1, A_2\}$, i.e.,*

$$\mathrm{cr}(X, A_1, f(X), A_2) = -1.$$

(iii) *In homogeneous coordinates with $A_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ the involution f is of the form*

$$f = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

or in affine coordinates

$$f(u) = -u \qquad \text{for } u \in \mathbb{C}$$

Proof. Geometry 1. ☐

Proposition 7.9.

(i) *A projective involution $f : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ has either two real fixed points or two imaginary conjugate fixed points (no real fixed points).*

(ii) *If f has two imaginary conjugate fixed points one can always choose homogeneous coordinates such that $A = \left[\begin{smallmatrix}i\\1\end{smallmatrix}\right]$ and $\bar{A} = \left[\begin{smallmatrix}-i\\1\end{smallmatrix}\right]$, and then f is of the form*

$$f = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

or in affine coordinates

$$f(u) = -\frac{1}{u} \qquad \text{for } u \in \mathbb{R}$$

Proof. See Proposition 7.12. ☐

7.6.2 General involutions

Proposition 7.10.

(i) A projective involution $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ has exactly two disjoint fixed point sets $L_1, L_2 \subset \mathbb{C}\mathrm{P}^n$, which are projective subspaces, and span the whole space, i.e.,

$$L_1 \cap L_2 = \varnothing, \qquad L_1 \wedge L_2 = \mathbb{C}\mathrm{P}^n.$$

(ii) If $\dim L_1 = k_1$ and $\dim L_2 = k_2$ one can choose homogeneous coordinates such that

$$f = [\mathrm{diag}(\underbrace{-1, \dots, -1}_{k_1+1}, \underbrace{1, \dots, 1}_{k_2+1})].$$

Proof.

(i)
- Fixed point spaces of $f = [F]$ correspond to eigenspaces of F and are therefore projective subspaces.
- For a point $X \in \mathbb{C}\mathrm{P}^n$ which is not a fixed point the line $\ell := X \wedge f(X)$ is preserved: $f(\ell) = \ell$, and the restriction $f\big|_\ell : \ell \to \ell$ is an involution on a complex projective line. By Proposition 7.8, $f\big|_\ell$ has two distinct fixed points, and thus f has at least two disjoint fixed point spaces.
- There is an invariant line through every point and every invariant line contains two fixed points. Therefore,

$$\mathbb{C}\mathrm{P}^n = \bigcup_{X,Y \text{ f.p.}} X \wedge Y,$$

  i.e., the fixed point spaces span the whole space.
- Assume there are three disjoint fixed point spaces, and choose a point P_1, P_2, P_3 on each fixed point space and let $\Pi := P_1 \wedge P_2 \wedge P_3$ be the plane through them. Then, $f\big|_\Pi : \Pi \to \Pi$ is an involution on a complex projective plane with exactly three fixed points, which contradicts Lemma 7.11.

(ii) Exercises.

□

Lemma 7.11. *There is no projective involution $f : \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ with exactly three fixed points.*

Proof. Choose homogeneous coordinates such that the three fixed points have coordinates $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. Then, a representative matrix of $f = [F]$ is of the form $F = \mathrm{diag}(a, b, c)$ with distinct $a, b, c \in \mathbb{C}$. Since f is an involution F must satisfy

$$F^2 = \mathrm{diag}(a^2, b^2, c^2) = \lambda I \qquad \text{for some } \lambda \neq 0,$$

and thus, $a^2 = b^2 = c^2$, which contradicts that a, b, c are distinct. □

Vice versa, a projective involution is uniquely determined by the two fixed point spaces $L_1, L_2 \subset \mathbb{C}\mathrm{P}^n$:

- For every point $X \in \mathbb{C}\mathrm{P}^n$ which is not a fixed point there is a unique line ℓ through X which intersects L_1 and L_2.
- On the line ℓ the involution is is determined by Proposition 7.8 (ii).

Proposition 7.12.

(i) For projective involution $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ the two fixed point spaces $L_1, L_2 \subset \mathbb{C}\mathrm{P}^n$ of its complexification are either both real or $\bar{L}_1 = L_2$.

(ii) If $\bar{L}_1 = L_2$ then $\dim L_1 = \dim L_2 =: k$, and one can choose homogeneous coordinates such that

$$f = \left[\operatorname{diag}\left(\underbrace{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}_{k+1}\right)\right].$$

Proof. Exercises. □

Remark 7.5. Note that the case $\bar{L}_1 = L_2$ of imaginary fixed points can only occur in $\mathbb{R}\mathrm{P}^n$ with n odd.

7.7 Correlations

A projective transformation $f : \mathrm{P}(V) \to \mathrm{P}(V)^*$ is called a *projective correlation.*

A projective correlation yields an identification of a projective space and its dual space.[2] By duality we can also interpret this as a map from points in $\mathrm{P}(V)$ to hyperplanes in $\mathrm{P}(V)$, or more generally, a map from k-planes $K \subset \mathrm{P}(V)$ to $(n-k-1)$-planes in $\mathrm{P}(V)$:

$$K \mapsto K^\perp := f(K)^\star.$$

$K^\perp$ is called the *polar subspace* of K (with respect to the correlation f).

Proposition 7.13. *Let $f : \mathrm{P}(V) \to \mathrm{P}(V)^*$ be a correlation. Then for any two projective subspaces $K_1, K_2 \subset \mathrm{P}(V)$:*

$$K_1 \subset K_2 \quad \Leftrightarrow \quad K_2^\perp \subset K_1^\perp,$$

and

$$(K_1 \wedge K_2)^\perp = K_1^\perp \cap K_2^\perp,$$
$$(K_1 \cap K_2)^\perp = K_1^\perp \wedge K_2^\perp.$$

Proof. Exercises. □

A correlation is induced by a linear transformation $F : V \to V^*$, which in turn can be identified with a bilinear form $b : V \times V \to \mathbb{F}$ by

$$F(x) = b(x, \cdot) \qquad \text{for } x \in V.$$

Thus, the correlation f can be identified with a bilinear form $[b]$ up to a non-zero scalar multiple, and we obtain

$$K^\perp = \{[x] \in \mathrm{P}(V) \mid b(x, y) = 0 \text{ for all } [y] \in K\}.$$

This seems to induce a relation similar to duality but entirely on the primal space. But in general it lacks the involutory property of duality: $(K^\perp)^\perp \neq K$.

[2]Recall that there is no canonical identification of this kind.

Proposition 7.14. *A projective correlation $f : \mathrm{P}(V) \to \mathrm{P}(V)^*$ satisfies $(K^\perp)^\perp = K$ for all projective subspaces $K \subset \mathrm{P}(V)$ if and only if it is an involution:*

$$f^* \circ f = \mathrm{id}\,.$$

Proof. Exercises. □

Let $\mathcal{B}$ be a basis of V and $\mathcal{B}^*$ the corresponding dual basis of V^*. In these bases a correlation is represented by the matrix $F \in \mathbb{F}^{(n+1)\times(n+1)}$.[3] In a new basis given by the transformation of homogeneous coordinates $[\tilde{x}] = [Ax]$ and the corresponding of homogeneous coordinates in the dual space $[\tilde{\alpha}] = [A^{-\intercal}\alpha]$ a representative matrix $\tilde{F}$ of f is given by

$$\tilde{F} = A^{-\intercal} F A^{-1}.$$

Proposition 7.15. *A projective correlation $f : \mathrm{P}(V) \to \mathrm{P}(V)^*$ is an involution if and only if its representative bilinear form b is symmetric or skew-symmetric, or equivalently, if its representative matrix in a basis and its dual basis satisfies*

$$F = F^\intercal \qquad \text{or} \quad F = -F^\intercal.$$

Proof. By Proposition 7.14 the matrix F must satisfy

$$F^{-\intercal} F = \lambda I \qquad \text{for some scalar } \lambda \neq 0.$$

Since F is invertible we obtain $F = \lambda F^\intercal$ and by transposing $F^\intercal = \lambda F$. Together, $F = \lambda^2 F$ and thus $\lambda^2 = 1$. □

In the case

- $F = F^\intercal$, the correlation $[F]$ is called a *polarity*,
- $F = -F^\intercal$, the correlation $[F]$ is called a *null-polarity*,

7.7.1 Null-polarities

(Non-degenerate) null-polarities only exist in $\mathbb{F}\mathrm{P}^n$ with n odd since non-singular skew-symmetric matrices only exist in even dimensions. They correspond to skew-symmetric bilinear forms.

Proposition 7.16. *Let $f : \mathbb{F}\mathrm{P}^n \to (\mathbb{F}\mathrm{P}^n)^*$ be a null-polarity. Then every point $X \in \mathbb{F}\mathrm{P}^n$ is contained in its polar hyperplane:*

$$X \in X^\perp$$

Proof. Exercises. □

Proposition 7.17. *Up to a choice of homogeneous coordinates a null-polarity $f : \mathbb{F}\mathrm{P}^n \to (\mathbb{F}\mathrm{P}^n)^*$ can always be represented by*

$$f = \left[\mathrm{diag}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\right].$$

[3] This is the same matrix that represents the corresponding bilinear form. If $\mathcal{B} = (e_1, \ldots e_{n+1})$, then

$$F_{ij} = b(e_i, e_j).$$

Example 7.2. In $\mathbb{R}\mathrm{P}^3$ the null-polarity $f : \mathbb{R}\mathrm{P}^3 \to (\mathbb{R}\mathrm{P}^3)^*$ distinguishes the set of self-polar lines $\ell = \ell^\perp$, which is called a *linear line complex*.

Let $P_1, P_2, P_3, P_4 \in \mathbb{R}\mathrm{P}^3$ be the vertices of a tetrahedron. Then $P_1^\perp, P_2^\perp, P_3^\perp, P_4^\perp$ are the planes of a second tetrahedron, such that the vertices of the first tetrahedron lie on the planes of the second and vice versa. Such a pair is called a *Möbius pair of tetrahedra*.

7.7.2 Polarities

Polarities correspond to symmetric bilinear forms. Symmetric bilinear forms on the other hand correspond to quadrics.

Part II
Quadrics

> Für den Aufbau der nichteuklidischen Geometrie ist die Theorie der *Gebilde zweiter Ordnung* von grundlegender Bedeutung. Diese Gebilde sind durch die Forderung bestimmt, daß die homogenen Koordinaten ihrer Punkte eine gegebene Gleichung zweiten Grades erfüllen sollen. (Vorlesungen über nichteuklidische Geometrie – Klein)

8 Quadrics in projective space

8.1 Bilinear forms

Let V be a vector space of dimension $n+1$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *bilinear form* is a map

$$b : V \times V \to \mathbb{F}$$

which is linear in both arguments.

A bilinear form b can be identified with a linear map

$$B : V \to V^*, \qquad x \mapsto b(x, \cdot).$$

This yields an isomorphism between the two vector spaces of bilinear forms over V and linear maps from V to V^*, which are therefore both of dimension $(n+1)^2$. We define the *rank* and *kernel* of a bilinear form by its associated linear map:

$$\operatorname{rk} b := \operatorname{rk} B, \qquad \ker b := \ker B.$$

8.1.1 Coordinate representation

Let $e_1, \ldots, e_{n+1}$ be a basis of V. Then the matrix $Q = (q_{ij}) \in \mathbb{F}^{(n+1)\times(n+1)}\}$

$$q_{ij} := b(e_i, e_j) \qquad \text{for } i, j = 1, \ldots, n+1$$

is called the *representative matrix*, or *Gram matrix*, of the bilinear form b. It coincides with the representative matrix of the corresponding linear map B in the basis $e_1, \ldots, e_{n+1}$ and its dual basis.

For two coordinate vectors $x = \sum_i x_i e_i$, $y = \sum_i y_i e_i \in V$ one finds

$$b(x, y) = x^\intercal Q y,$$

while a change of coordinates $\tilde{x} = Ax$ with $A \in \mathrm{GL}(n+1, \mathbb{F})$ acts on the representative matrix as

$$\tilde{Q} = A^{-\intercal} Q A^\intercal.$$

8.1.2 Symmetric bilinear forms and quadratic forms

A bilinear for is called *symmetric* if

$$b(x, y) = b(y, x) \qquad \text{for } x, y \in V,$$

or equivalenty, if its representative matrix is symmetric

$$Q^{\mathsf{T}} = Q.$$

The space of symmetric bilinear forms $\mathrm{Sym}(V)$ is a linear subspace of dimension

$$\dim \mathrm{Sym}(V) = \frac{(n+1)(n+2)}{2}.$$

A symmetric bilinear form $b(\cdot, \cdot)$ defines a corresponding *quadratic form* $b(\cdot)$

$$b(x) := b(x, x) \qquad \text{for } x \in V.$$

Vice versa, a quadratic form uniquely determines its bilinear form (polarization identity)

$$2b(x, y) = b(x + y) - b(x) - b(y),$$

and thus, the vector spaces of symmetric bilinear forms on V and quadratic forms on V are isomorphic.

8.2 Quadrics

The zero set of a non-zero quadratic form defines a *quadric* on $\mathrm{P}(V)$

$$\mathcal{Q}_b := \{[x] \in \mathrm{P}(V) \mid b(x) = 0\}.$$

A point $[x] \in \mathrm{P}(V)$ is called a *singular point* of the quadric $\mathcal{Q}_b$ if $x \in \ker b$.

Proposition 8.1. *Let $\mathcal{Q}_b \subset \mathrm{P}(V)$ be a quadric with $\operatorname{rk} b = k$. Then its singular points are contained in $\mathcal{Q}_b$ and constitute a projective subspace of dimension $n - k$.*

Proof. Exercise. □

The quadric $\mathcal{Q}_b$ is called *non-degenerate* if it has no singular points, or equivalently, if b has full rank.

A non-zero scalar multiple of b defines the same quadric:

$$\mathcal{Q}_b = \mathcal{Q}_{\lambda b} \qquad \text{for } \lambda \neq 0.$$

Proposition 8.2. *If $\mathbb{F} = \mathbb{C}$ two quadrics coincide if and only if their bilinear forms coincide up to a non-zero scalar multiple, i.e.,*

$$\mathcal{Q}_b = \mathcal{Q}_{\tilde{b}} \qquad \Leftrightarrow \qquad b = \lambda \tilde{b} \quad \textit{for some } \lambda \neq 0.$$

Proof. Geometry 1. □

Remark 8.1. For $\mathbb{F} = \mathbb{R}$ the statement remains true for non-empty quadrics that do not solely consist of singular points, or alternatively, for all quadrics $\mathcal{Q} \subset \mathrm{P}(V)$ if we consider their complexification

$$\mathcal{Q}_{\mathbb{C}} := \{[x] \in \mathrm{P}(V)_{\mathbb{C}} \mid b(x, x) = 0\}.$$

Thus, we can identify the space of quadrics with the projective space $\mathrm{P}\,\mathrm{Sym}(V)$. Its dimension is given by

$$\dim \mathrm{P}\,\mathrm{Sym}(V) = \dim \mathrm{Sym}(V) - 1 = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}.$$

and the coefficients

$$q_{ij} = b(e_i, e_j), \qquad \text{for } j \leqslant i$$

can be taken as homogenous coordinates on the space of quadrics.

Correspondingly, $\frac{n(n+3)}{2}$ generic points in $\mathrm{P}(V)$ determine a unique quadric. We specify "generic points" more precisely in the case $n = 2$:

Proposition 8.3. *Let P_1, P_2, P_3, P_4, P_5 be five points in $\mathbb{F}\mathrm{P}^2$, then there exists a conic through $P_1, \ldots, P_5$. Moreover*

- *If no four points lie on a line, the conic is unique.*
- *If no three points lie on a line, the conic is non-degenerate.*

Proof. Geometry 1. □

Lemma 8.4. *If three collinear points are on a quadric, then the quadric contains the whole line.*

Proof. Geometry 1. □

Example 8.1. Consider three skew lines in $\mathbb{R}\mathrm{P}^3$. Then there exists a one-parameter family of lines which intersect all three of them. Take any three of those lines. Then the 9 points of intersection determine a unique quadric in $\mathbb{R}\mathrm{P}^3$, which contains the whole family.

A projective transformation $f : \mathrm{P}(V) \to \mathrm{P}(W)$ induces a corresponding projective transformation on the space of quadrics

$$\mathrm{P}\,\mathrm{Sym}(V) \to \mathrm{P}\,\mathrm{Sym}(W), \qquad [b] \mapsto \left[b\left(f^{-1}(\cdot), f^{-1}(\cdot)\right)\right].$$

In homogeneous coordinates this transformation is induced by

$$Q \mapsto F^{-\intercal} Q F^{-1}.$$

8.2.1 Projective classification of quadrics

We want to classify quadrics up to projective automorphisms. Thus, two quadrics $\mathcal{O}_b, \mathcal{Q}_{\tilde{b}} \subset \mathbb{F}\mathrm{P}^n$ are called projectively equivalent if

$$f(\mathcal{Q}_b) = \mathcal{Q}_{\tilde{b}}$$

for some $f \in \mathrm{PGL}(n+1.\mathbb{F})$, or in homogeneous coordinates

$$\tilde{Q} = \lambda\, F^{-\intercal} Q F^{-1} \qquad \text{for some } \lambda \neq 0.$$

Thus, the projective classification of quadrics is based on the diagonalization of symmetric bilinear forms.

Proposition 8.5. *Let b be a symmetric bilinear form on V. Then there exists a basis $(e_1, \ldots, e_{n+1})$ of V such that*

$$b(e_i, e_j) = 0 \quad \text{if} \quad i \neq j.$$

Then (up to reordering) this basis further satisfies

$$\begin{aligned} b(e_i, e_i) &\neq 0 \quad \text{if} \quad 1 \leqslant i \leqslant r, \\ b(e_i, e_i) &= 0 \quad \text{if} \quad r+1 \leqslant i \leqslant n+1, \end{aligned}$$

where $r = \operatorname{rk} b$ and the basis vectors $e_{r+1}, \ldots, e_{n+1}$ form a basis of $\ker b$.

(i) If $\mathbb{F} = \mathbb{C}$ there exists a basis that additionally satisfies

$$b(e_i, e_i) = 1 \quad \text{if} \quad 1 \leqslant i \leqslant r.$$

(ii) If $\mathbb{F} = \mathbb{R}$ there exists a basis that additionally satisfies

$$\begin{aligned} b(e_i, e_i) &= 1 \quad \text{if} \quad 1 \leqslant i \leqslant p, \\ b(e_i, e_i) &= -1 \quad \text{if} \quad p+1 \leqslant i \leqslant r, \end{aligned}$$

where the number p is uniquely determined.

Proof. Geometry 1. □

The tuple $(p, q := r - p, t := n + 1 - r)$ is called the *signature* of the symmetric bilinear form b and is also written as

$$(\underbrace{+\cdots+}_{p}\underbrace{-\cdots-}_{q}\underbrace{0\cdots 0}_{t}).$$

For a quadric its signature is defined up to the equivalence

$$(p, q, t) \quad \sim \quad (q, p, t).$$

Theorem 8.6.

(i) Two quadrics in $\mathbb{C}\mathrm{P}^n$ are projectively equivalent if and only if they have the same rank.

(ii) Two quadrics in $\mathbb{R}\mathrm{P}^n$ are projectively equivalent if and only if they have the same signature.

Proof. Geometry 1. □

Thus, there are $\left\lfloor \frac{r}{2} \right\rfloor$ projectively different quadrics in $\mathbb{R}\mathrm{P}^n$ of rank r.

Quadrics in $\mathbb{R}\mathrm{P}^1$

- $(++)$ *empty quadric.* By complexification these are two complex conjugate points.
- $(+-)$ *two points.*
- $(+0)$ *one (double) point.*

Quadrics in $\mathbb{R}\mathrm{P}^2$ (conics)

- $(+\,+\,+)$ *empty conic.* By complexification this is an imaginary conic.
- $(+\,+\,-)$ *oval conic.* In affine coordinates this conic is an ellipse, a hyperbola, or a parabola.
- $(+\,+\,0\,)$ *point.* By complexification these are two imaginary lines that intersect in a real point.
- $(+\,-\,0\,)$ *pair of lines.*
- $(+\,0\,0\,)$ *one (double) line.*

Quadrics in $\mathbb{R}\mathrm{P}^3$

- $(+\,+\,++)$ *empty quadric.* By complexification this is an imaginary quadric.
- $(+\,+\,+-)$ *oval quadric.* In affine coordinates this quadric is an ellipsoid, a 2-sheeted hyperboloid, or an elliptic paraboloid.
- $(+\,+\,--)$ *toric quadric.* In affine coordinates this quadric is a 1-sheeted hyperboloid or a hyperbolic paraboloid.
- $(+\,+\,+0\,)$ *point.* By complexification this is a imaginary cone with real vertex.
- $(+\,+\,-0\,)$ *cone.* In affine coordinates this quadric is a cone or cylinder.
- $(+\,+\,0\,0\,)$ *line.* By complexification these are two imaginary planes intersecting in a real line.
- $(+\,-\,0\,0\,)$ *two planes.*
- $(+\,0\,0\,0\,)$ *one (double) plane.*

8.2.2 Quadrics on projective subspaces

Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric of signature (p, q, t). If $\mathcal{Q}$ is degenerate, i.e., $t \geqslant 1$, it can be decomposed into its singular subspace and a non-degenerate quadric on a complementary subspace.

Proposition 8.7. *Let $K_0 := \mathrm{P}(\ker b)$ and $K_1 \subset \mathbb{R}\mathrm{P}^n$ a complementary projective subspace, i.e.,*

$$K_0 \cap K_1 = \varnothing, \qquad K_0 \wedge K_1 = \mathbb{R}\mathrm{P}^n.$$

Then the restriction of $\mathcal{Q}$ onto K_1 is a (non-degenerate) quadric of signature (p, q). Furthermore,

$$\mathcal{Q} = \bigcup_{X \in K_0,\, Y \in K_1} X \wedge Y.$$

Proof. Geometry 1. □

Example 8.2. Consider a quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^3$ with signature $(+\,+\,-0)$. In affine coordinates, it may be given by a cone where the vertex P is the singular point of $\mathcal{Q}$. Intersecting $\mathcal{Q}$ with a plane not through P yields a non-degenerate conic with signature $(+\,+\,-)$, e.g., an ellipse. The cone $\mathcal{Q}$ can be recovered by joining P with every point on the ellipse.

Let $K \subset \mathbb{R}\mathrm{P}^n$ be a projective subspace. If $K \subset \mathcal{Q}$ then K is called *isotropic*. Otherwise the restriction

$$\mathcal{Q} \cap K = \{[x] \in K \mid b(x,x) = 0\}$$

defines a quadric on K of signature $(\tilde{p}, \tilde{q}, \tilde{t})$, which is called the *signanure of K* (with respect to $\mathcal{Q}$). We associate the isotropic case with $(0, 0, \tilde{t})$.

Proposition 8.8. *A projective subspace K can assume any signature $(\tilde{p}, \tilde{q}, \tilde{t})$ satisfying*

$$\tilde{p} \leqslant p, \qquad \tilde{q} \leqslant q, \qquad \tilde{t} \leqslant t + \min\{p - \tilde{p},\ q - \tilde{q}\}.$$

In particular, the maximal dimension of isotropic subspaces contained in $\mathcal{Q}$ is

$$t + \min\{p,\ q\} - 1.$$

Proof. Geometry 1. □

Example 8.3. Consider a quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^3$ with signature $(+ + --)$. In affine coordinates, it may be given by a one-sheeted hyperboloid. By Proposition 8.8 it contains isotropic subspaces of dimension 1. Indeed, $\mathcal{Q}$ is a ruled surface and contains two one parameter families of lines.

8.3 Polarity

Let $\mathcal{Q}_b \subset \mathbb{F}\mathrm{P}^n$ be a quadric. For a projective subspace $\mathrm{P}(U) \subset \mathbb{F}\mathrm{P}^n$ its *polar subspace* is defined by

$$\mathrm{P}(U)^{\perp} := \{[x] \in \mathbb{F}\mathrm{P}^n \mid b(x,y) = 0 \text{ for all } y \in U\}.$$

Proposition 8.9. *Let $X = [x], Y = [y] \in \mathbb{F}\mathrm{P}^n$ be two points. Then*

$$X \in Y^{\perp} \qquad \Leftrightarrow \qquad Y \in X^{\perp} \qquad \Leftrightarrow \qquad b(x,y) = 0$$

Two such points, which lie in its respective polar subspaces, are called *conjugate*.

Proposition 8.10. *Let $X, Y \in \mathbb{R}\mathrm{P}^n$ be two conjugate points such that the line $X \wedge Y$ intersects the quadric $\mathcal{Q}$ in two (possibly imaginary) points F_1, F_2. Then the pair $\{X, Y\}$ separates the pair $\{F_1, F_2\}$ harmonically, i.e.,*

$$\mathrm{cr}(X, F_1, Y, F_2) = -1.$$

Proof. Geometry 1. □

If the quadric is non-degenerate, the symmetric bilinear form induces a projective correlation

$$\pi : \mathbb{F}\mathrm{P}^n \to (\mathbb{F}\mathrm{P}^n)^*, \qquad [x] \mapsto [b(x,\cdot)]$$

which is a *polarity*, and

$$K^{\perp} = \pi(K)^{\star}.$$

Thus, in this case, the dimensions of two polar subspaces satisfy

$$\dim K + \dim K^{\perp} = n - 1.$$

A refinement of this statement, which includes the signatures of the two polar subspaces, is captured in the following Proposition:

Proposition 8.11. *Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a non-degenerate quadric of signature (p, q). Then the signature $(\tilde{p}, \tilde{q}, \tilde{t})$ of a subspace $\mathrm{P}(U) \subset \mathbb{RP}^n$ and the signature $(\tilde{p}_\perp, \tilde{q}_\perp, \tilde{t}_\perp)$ of its polar subspace $\mathrm{P}(U)^\perp$ with respect to $\mathcal{Q}$ satisfy*

$$p = \tilde{p} + \tilde{p}_\perp + \tilde{t}, \quad q = \tilde{q} + \tilde{q}_\perp + \tilde{t}, \quad \tilde{t} = \tilde{t}_\perp.$$

In particular, $\tilde{t} \leqslant \min\{p, q\}$.

Proof. Geometry 1. □

8.4 Dual quadrics

> Under this dual correspondence, the points of a curve correspond to a collection of straight lines that in general envelop a second curve as tangents. A more detailed study reveals that the family of straight lines corresponding dually to the points of a conic always envelops another conic. (Geometry and the Imagination – Hilbert, Cohn-Vossen)

Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric of signature (p, q, t) and $\mathcal{V} = \mathrm{P}(\ker b)$ its singular points. For a point $X \in \mathcal{Q} \backslash \mathcal{V}$ the *tangent hyperplane* of $\mathcal{Q}$ at X is given by its polar hyperplane $X^\perp$. The tangent plane has signature $(p-1, q-1, t+1)$ and intersects the quadric in exactly the isotropic subspaces through X.

Thus, the set of tangent planes of a quadric is given by

$$\mathrm{T}\mathcal{Q} = \left\{ X^\perp \subset \mathbb{RP}^n \;\middle|\; X \in \mathcal{Q} \backslash \mathcal{V} \right\},$$

By duality each tangent plane corresponds to a point in the dual space $(\mathbb{RP}^n)^*$ We define the *dual quadric* of $\mathcal{Q}$ by

$$\mathcal{Q}^\star := \left\{ K^\star \in (\mathbb{RP}^n)^* \;\middle|\; K \in \mathrm{T}\mathcal{Q} \right\} = \{ \pi(X) \mid X \in \mathcal{Q} \backslash \mathcal{V} \}.$$

Proposition 8.12.

(i) Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a non-degenerate quadric of signature (p, q) with representative matrix Q. Then its dual quadric $\mathcal{Q}^\star \subset (\mathbb{RP}^n)^*$ is a non-degenerate quadric of signature (p, q) with representative matrix Q^{-1}.

(ii) Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a degenerate quadric of signature (p, q, t) with singular points $\mathcal{V}$. Then its dual quadric $\mathcal{Q}^\star \subset (\mathbb{RP}^n)^*$ is entirely contained in the projective subspace $\mathcal{V}^\star$ of dimension $n - t - 1$:

$$\mathcal{Q}^\star \subset \mathcal{V}^\star.$$

On $\mathcal{V}^\star$ it constitutes a non-degenerate quadric of signature (p, q).

Proof. Exercises. □

9 The Klein Erlangen program

> Among the advances of the last fifty years in the field of geometry, the development of projective geometry occupies the first place. Although it seemed at first as if the so-called metrical relations were not accessible to this treatment,

> as they do not remain unchanged by projection, we have nevertheless learned recently to regard them also from the projective point of view, so that the projective method now embraces the whole of geometry. But metrical properties are then to be regarded no longer as characteristics of the geometrical figures per se, but as their relations to a fundamental configuration, the imaginary circle at infinity common to all spheres. (Vergleichende Betrachtungen über neue geometrische Forschungen (1872) – Felix Klein – english translation)

In the *Klein Erlangen program* Euclidean and non-Euclidean geometries are considered as subgeometries of projective geometries. Their transformation groups are regarded as subgroups of the projective linear group. In particular as subgroups preserving a quadric, i.e., projective orthogonal groups. The cross-ratio, which is the invariant of projective geometry, together with the quadric yields the invariants of the subgeometries.

9.1 Projective orthogonal groups

Let b be symmetric bilinear form on $\mathbb{R}^{n+1}$. Then $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is *orthogonal* with respect to b if

$$b(F(x), F(y)) = b(x, y) \quad \text{for all } x, y \in \mathbb{R}^{n+1}.$$

The projective automorphism $f = [F] : \mathbb{RP}^n \to \mathbb{RP}^n$ induced by the orthogonal transformation F maps the quadric $\mathcal{Q}_b$ to itself: $f(\mathcal{Q}_b) = \mathcal{Q}_b$.

Proposition 9.1. *Let $\mathcal{Q}_b \subset \mathbb{RP}^n$ be a quadric of signature (p, q, t) with $p \neq q$ and $pq \neq 0$, then any projective automorphism that maps $\mathcal{Q}_b$ to itself comes from a linear map which is orthogonal with respect to b.*

Proof. Geometry 1. □

The *group of orthogonal transformations* for a bilinear form of signature (p, q, t) is denoted by $\mathrm{O}(p, q, t)$. The corresponding group of projective automorphisms $\mathrm{PO}(p, q, t) \subset \mathrm{PGL}(n+1)$ is called the *projective orthogonal group*. Hence, under the assumption of non-neutral signature, this is the group of projective transformations mapping a quadric to itself.

Remark 9.1.

(i) In case of neutral signature, i.e. $p = q$, there exist projective automorphisms preserving the quadric, which are not induced by an orthogonal transformation. Those are transformations that exchange the two sides of the quadric.

(ii) In the case $pq = 0$ the statement remains true if we consider the complexification of $\mathcal{Q}_b$.

Proposition 9.2. *A projective automorphism $f : \mathbb{RP}^n \to \mathbb{RP}^n$ preserves a quadric $\mathcal{Q} \subset \mathbb{RP}^n$ if and only if its dual transformation $f^* : (\mathbb{RP}^n)^* \to (\mathbb{RP}^n)^*$ preserves its dual quadric $\mathcal{Q}^\star \subset (\mathbb{RP}^n)^*$.*

Proof. Exercise. □

9.1.1 Projective orthogonal involutions

Proposition 9.3. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a non-degenerate quadric, and $f : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ a projective involution preserving $\mathcal{Q}$. Then its two fixed point spaces $L_1, L_2 \subset \mathbb{R}\mathrm{P}^n$ are polar subspaces with respect to $\mathcal{Q}$, i.e.,*

$$L_1^\perp = L_2.$$

Proof. Follows from Proposition 8.10. □

Thus, any point $[q] \in \mathbb{F}\mathrm{P}^n$ induces an involution that fixes $[q]$ and its polar hyperplane $[q]^\perp$. It is given by

$$\sigma_q : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n, \qquad [x] \mapsto \left[x - 2\frac{b(x,q)}{b(q,q)}q\right]$$

and called *reflection in the hyperplane* $[x]^\perp$.

By the *theorem of Cartan and Dieudonné* every projective orthogonal transformation can be decomposed into a finite number of reflections in hyperplanes.

Theorem 9.4. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a non-degenerate quadric of signature (p, q). Then each element of the corresponding projective orthogonal group* $\mathrm{PO}(r, s)$ *is the composition of at most $n + 1$ reflections in hyperplanes.*

Proof. Geometry 1. □

9.1.2 Projective orthogonal group in the plane

In the plane a projective orthogonal transformation can be specified by prescribing the images of three points on the conic.

Proposition 9.5. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^2$ be a non-empty non-degenerate conic. Let $X_1, X_2, X_3 \in$ quadric be a triple of distinct points, and $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \in$ quadric another triple of distinct points.*

Then there exists a unique projective automorphism $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ that preserves the conic, i.e., $f(\mathcal{Q}) = \mathcal{Q}$, and

$$\tilde{X}_i = f(X_i) \qquad \textit{for } i = 1, 2, 3.$$

Proof. Geometry 1. □

9.2 Cayley-Klein distance

Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric with corresponding bilinear form $b\cdot\cdot$. Then we denote by

$$K_{\mathcal{Q}}(X, Y) := \frac{b(x,y)^2}{b(x,x)\, b(y,y)}$$

the *Cayley-Klein distance* of any two points $X = [x], Y = [y] \in \mathbb{R}\mathrm{P}^n \backslash \mathcal{Q}$ that are not on the quadric. In the presence of a Cayley-Klein distance the quadric $\mathcal{Q}$ is called the *absolute quadric.*

Remark 9.2. The name Cayley-Klein distance, or Cayley-Klein metric, is usually assigned to a metric derived from above quantity. Nevertheless, we prefer to assign it to this basic quantity associated with an arbitrary quadric.

The Cayley-Klein distance is projectively well-defined, in the sense that it depends neither on the choice of the bilinear form corresponding to the quadric $\mathcal{Q}$ nor on the choice of homogeneous coordinate vectors for the points X and Y.

Proposition 9.6. *The Cayley-Klein distance is invariant under projective automorphisms that preserve the absolute quadric.*

Proof. Exercise. □

The projective automorphisms preserving the absolute quadric are also called *isometries.*

The Cayley-Klein distance can be expressed in terms of the cross-ratio in the following way:

Proposition 9.7. *Let $X, Y \in \mathbb{RP}^n \backslash \mathcal{Q}$ and $P, Q \in \mathcal{Q}$ be the two (possibly imaginary) points of intersection of the line $X \wedge Y$ with $\mathcal{Q}$ such that the pair $\{X, Y\}$ separates the pair $\{P, Q\}$. Then*

$$K_{\mathcal{Q}}(X, Y) = \cosh^2\left(\frac{1}{2}\log \mathrm{cr}(Y, P, X, Q)\right).$$

Proof. Geometry 1. □

A *Cayley-Klein space* is usually considered to be one side of the quadric together with a metric or pseudo-metric derived from the Cayley-Klein distance, or equivalently, together with the transformation group preserving the quadric.

9.3 Hyperbolic geometry

Let $\langle \cdot, \cdot \rangle$ be the standard non-degenerate bilinear form of signature $(n, 1)$, i.e.

$$\langle x, y \rangle := x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

for $x, y \in \mathbb{R}^{n+1}$, and denote by $\mathcal{S} \subset \mathbb{RP}^n$ the corresponding quadric. We identify the "inside" of $\mathcal{S}$ with the n-dimensional *hyperbolic space*

$$\mathcal{H} := \mathcal{S}^- := \{[x] \in \mathbb{RP}^n \mid \langle x, x \rangle < 0\}.$$

For two points $X, Y \in \mathcal{H}$ one has $K_{\mathcal{S}}(X, Y) \geqslant 1$, and the quantity d given by

$$K_{\mathcal{S}}(X, Y) = \cosh^2 d(X, Y)$$

defines a metric on $\mathcal{H}$ of constant negative sectional curvature. Geodesics are given by intersections of projective lines in $\mathbb{RP}^n$ with $\mathcal{H}$. The corresponding group of isometries is given by $\mathrm{PO}(n, 1)$ and called the group of *hyperbolic motions.* The absolute quadric $\mathcal{S}$ consists of the points at (metric) infinity.

9.4 Elliptic geometry

For $x, y \in \mathbb{R}^{n+1}$ we denote by

$$\langle x, y\rangle := x_1y_1 + \dots x_ny_n + x_{n+1}y_{n+1}$$

the standard (positive definite) scalar product on $\mathbb{R}^{n+1}$, i.e. the standard non-degenerate bilinear form of signature $(n+1, 0)$. The corresponding quadric $\mathcal{O} \subset \mathbb{RP}^n$ is empty, yet we consider its comlpexification which is purely imaginary. Now we identify $\mathbb{RP}^n$ with the n-dimensional *elliptic space*. For two points $X, Y \in \mathbb{RP}^n$ one always has $0 \leqslant K_{\mathcal{O}}(\boldsymbol{x}, \boldsymbol{y}) \leqslant 1$ and the quantity d given by

$$K_{\mathcal{O}}(X, Y) = \cos^2 d(X, Y)$$

defines a metric on $\mathcal{E}$ of constant positive sectional curvature. Geodesics are given by projective lines, and the corresponding group of isometries is given by $\mathrm{PO}(n+1)$ and called the group of *elliptic motions*.

Remark 9.3. Elliptic geometry is spherical geometitry with anti-podal points identified.

9.5 Euclidean geometry

Euclidean geometry is induced by a degenerate quadric on the dual space. We look at the corresponding construction in the 2-dimensional case, i.e., the Euclidean plane.

Let $\langle\cdot,\cdot\rangle$ be the standard degenerate bilinear form of signature $(2, 0, 1)$ on the dual space $(\mathbb{R}^3)^*$, i.e.,

$$\langle x, y\rangle := x_1y_1 + x_2y_2$$

for $x, y \in (\mathbb{R}^3)^*$, and denote by $\mathcal{C} \subset \mathbb{RP}^2$ the corresponding quadric.

The equation

$$x_1^2 + x_2^2 = 0$$

for the quadric $\mathcal{C}$ has exactly one real solution $M = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]$. From

$$x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2) = 0$$

we see, that the complex solutions constitute two complex conjugate lines

$$\ell = \left\{[x] \in \left(\mathbb{RP}^2\right)^* \mid x_1 + ix_2 = 0\right\}, \qquad \bar{\ell} = \left\{[x] \in \left(\mathbb{RP}^2\right)^* \mid x_1 - ix_2 = 0\right\},$$

which intersect in this real point: $\ell \cap \bar{\ell} = M$.

By duality $M^\star$ is the line at infinity in affine coordinates on

$$\mathbb{RP}^2 = \mathbb{R}^2 \cup M^\star,$$

the two complex conjugate lines become two complex conjugate imaginary points

$$\ell^\star = \left[\begin{smallmatrix}1\\i\\0\end{smallmatrix}\right], \qquad \bar{\ell}^\star = \left[\begin{smallmatrix}1\\-i\\0\end{smallmatrix}\right].$$

on the line at infinity $M^\star$. Thus,

$$\mathcal{C} = \left\{\left[\begin{smallmatrix}1\\i\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-i\\0\end{smallmatrix}\right]\right\}$$

Projective transformations on $(\mathbb{RP}^2)^*$ that preserve $\mathcal{C}$ are of the form

$$[F] = \left[\begin{array}{c|c} A & 0 \\ \hline c^\intercal & d \end{array}\right]$$

with $A \in \mathrm{O}(2)$, $c \in \mathbb{R}^2$, $d \neq 0$. The dual transformations on $\mathbb{RP}^2$ (that preserve $\mathcal{C}^\star$) are of the form

$$[F]^\star = \left[\begin{array}{c|c} A^{-\intercal} & -\frac{1}{d}A^{-\intercal}c \\ \hline 0 & \frac{1}{d} \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{c} \\ \hline 0 & \tilde{d} \end{array}\right]$$

with $\tilde{A} \in \mathrm{O}(2)$, $\tilde{b} \in \mathbb{R}^2$, $\tilde{d} \neq 0$. These are *similarity transformations*, i.e., Euclidean motions and scalings, and they preserve the ratio of Euclidean distances.

Thus, we have reconstructed similarity geometry.

9.5.1 Angles in Euclidean geometry

Consider two lines in the Euclidean plane

$$\ell = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; n_1 x + n_2 y = d\right\}, \quad \tilde{\ell} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \tilde{n}_1 x + \tilde{n}_2 y = \tilde{d}\right\}.$$

with normal vectors $n = (n_1, n_2)$, $\tilde{n} = (\tilde{n}_1, \tilde{n}_2)$. Then the homogeneous coordinates of the corresponding points in the dual space are given by

$$\ell^\star = [m] = \begin{bmatrix} n_1 \\ n_2 \\ -d \end{bmatrix}, \qquad \tilde{\ell}^\star = [\tilde{m}] = \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ -\tilde{d} \end{bmatrix},$$

and their Cayley-Klein distance in the dual space by

$$K_{\mathcal{C}^\star}\left(\ell^\star, \tilde{\ell}^\star\right) = \frac{\langle m, \tilde{m}\rangle^2}{\langle m, m\rangle \langle \tilde{m}, \tilde{m}\rangle} = \frac{\langle n, \tilde{n}\rangle_{\mathbb{R}^2}^2}{\langle n, n\rangle_{\mathbb{R}^2} \langle \tilde{n}, \tilde{n}\rangle_{\mathbb{R}^2}} = \cos^2\varphi,$$

where φ is the intersection angle of the two lines. In particular

$$\cos^2\varphi = 0 \quad \Leftrightarrow \quad \langle m, \tilde{m}\rangle = 0.$$

Proposition 9.8. *Two Euclidean lines ℓ, ℓ are orthogonal if and only if the two points $\ell^\star$, $\tilde{\ell}^\star$ are conjugate with respect to $\mathcal{C}^\star$.*

9.5.2 Circles in Euclidean geometry

Proposition 9.9. *A non-degenerate conic is a circle if and only if it contains the two points from $\mathcal{C}$. Furthermore, the two imaginary tangent lines of a circle in the two points from $\mathcal{C}$ intersect in its center.*

Proof. A non-degenerate conic can be diagonalized by a Euclidean transformation:

$$Q = \mathrm{diag}(a, b, -1).$$

It contains the point $Z = [1, i, 0]$ (and therefore also $\bar{Z}$) if and only if

$$a = b,$$

which describes a circle.

By a similarity transformation the representative matrix of a circle can always be brought to the form

$$C = \operatorname{diag}(1, 1, -1).$$

The tangent line in the point Z is given by

$$x_1 + ix_2 = 0.$$

It intersects its complex conjugate line in the point $[0, 0, 1]$ which is the center of the circle. $\square$

We correspondingly call the two points of $\mathcal{C}$ the two *circle points* of similarity geometry.

10 Pencils of quadrics

A projective subspace in the space of quadrics $\mathrm{P}\,\mathrm{Sym}(V)$ is called a *linear system of quadrics*. A linear system of quadrics is called *degenerate* if it solely consists of degenerate quadrics.

Example 10.1. Let $[x] \in \mathrm{P}(V)$ be a point. Then all quadrics through $[x]$ form a non-degenerate linear system of quadrics of codimension 1, since

$$b(x, x) = 0$$

is one (non-trivial) linear equation in b.

Similarly, all quadrics through k generic points in $\mathrm{P}(V)$ form a linear system of quadrics of codimension k.

A linear system of quadrics of dimension 1, i.e. a line in $\mathrm{P}\,\mathrm{Sym}(V)$, is called a *pencil of quadrics*. Any two quadrics $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathrm{P}\,\mathrm{Sym}(V)$ span a pencil, which is given in homogeneous coordinates by

$$\mathcal{Q}_1 \wedge \mathcal{Q}_2 = [\lambda Q_1 + \mu Q_2]_{[\lambda,\mu] \in \mathbb{F}\mathrm{P}^1}.$$

The degenerate quadrics of the pencil are characterized by the equation

$$\det(\lambda Q_1 + \mu Q_2) = 0.$$

If the pencil is not degenerate, this is a non-trivial homogeneous polynomial equation in λ and μ of order $n+1$ and thus has at most $n+1$ solutions. If $\mathbb{F} = \mathbb{C}$ it has exactly $n+1$ solutions counting multiplicities.

In affine coordinates ($\lambda = 1$) the equation becomes

$$\det(Q_1 + \mu Q_2) = 0, \tag{2}$$

which now is a polynomial equation in μ of order at most $n+1$. If $\mathbb{F} = \mathbb{C}$ it again has exactly $n+1$ solutions counting multiplicities and allowing $\mu = \infty$ as a solution.

Lemma 10.1. *The multiplicity of a root of equation* (2)*, which corresponds to a degenerate quadric, is independent of the choice of basis Q_1, Q_2, or equivalently, independent under projective transformations on the line $\mathcal{Q}_1 \wedge \mathcal{Q}_2$.*

Proof. Exercises. $\square$

Thus, we can invariantly assign the multiplicities of the roots to the degenerate quadrics of the pencil and obtain:

Proposition 10.2. *A non-degenerate pencil of quadrics contains at most $n+1$ degenerate quadrics. If $\mathbb{F} = \mathbb{C}$ the multiplicities of the degenerate quadrics add up to $n+1$.*

A point which lies on two (and thus on every) quadrics of the pencil is called a *base point* of the pencil.

Proposition 10.3. *Let $\mathcal{P}$ be a pencil of quadrics. Let X be a point and H a hyperplane containing the point X. If two quadrics from $\mathcal{P}$ are tangent to H in X, then X is a base point of $\mathcal{P}$ and all quadrics from $\mathcal{P}$ are tangent to H in X.*

Proof. Exercise. □

Proposition 10.4. *Let $\mathcal{P}$ be a pencil of quadrics. Let H a hyperplane tangent to two quadrics of $\mathcal{P}$ in the two points X, Y. Then X and Y are conjugate with respect to all quadrics in the pencil.*

Proof. Exercise. □

10.1 Pencils of conics

The space of conics in $\mathbb{F}\mathrm{P}^2$ is a 5-dimensional projective space

$$\mathrm{P}\,\mathrm{Sym}(\mathbb{F}^3) \cong \mathbb{F}\mathrm{P}^5.$$

In homogeneous coordinates $[x] = [x_1, x_2, x_3]$ on $\mathbb{F}\mathrm{P}^2$ and the corresponding homogeneous coordinates $\mathcal{Q} = [q_{11}, q_{22}, q_{33}, q_{12}, q_{23}, q_{13}]$ on the space of conics the equation for the point $[x]$ lying on the conic $\mathcal{Q}$ is given by

$$q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{12}x_1x_2 + q_{23}x_2x_3 + q_{13}x_1x_3 = 0.$$

Example 10.2. Let $X_1, X_2, X_3, X_4 \in \mathbb{F}\mathrm{P}^2$ be four points in general position. Consider the set $\mathcal{P}$ of all conics containing these four points. Then by Proposition 8.3 there exists exactly one conic in $\mathcal{P}$ through every point in the plane except X_1, X_2, X_3, X_4, which are the only common points of all conics in $\mathcal{P}$. We will show that $\mathcal{P}$ is a pencil of conics with base points X_1, X_2, X_3, X_4 and three degenerate conics of rank 2.

By choosing homogeneous coordinates such that

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

the representative matrices $Q = (q_{ij})_{1 \leqslant i \leqslant j \leqslant 3}$ for the conics in $\mathcal{P}$ must satisfy

$$\begin{aligned} q_{11} + q_{22} + q_{33} + 2q_{12} + 2q_{23} + 2q_{13} &= 0 \\ q_{11} + q_{22} + q_{33} - 2q_{12} + 2q_{23} - 2q_{13} &= 0 \\ q_{11} + q_{22} + q_{33} + 2q_{12} - 2q_{23} - 2q_{13} &= 0 \\ q_{11} + q_{22} + q_{33} - 2q_{12} - 2q_{23} + 2q_{13} &= 0 \end{aligned}$$

By subtracting equations we obtain

$$q_{12} + q_{13} = 0, \quad q_{13} - q_{23} = 0, \quad q_{12} - q_{13} = 0,$$

which implies $q_{12} = q_{13} = q_{23} = 0$. By adding up all four equations we additionally obtain

$$q_{11} + q_{22} + q_{33} = 0$$

Thus, every conic in $\mathcal{P}$ is given by

$$Q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda - \mu \end{pmatrix}$$

for some $[\lambda, \mu] \in \mathbb{F}\mathrm{P}^1$, which describes a one-dimensional projective supbspace in $\mathbb{F}\mathrm{P}^5$ and thus a pencil of conics. The equations of the conics in this pencil are given by

$$\lambda(x_1^2 - x_3^2) + \mu(x_2^2 - x_3^2) = 0.$$

Its degenerate conics are given by

$$x_1^2 - x_3^2 = 0, \quad x_2^2 - x_3^2 = 0, \quad x_1^2 - x_2^2 = 0,$$

which each consists of a pair of opposite lines from the complete quadrangle defined by the four base points. They all have multiplicity 1.

Note that the diagonal triangle

$$A = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], \quad B = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right], \quad C = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$$

of the complete quadrangle of base points is a *polar triangle* for all conics of the pencil, i.e., each point is the pole of the opposite line.

Not every pencil of conics is given by all conics through four given points as the next example shows.

Example 10.3. Let $X_1, X_2 \in \mathbb{F}\mathrm{P}^2$ be two (distinct) points and $\ell_1, \ell_2 \subset \mathbb{F}\mathrm{P}^2$ two (distinct) lines such that X_1 lies on ℓ_1 and X_2 lies on ℓ_2. Consider the set $\mathcal{P}$ of conics which are tangent to ℓ_1 in X_1 and to ℓ_2 in X_2. We will show that $\mathcal{P}$ is a non-degenerate pencil with base points X_1, X_2 and two degenerate conics, one of rank 2 and multiplicity 2 and one of rank 1 and multiplicity 1.

Choose homogeneous coordinates such that

$$X_1 = \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right], \quad X_2 = \left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right], \quad \ell_1 = \left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]^\star, \quad \ell_2 = \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right]^\star.$$

The two tangency conditions are given by

$$Q\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right), \qquad Q\left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right),$$

which yields

$$\begin{aligned} q_{12} + q_{23} &= 0 \\ q_{12} + q_{23} &= 0 \\ q_{11} + q_{33} + 2q_{13} &= 0 \\ q_{11} - q_{33} + 2q_{13} &= 0, \end{aligned}$$

or equivalently,

$$q_{12} = q_{23} = q_{13} = 0, \qquad q_{11} = q_{33}.$$

Thus, all conics from $\mathcal{P}$ are given by

$$Q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

for some $[\lambda, \mu] \in \mathbb{F}\mathrm{P}^1$. The equations of the conics in this pencil are given by

$$\lambda(x_1^2 - x_3^2) + \mu x_2^2 = 0.$$

Its degenerate quadrics are given by

$$x_1^2 - x_3^2 = 0,$$

which has multiplicity 1 and consists of the two lines ℓ_1, ℓ_2, and

$$x_2 = 0,$$

which has multiplicity 2 and consists of the (double) line $X_1 \wedge X_2$.

We have found two examples of non-degenerate pencils of conics. We continue by investigating the possible numbers of base points.

Proposition 10.5. *A non-degenerate pencil of conics in* $\mathbb{F}\mathrm{P}^2$ *has at most* 4 *base points.*

Proof. Let

$$\mathcal{P} = \mathcal{Q}_1 \wedge \mathcal{Q}_2$$

be a non-degenerate pencil of conics. Since $\mathcal{P}$ is non-degenerate we may assume $\mathcal{Q}_1$ is non-degenerate and choose homogeneous coordinates in which its equation is given by

$$x_1^2 - x_2 x_3 = 0.$$

Note that while in the case $\mathbb{F} = \mathbb{C}$ this normalization is always possible, for $\mathbb{F} = \mathbb{R}$ it is possible if and only if the pencil contains at least one conic of signature $(+ + -)$.

The conic $\mathcal{Q}_2$ is given by

$$q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{12}x_1x_2 + q_{23}x_2x_3 + q_{13}x_1x_3 = 0.$$

The point

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathcal{Q}_1$$

is the only point of $\mathcal{Q}_1$ on the line $x_3 = 0$. By Proposition 9.5 we can further assume that

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \mathcal{Q}_2,$$

or equivalently,

$$q_{22} \neq 0,$$

we can introduce affine coordinates

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}$$

without having any base points on the line at infinity. In affine coordinates the two equations for $\mathcal{Q}_1, \mathcal{Q}_2$ are give by

$$\begin{aligned} &y = x^2 \\ &q_{11}x^2 + q_{22}y^2 + 2q_{12}xy + 2q_{23}y + 2q_{13}x + q_{33} = 0. \end{aligned} \tag{3}$$

Substituting the first equation into the second we obtain

$$q_{22}x^4 + 2q_{12}x^3 + (q_{11} + 2q_{23})x^2 + 2q_{13}x + q_{33} = 0. \tag{4}$$

and every solution of (4) corresponds to exactly one solution of (3). Furthermore equation (4) has up to 4 solutions. □

Lemma 10.6. *The multiplicity of a root of equation* (4)*, which corresponds to a base point, is independent of the choice of basis Q_1, Q_2, or equivalently, independent under projective transformations on the line $\mathcal{Q}_1 \wedge \mathcal{Q}_2$.*

Thus, we can invariantly assign the multiplicities of the roots to the base points of the pencil and since equation (4) has exactly 4 solutions counting multiplicities if $\mathbb{F} = \mathbb{C}$, we obtain

Proposition 10.7. *A non-degenerate pencil of conics in $\mathbb{C}\mathrm{P}^2$ has exactly* 4 *base points counting multiplicities. In particular, it always has at least one base point.*

Thus, for a non-degenerate pencil in $\mathbb{C}\mathrm{P}^2$ there are exactly five possible cases, which we denote as follows:

(I) four simple base points $(1, 1, 1, 1)$

(II) one double and two simple base points $(2, 1, 1)$

(III) two double base points $(2, 2)$

(IV) one triple and one simple base point $(3, 1)$

(V) one quadruple base point (4)

This list is exclusive and exhaustive in $\mathbb{C}\mathrm{P}^2$, i.e., each pencil is of exactly one of the given *types*. We have already encountered pencils of type I in Example 10.2. while the pencils in Example 10.3 must be of type III due to the symmetry of the base points. Thus, the two base points have multiplicity 2. Intuitively, a double base point may be thought of as two simple base points brought together, where in the limit a conic through these two points becomes tangent to a fixed line.

Proposition 10.8. *For a non-degenerate pencil $\mathcal{P}$ of conics in $\mathbb{F}\mathrm{P}^2$ with a base point X the following statements are equivalent:*

(i) All conics in $\mathcal{P}$ are tangent to a common line at X.

(ii) The base point X has multiplicity at least 2.

Remark 10.1. From the classification Theorem 10.9 we will see that this is further equivalent to the pencil $\mathcal{P}$ containing a degenerate conic of multiplicity at least 2.

Proof. We use the same normalization for the two conics $\mathcal{Q}_1, \mathcal{Q}_2$ that span the pencil as in the proof of Proposition 10.5, and additionally assume, by Proposition 9.5, that the base point X has homogeneous coordinates

$$X_1 = \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right].$$

Thus $x = 0$ is a solution to (4), or equivalently,

$$q_{33} = 0.$$

Its multiplicity is at least 2 if and only if

$$q_{13} = 0.$$

On the other hand by Proposition 10.3 statement (i) is equivalent to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ sharing the same tangent line in X_1, i.e.,

$$\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] = \left[Q_1\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\right] = \left[Q_2\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)\right] = \left[\begin{smallmatrix}q_{13}\\q_{23}\\0\end{smallmatrix}\right],$$

which again is equivalent to $q_{13} = 0$. ☐

10.1.1 Classification of pencils of conics in $\mathbb{C}\mathrm{P}^2$

We are now ready to prove the following classification result for pencils of concis in $\mathbb{C}\mathrm{P}^2$.

Theorem 10.9. *Two non-degenerate pencils of conics in $\mathbb{C}\mathrm{P}^2$ are projectively equivalent if and only if they are of the same type.*

Furthermore, the degenerate conics, their multiplicities, and a normal form for each type are as stated in Table 19.

Type	Base points	Deg. conics	Normal form
I	$1, 1, 1, 1$	$\times, \times, \times$	$\lambda(x_1^2 - x_3^2) + \mu(x_2^2 - x_3^2) = 0$
II	$2, 1, 1$	$2\times, \times$	$\lambda(x_1^2 - x_2^2) + \mu x_2(x_2 - x_3) = 0$
III	$2, 2$	$2\parallel, \times$	$\lambda(x_1^2 - x_3^2) + \mu x_2^2 = 0$
IV	$3, 1$	$3\times$	$\lambda(x_1^2 - x_2x_3) + \mu x_1x_2 = 0$
V	4	$3\parallel$	$\lambda(x_1^2 - x_2x_3) + \mu x_2^2 = 0$

Figure 19. The classification of pencils of conics in $\mathbb{C}\mathrm{P}^2$. The two types of degenerate conics are two lines ($\times$), and a double line ($\parallel$).

Remark 10.2. From Table 19 we see that the types of pencils can also be characterized by the number and rank of their degenerate conics.

Proof. To prove the projective equivalence of two pencils of the same type, we derive the normal forms for each type as given in Table 19. The degenerate conics and its multiplicities are then easily derived from it.

Type I) The pencil has exactly 4 base points. Since the pencil is non-degenerate no three of those base points lie on a line. Therefore, the 4 base points are in general position and the case is entirely captured by Example 10.2. There the normal form has already been derived.

TYPE III) The pencil has exactly 2 base points each of multiplicity 2. By Proposition 10.8, all conics from the pencil are tangent to a common line in each of the two base points. Thus, this type is entirely captured by Example 10.3, where the normal form has already been derived.

For the remaining cases we use the same normalization for the two conics $\mathcal{Q}_1, \mathcal{Q}_2$ that span the pencil as in the proof of Proposition 10.5. Additionally, we normalize the base point X_1 which has the greatest muliplicity to have homogeneous coordinates

$$X_1 = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right].$$

Thus $x = 0$ is a solution to (4), or equivalently,

$$q_{33} = 0.$$

TYPE II) The point X_1 is a double base point and thus we have

$$q_{33} = 0, \quad q_{13} = 0, \quad q_{11} + 2q_{23} \neq 0.$$

We normalize the other two (simple) base points to have homogeneous coordinates

$$X_2 = \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right], \quad X_3 = \left[\begin{smallmatrix} -1 \\ 1 \\ 1 \end{smallmatrix}\right].$$

The conditions $X_2, X_3 \in \mathcal{Q}_2$ yield

$$\begin{aligned} q_{11} + q_{22} + 2q_{12} + 2q_{23} &= 0, \\ q_{11} + q_{22} - 2q_{12} + 2q_{23} &= 0, \end{aligned}$$

or equivalently,

$$q_{12} = 0, \qquad q_{22} = -q_{11} - 1q_{23}.$$

Thus, the conic $\mathcal{Q}_2$ is of the form

$$Q_2 = \begin{pmatrix} q_{11} & 0 & 0 \\ 0 & -q_{11} - 2q_{23} & q_{23} \\ 0 & q_{23} & 0 \end{pmatrix},$$

which already describes a pencil of conics containing $\mathcal{Q}_1$. By choosing new representatives on the line $\mathcal{Q}_1 \wedge \mathcal{Q}_2$ we can rewrite its equation as

$$\lambda(x_1^2 - x_2^2) + \mu x_2(x_2 - x_3) = 0.$$

TYPE IV) The point X_1 is a triple base point and thus we have

$$q_{33} = 0, \quad q_{13} = 0, \quad q_{11} + 2q_{23} = 0, \quad q_{12} \neq 0.$$

We normalize the second (simple) base point to have homogeneous coordinates

$$X_2 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right],$$

and thus,

$$q_{22} = 0.$$

Thus, the conic $\mathcal{Q}_2$ is of the form

$$Q_2 = \begin{pmatrix} q_{11} & q_{12} & 0 \\ q_{12} & 0 & -\frac{q_{11}}{2} \\ 0 & -\frac{q_{11}}{2} & 0 \end{pmatrix},$$

which already describes a pencil of conics containing $\mathcal{Q}_1$ and has the equation

$$\lambda(x_1^2 - x_2x_3) + \mu x_1 x_2 = 0.$$

TYPE V) The point X_1 is a quadruple base point and thus we have

$$q_{33} = 0, \quad q_{13} = 0, \quad q_{11} + 2q_{23} = 0, \quad q_{12} = 0, \quad q_{22} \neq 0.$$

Thus, the conic $\mathcal{Q}_2$ is of the form

$$Q_2 = \begin{pmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & -\frac{q_{11}}{2} \\ 0 & -\frac{q_{11}}{2} & 0 \end{pmatrix},$$

which already describes a pencil of conics containing $\mathcal{Q}_1$ and has the equation

$$\lambda(x_1^2 - x_2x_3) + \mu x_2^2 = 0.$$

□

A pencil of conics is called *diagonalizable* if there exist homogenous coordinates in which two of its conics (and therefore all) are represented by diagonal matrices. Note that the two normal forms for type I and III in Table 19 are diagonal, while all the others are not. Indeed, diagonalizability characterizes the pencils of these two types.

Proposition 10.10. *A non-degenerate pencil of conics in $\mathbb{C}\mathrm{P}^2$ is diagonalizable if and only if it is of type I or III.*

Proof. Exercises. □

10.1.2 Classification of pencils of conics in $\mathbb{R}\mathrm{P}^2$

Under the embedding $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{C}\mathrm{P}^2$ and the complexification of all conics in a pencil the number of base points (counting multiplicities) of a real pencil is still 4 as asserted by Proposition 10.7. Yet some base points may be imaginary, which always come in complex conjugate pairs. Thus, some of the complex cases split into multiple real cases:

(Ia) four simple real base points $(1, 1, 1, 1)$

(Ib) two simple real base points and a pair of simple imaginary base points $(1, 1, (1, \bar{1}))$

(Ic) two pairs of simple imaginary base points $((1, \bar{1}), (1, \bar{1}))$

(IIa) one double and two simple real base points $(2, 1, 1)$

(IIb) one double real base point and a pair of simple imaginary base points $(2, (1, \bar{1}))$

(IIIa) two double real base points $(2, 2)$

(IIIb) a pair of double imaginary base points $(2, \bar{2})$

(IV) one triple and one simple real base point $(3, 1)$

(V) one quadruple real base point (4)

Theorem 10.11. *Two non-degenerate pencils of conics in $\mathbb{RP}^2$ are projectively equivalent if and only if they are of the same (real) type.*

Furthermore, the degenerate conics, their multiplicities, and a normal form for each type are as stated in Table 20.

Type	base points	# real	Deg. conics	Roots	Normal forms
Ia	$1, 1, 1, 1$	4	$\times, \times, \times$	$1, 1, 1$	$\lambda(x_1^2 - x_3^2) + \mu(x_2^2 - x_3^2) = 0$
Ib	$1, 1, (1, \bar{1})$	2	$\times, \circ, \bar{\circ}$	$1, (1, \bar{1})$	$\lambda(x_1^2 + x_2^2 - x_3^2) + \mu x_2 x_3$
Ic	$(1, \bar{1}), (1, \bar{1})$	0	$\times, \bullet, \bullet$	$1, 1, 1$	$\lambda(x_1^2 + x_2^2 + x_3^2) + \mu x_1 x_3 = 0$
IIa	$2, 1, 1$	3	$2\times, \times$	$2, 1$	$\lambda(x_1^2 - x_2^2) + \mu x_2(x_2 - x_3) = 0$
IIb	$2, (1, \bar{1})$	1	$2\bullet, \times$	$2, 1$	$\lambda(x_1^2 + x_2^2) + \mu x_2 x_3 = 0$
IIIa	$2, 2$	2	$2\Vert, \times$	$2, 1$	$\lambda(x_1^2 - x_3^2) + \mu x_2^2 = 0$
IIIb	$(2, \bar{2})$	0	$2\Vert, \bullet$	$2, 1$	$\lambda(x_1^2 + x_2^2) + \mu x_3^2 = 0$
IV	$3, 1$	2	$3\times$	3	$\lambda(x_1^2 - x_2 x_3) + \mu x_1 x_2 = 0$
V	4	1	$3\Vert$	3	$\lambda(x_1^2 - x_2 x_3) + \mu x_2^2 = 0$

Figure 20. The classification of real pencils of conics. There exist four different types of degenerate conics. ($\times$) Two real intersecting lines. ($\circ$) Two non-intersecting complex lines. ($\bullet$) Two complex conjugate lines which intersect in a real point. ($\Vert$) A real double line.

Proof. We derive normal forms for the missing real cases.

Type Ib) The pencil has 4 base points two of which are imaginary. We choose homogeneous coordinates such that the base points are given by (Exercises)

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \bar{X}_3 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

Then the representative matrix of a conic through these four points must satisfy

$$\begin{aligned} q_{11} + q_{33} + 2q_{13} &= 0, \\ q_{11} + q_{33} - 2q_{13} &= 0, \\ q_{11} - q_{22} + 2iq_{12} &= 0, \\ q_{22} - q_{22} - 2iq_{12} &= 0. \end{aligned}$$

or equivalently,

$$q_{12} = q_{13} = 0, \quad q_{11} = q_{22} = -q_{33}.$$

Thus, the equation of the pencil is given by

$$\lambda(x_1^2 + x_2^2 - x_3^2) + \mu x_2 x_3.$$

Note that in affine coordinates ($x_3 = 1$) the points X_3 and $\bar{X}_3$ are the two circle points at infinity. Thus, all conics in this pencil are circles. In particular all circles through the two points X_1 and X_2.

Type Ic) The pencil has 4 base points all of which are imaginary. We choose homogeneous coordinates such that the base points are given by

$$X_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \bar{X}_1 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}.$$

Then the representative matrix of a conic through these four points must satisfy

$$\begin{aligned} q_{11} - q_{22} + 2iq_{12} &= 0, \\ q_{11} - q_{22} - 2iq_{12} &= 0, \\ q_{22} - q_{33} + 2iq_{23} &= 0, \\ q_{22} - q_{33} - 2iq_{23} &= 0. \end{aligned}$$

or equivalently,

$$q_{12} = q_{23} = 0, \quad q_{11} = q_{22} = q_{33}.$$

Thus, the equation of the pencil is given by

$$\lambda(x_1^2 + x_2^2 + x_3^2) + \mu x_1 x_3 = 0.$$

Type IIb) The pencil has 3 base points, one double real base point, and a pair of simple imaginary base points. We normalize the double real base point to have homogeneous coordinates

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then,

$$q_{33} = q_{13} = 0, \qquad q_{11} + 2q_{23} \neq 0.$$

We further normalize the pair of simple imaginary base points to

$$X_2 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}.$$

Thus,

$$q_{12} = 0, \qquad q_{11} = q_{22},$$

and the equation of the pencil is given by

$$\lambda(x_1^2 + x_2^2) + \mu x_2 x_3 = 0.$$

This pencil consists of all circles tangent to the line $y = 0$ in the point $(0, 0)$.

Type IIIb) This pencil has one pair of double imaginary base points. We normalize those to be

$$X_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad \bar{X}_1 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix},$$

and the two complex conjugate imaginary tangent lines ℓ_1, $\bar{\ell}_1$ to contain the point $[0, 0, 1]$. Then,

$$\ell_1 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}^\star, \quad \bar{\ell}_1 = \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}^\star.$$

Together this leads to the equation

$$\lambda(x_1^2 + x_2^2) + \mu x_3^2 = 0.$$

In affine coordinates, this is the family of concentric circles with cnenter $(0, 0)$. $\square$

Proposition 10.12. *A non-degenerate pencil of conics in* $\mathbb{RP}^2$ *is diagonalizable if and only if it is of type Ia, Ic, IIIa, or IIIb.*

Proof. Exercises. $\square$

Remark 10.3. Diagonalizability corresponds to certain symmetries of the pencil. A pencil is diagonal if and only if every conic in the pencil is invariant under the three involutions given by $[\mathrm{diag}(\pm 1, \pm 1, \pm 1)]$, where the invariance with respect to any two implies the invariance with respect to the third. In affine coordinates ($x_3 = 1$) this corresponds to the symmetry with respect to reflection in the coordinate axes.

11 Dual pencils of quadrics

Let $(\mathcal{Q}_\lambda)_{\lambda \in \mathbb{FP}^1}$ be a pencil of quadrics in $\mathbb{FP}^n$. Then the family of dual quadrics $(\mathcal{Q}_\lambda^\star)_{\lambda \in \mathbb{FP}^1}$ is called *a dual pencil of quadrics* in $(\mathbb{FP}^n)^*$.

While a pencil of quadrics is a linear family, a dual pencil is generically a family of degree n. Indeed, the quadrics from the primal pencil can be represented as

$$Q_\lambda = Q_1 + \lambda Q_2$$

with two symmetric matrices Q_1, Q_2. Then the (non-degenerate) quadrics from the dual pencil can be represented by the inverse matrices

$$Q_\lambda^{-1} = (Q_1 + \lambda Q_2)^{-1}.$$

The inverse matrices contain the first minors of Q_λ, at least one of which is of degree n.

From this we can also conclude, that generically there are n quadrics in a pencil of quadrics in $\mathbb{CP}^n$ that touch a given hyperplane. If $n \geqslant 2$ the touching points of common hyperplane are conjugate points with respect to any quadric from the pencil by Proposition 10.4.

11.1 Dual pencils of conics

Dual pencils of conics in $\mathbb{FP}^2$ can be classified by the type of the corresponding primal pencil. Thus, we assign the same symbols. Yet their geometric description is different and obtained by dualization.

In particular, in $\mathbb{CP}^2$ the degenerate conics of a dual pencil are either two points or one (double) point. The first three types are geometrically described in the following way:

TYPE I) Consists of all conics tangent to four given lines. The degenerate dual conics are given by the three pairs of opposite points of the corresponding complete quadrilateral.

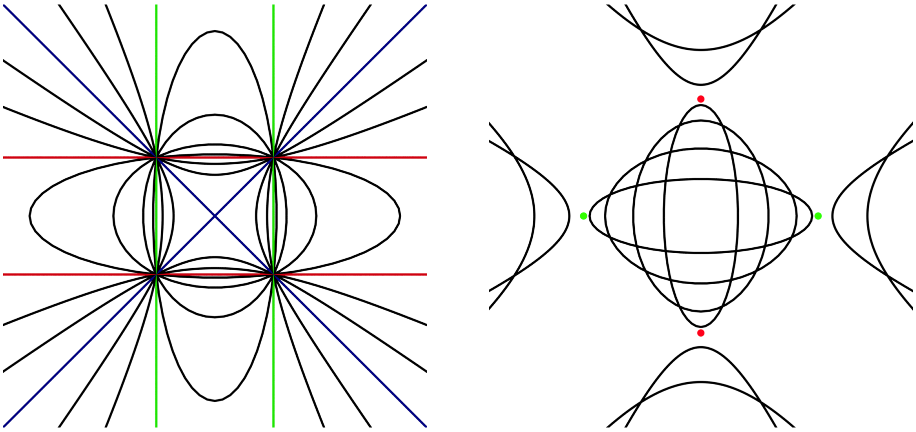


Figure 21. *Left:* Primal pencil of conics of type I. *Right:* Dual pencil of conics of type I.

TYPE II) Consists of all conics tangent to three given lines with one prescribed point of tangency on one of those lines. The degenerate dual conics are two pairs of points.

TYPE III) Consists of all conics tangent to two lines with one prescribed point of tangency on each line. Thus it geometrically describes the same kind of family as a primal pencil of type III. Its degenerate dual conics are one pair of points and a double point.

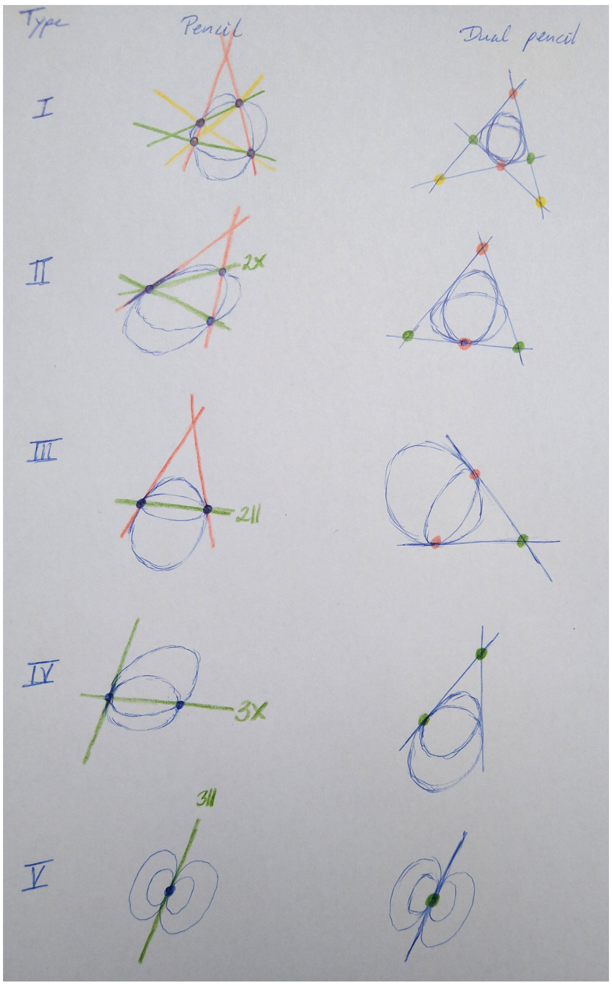


Figure 22. Primal pencils of types I-V and the corresponding dual pencils.

In $\mathbb{RP}^2$ we obtain additional cases, characterized by containing different kind of imaginary degenerate dual conics.

11.2 Confocal conics

Proposition 11.1. *Let $a > b$. The family of confocal conics (confocal ellipses and hyperbolas)*

$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} = 1\right\}$$

for $\lambda \in \mathbb{R}$ is a dual pencil of type Ic.

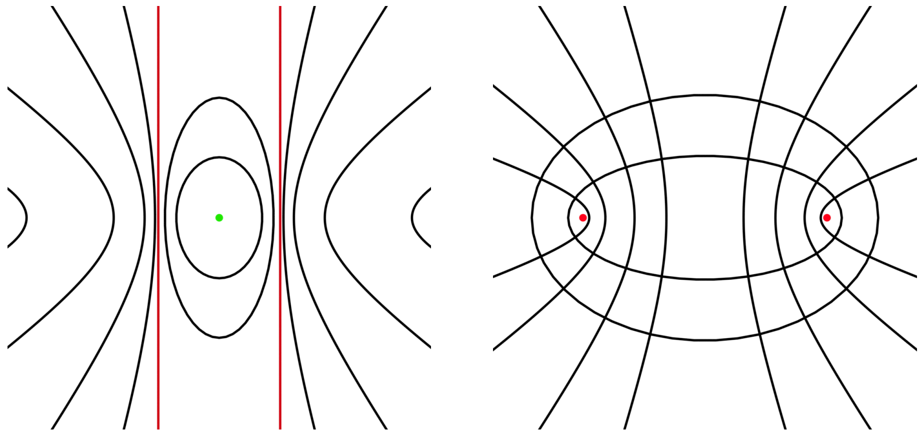


Figure 23. *Left:* Primal pencil of conics of type Ic. *Right:* Dual pencil of conics of type Ic.

Proof. Embedding $\mathbb{R}^2 \subset \mathbb{RP}^2$ and homogenizing the confocal conics equation one obtains

$$\frac{x_1^2}{a+\lambda} + \frac{x_2^2}{b+\lambda} - x_3^2 = 0.$$

where $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$. Dualizing yields

$$(a+\lambda)x_1^2 + (b+\lambda)x_2^2 - x_3^2 = 0,$$

which indeed describes a pencil of conics. To determine the type we find its degenerate conics. The roots of

$$\det \operatorname{diag}(a+\lambda, b+\lambda, -1) = -(a+\lambda)(b+\lambda) = 0$$

are given by $\lambda = -a, -b, \infty$. Since there are three distinct roots the pencil is of type I.

$\lambda = -a$: The equation of the degenerate conic is given by

$$(a-b)x_2^2 + x_3^2 = 0.$$

Since $a - b > 0$ these are two complex conjugate imaginary lines $x_3 = \pm i\sqrt{a-b}\,x_2$, or dually two complex conjugate imaginary points, which lie on the y-axis.

$\lambda = -b$: The equation of the degenerate conics is given by

$$(a-b)x_1^2 - x_3^2 = 0.$$

These are two real lines $x_3 = \pm\sqrt{a-b}\,x_1$, or dually the two foci of the confocal familly, which both lie on the x-axis.

$\lambda = \infty$: The equation of the degenerate conics is given by

$$x_1^2 + x_2^2 = 0.$$

These again are two complex conjugate imaginary lines $x_1 = \pm i x_2$, which dually correspond to the two circle points of similarity geometry.

The degenerate conics are one pair of real lines and two pairs of complex conjugate imaginary lines. Thus, the pencil, and the dual pencil of confocal conics, are of type Ic. □

One similarly proves:

Proposition 11.2.

(i) The family of confocal parabolas

$$\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2 = \lambda^2 + 2\lambda y\right\}$$

for $\lambda \in \mathbb{R}$ is a dual pencil of type IIb

(ii) The family of concentric circles is a dual pencil of type IIIb.

Proof. Exercises. □

Remark 11.1. Note that the family of concentric circles is also a primal pencil of type IIIb.

The families of confocal ellipses/hyperbolas, confocal parabolas and concentric circles all contain the two circle points of similarity geometry. On the other hand, by comparing with the classification in Table 20 we find that any dual pencil containing the two circle points must be of type Ic, IIb, or IIIb. Thus, we obtain the following corollary.

Corollary 11.3. *A dual pencil of conics is a family of confocal conics (including confocal parabolas and concentric circles) if and only if one its dual degenerate conics is the pair of circle points of similarity geometry.*

Proposition 11.4. *Any two intersecting conics from a family of confocal conics intersect orthogonally.*

Proof. The two tangents at a point of intersection dually correspond to the two touching points of a common tangent. By Proposition 10.4, these two points are conjugate with respect to every conic in the pencil, in particular, to the degenerate conic corresponding to the two circle points of similarity geometry. Thus, by Proposition 9.8, the two tangent lines intersect orthogonally. □

11.3 Graves-Chasles theorem

We are now prepared to proof the incidence theorem of Graves-Chasles and furthermore identify it as a Euclidean version of a general projective theorem on pencils of conics.

Theorem 11.5. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^2$ be a non-degenerate conic and V a complete quadrilateral whose sides touch the conic $\mathcal{Q}$. Then V circumscribes a circle if and only if there exists a conic confocal with $\mathcal{Q}$ through any pair of opposite vertices of V.*

Furthermore, in this case the tangent lines of the three conics confocal with $\mathcal{Q}$ in the three pairs of opposite vertices of V all intersect in one point, which is the center of the circle.

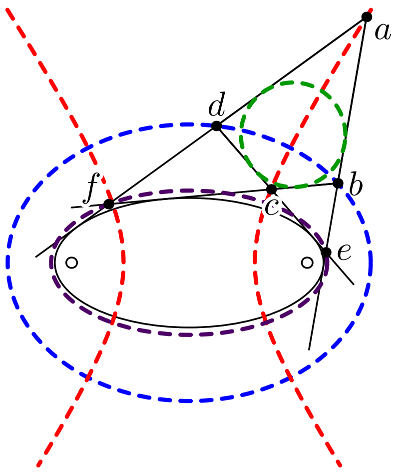


Figure 24. Graves–Chasles theorem.

Two points lying on a conic can be characterized in terms of dual pencils of conics in the following way:

Lemma 11.6. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^2$ be a non-degenerate conic and $\mathcal{D} = \{D_1, D_2\} \subset \mathbb{R}\mathrm{P}^2$ be a pair of points. Then the following statements are equivalent*

(i) $\mathcal{D} \subset \mathcal{Q}$.

(ii) The dual pencil spanned by $\mathcal{Q}$ and $\mathcal{D}$ is of type IIIa.

(iii) The dual pencil spanned by $\mathcal{Q}$ and $\mathcal{D}$ contains a double point as degenerate dual conic.

In this case, the double point on the pencil is the intersection of the two tangent lines of $\mathcal{Q}$ in $\mathcal{D}$.

Proof. Exercise. □

Taking the two points to be the pair of circle points we obtain the following characterization of circles in terms of dual pencils of conics.

Lemma 11.7. *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^2$ be a non-degenerate conic and $\mathcal{C} = \{Z, \bar{Z}\}$ be the pair of circle points of similarity geometry. Then the following statements are equivalent*

(i) $\mathcal{C} \subset \mathcal{Q}$, i.e. $\mathcal{Q}$ is a circle.

(ii) The dual pencil spanned by $\mathcal{Q}$ and $\mathcal{C}$ is of type IIIb.

(iii) The dual pencil spanned by $\mathcal{Q}$ and $\mathcal{C}$ contains a double point as degenerate dual conic.

In this case, the double point in the pencil is the center of the circle.

Proof. Exercise. □

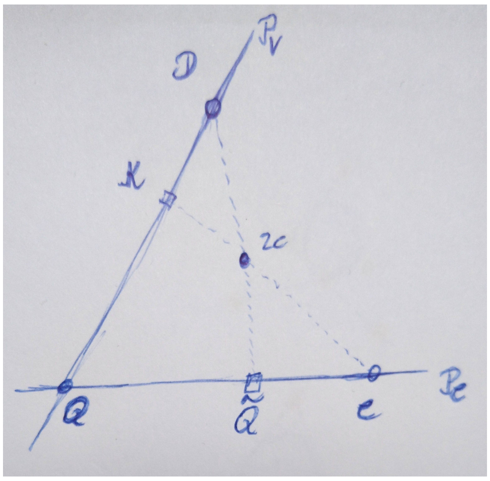


Figure 25. Proof of Graves-Chasles theorem

Proof of Theorem 11.5. Let $\mathcal{P}_\mathcal{Q}$ be the dual pencil of conics confocal with $\mathcal{Q}$, i.e. spanned by $\mathcal{Q}$ and $\mathcal{C}$.

Let $\mathcal{P}_V$ be the dual pencil of conics tangent to the four lines of the complete quadrilateral V. This is a dual pencil of type I. Its degenerate dual conics are the three pairs of opposite vertices of V. Let $\mathcal{D}$ be such a pair.

By assumption the pencil $\mathcal{P}_V$ contains the conic $\mathcal{Q}$. Thus, by duality, $\mathcal{P}_\mathcal{Q}$ and $\mathcal{P}_V$ constitute two intersecting lines in the space conics on $(\mathbb{RP}^2)^*$.

"⇒" Let $\mathcal{K}$ be a circle touching the four lines of V. Then $\mathcal{K} \in \mathcal{P}_V$ and, by Lemma 11.7, the dual pencil spanned by $\mathcal{K}$ and $\mathcal{C}$ contains a double point given by the center $2c$ of $\mathcal{K}$. Consider the dual pencil spanned by $\mathcal{D}$ and the double point $2c$. This is a dual pencil of type IIIa. It intersects $\mathcal{P}_\mathcal{Q}$ in point a $\tilde{\mathcal{Q}}$, which by Lemma 11.6 contains the two points $\mathcal{D}$.

"⇐" Let $\tilde{\mathcal{Q}}$ be a conic in $\mathcal{P}_\mathcal{Q}$ that contains the two points $\mathcal{D}$. Then, by Lemma 11.6, the dual pencil spanned by $\tilde{\mathcal{Q}}$ and $\mathcal{D}$ contains a double point $2c$. Consider the dual pencil spanned by $\mathcal{C}$ and the double point $2c$. This is a dual pencil of type IIIb. It intersects $\mathcal{P}_V$ in a point $\mathcal{K}$, which by Lemma 11.7 is a circle.

On one hand, by Lemma 11.6, the double point $2c$ is the intersection of the two tangents of $\tilde{\mathcal{Q}}$ in the two points $\mathcal{D}$, and on the other, by Lemma 11.7, it is the center of the circle $\mathcal{K}$. □

Note that the argument in both directions of the Graves-Chasles theorem is totally symmetric except for the fact the degenerate dual conic $\mathcal{D}$ on $\mathcal{P}_V$ consists of two real points, while the degenerate dual conic $\mathcal{C}$ on $\mathcal{P}_\mathcal{Q}$ consists of two complex conjugate imaginary points. Thus, by either taking the statement in the complex plane or considering both pairs of points to be real, we obtain the following projective version of the theorem. Furthermore, by dualizing this becomes a theorem on primal pencils of conics.

Theorem 11.8 (Projective Graves-Chasles)**.** *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two pencils of conics in $\mathbb{FP}^2$ which contain a common conic. Let $\mathcal{D}_1$ be a pair of lines contained as a degenerate conic in $\mathcal{P}_1$ and $\mathcal{D}_2$ be a pair of lines contained as a degenerate conic in $\mathcal{P}_2$.*

Then $\mathcal{P}_1$ contains a conic that touches the two lines $\mathcal{D}_2$ if and only if $\mathcal{P}_2$ contains a conic that touches the two lines $\mathcal{D}_1$.

Furthermore, in that case, the four touching points lie on a line.

Proof. Exercise. □

12 Incircular nets

[Böh1970] W. Böhm.
Verwandte Sätze über Kreisvierseitnetze,
Arch. Math. (Basel) **21** (1970) 326–330.

[AB2018] A.V. Akopyan, A.I. Bobenko.
Incircular nets and confocal conics,
Trans. AMS **370:4** (2018) 2825–2854.

Two families $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ of lines in the Euclidean plane are called an *incircular net* (IC-net) if for every $i, j \in \mathbb{Z}$ the four lines $\ell_i, \ell_{i+1}, m_j, m_{j+1}$ touch a common circle S_i such that the following *regularity condition* is satisfied:

(R) The line through the centers of $S_{ij}, S_{i+1,j+1}$ and the line through the centers of $S_{i+1,j}, S_{i,j+1}$ are the distinct angle bisectors of the lines ℓ_{i+1} and m_{j+1}.

Remark 12.1.

(i) The regularity condition further implies that the line connecting the two centers of S_{ij} and $S_{i+1,j}$ is the angle bisector of the lines ℓ_i and ℓ_{i+1}.

(ii) For any two diagonally neighboring circles $S_{ij}, S_{i+1,j+1}$ its two centers and the point $\ell_{i+1} \cap m_{j+1}$ lying on a line, which then must be the angle bisector of ℓ_{i+1} and m_{j+1}, is equivalent to the existence of an orientation of the two circles $S_{ij}, S_{i+1,j+1}$ and its common tangents ℓ_{i+1}, m_{j+1} in such a way that they are in oriented contact.

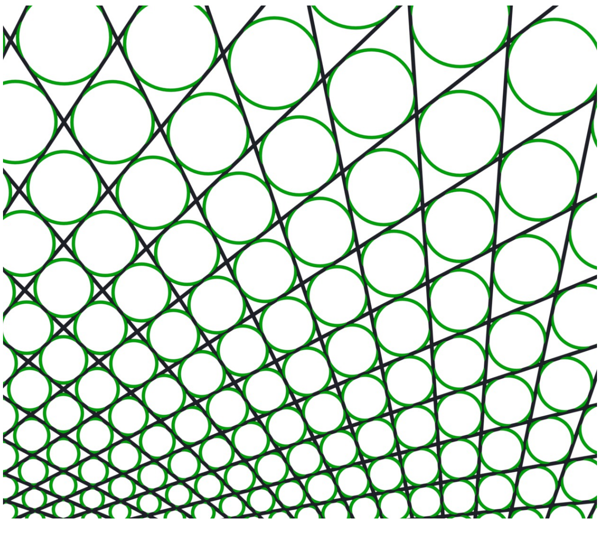


Figure 26. An example of an incircular net.

Lemma 12.1. *Let $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$ be six lines of an incircular net (with four incircles). Then the following holds:*

(i) All six lines touch a common conic $\mathcal{Q}$.

(ii) There is a conic confocal with $\mathcal{Q}$ through the three diagonal points $\ell_1 \cap m_1$, $\ell_2 \cap m_2$, $\ell_3 \cap m_3$.

(iii) The quadrilateral ℓ_1, ℓ_3, m_1, m_3 is circumscribed.

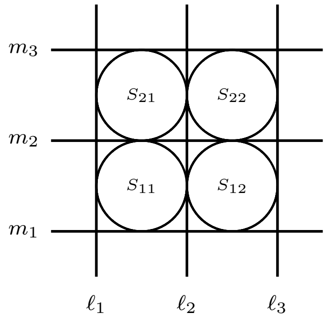


Figure 27. Small patch of an incircular net.

Proof.

(i) Let $\mathcal{Q}$ be the unique conic tangent to the five lines ℓ_1, ℓ_2, ℓ_3, m_1, m_2. Show that m_3 touches $\mathcal{Q}$ as well (Exercise).

(ii) By Theorem 11.5, there exists a conic $\mathcal{Q}_1$ confocal with $\mathcal{Q}$ through $\ell_1 \cap m_1$, $\ell_2 \cap m_2$, and a conic $\mathcal{Q}_2$ confocal with $\mathcal{Q}$ through $\ell_2 \cap m_2$, $\ell_3 \cap m_3$. Since they share the common point $\ell_2 \cap m_2$ they must coincide (due to the assumed regularity condition).

(iii) Follows from (ii) and Theorem 11.5.

□

These local properties carry over globally to incircular nets.

Proposition 12.2. *Let $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ be an incircular net.*

(i) All lines touch a common conic $\mathcal{Q}$.

(ii) The points of intersection along the diagonals lie on conics confocal with $\mathcal{Q}$.

(iii) All quadrilaterals $\ell_i, \ell_{i+k}, m_j, m_{j+k}$ are circumscribed.

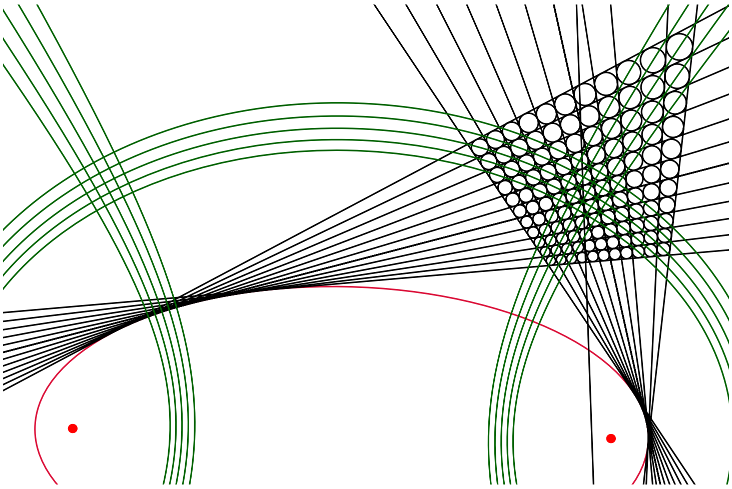


Figure 28. Properties of incircular nets.

Lemma 12.3. *Let $\ell_1, \ell_2, \ell_3, m_1, m_2, m_3$ be six lines touching a common conic. If the two incircles S_{11} and S_{22} exist such that the line trough their centers is a bisector of the two lines ℓ_1 and m_1, then there exists a circle $\tilde{S}_{11}$ touching ℓ_1, ℓ_3, m_1, m_3.*

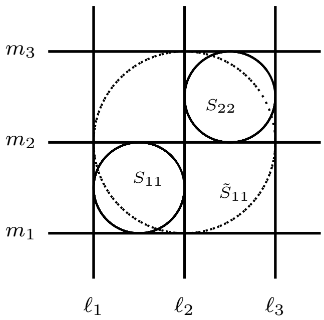


Figure 29. 3rd circle incidence theorem for six lines touching a conic.

Proof. Similar to Lemma 12.1 (ii) and (iii). □

Remark 12.2. By permutation of the lines one obtains that the existence of S_{11} and $\tilde{S}_{11}$ also implies the existence of S_{22}.

12.1 Elementary construction

We answer the question of whether incircular nets exist and how many degrees of freedom they have by the following elementary construction:

- Choose the lines $\ell_1, \ell_2, \ell_3, m_1$ arbitrarily.
- Generically there exist four circles touching the three lines ℓ_1, ℓ_2, m_1. Choose one of those as S_{11}.
- The choice for the circle S_{12} from the four circles touching the three lines ℓ_2, ℓ_3, m_1 is uniquely determined by the regularity condition.

- The line m_2 is uniquely determined as the reflection of m_1 in the line through the centers of S_{11} and S_{12} (see Remark 12.1 (i)).
- The circles S_{13}, S_{21}, S_{22} are uniquely determined.
- The line m_3 is uniquely determined.
- The circles S_{23}, S_{31}, S_{32} are uniquely determined.
- The lines ℓ_4 and m_4 are uniquely determined.

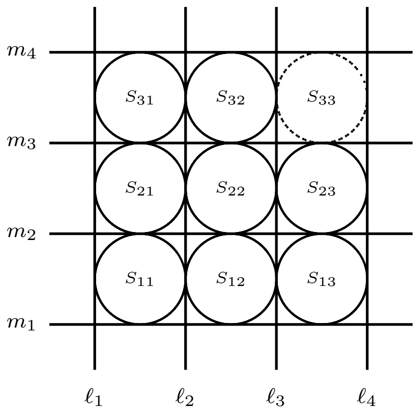


Figure 30. 9th circle incidence theorem.

Theorem 12.4 (9th circle incidence theorem)**.** *The lines ℓ_3, ℓ_4, m_3, m_4 touch a common circle S_{33}.*

Proof. Let $\mathcal{Q}$ be the unique conic touching the five lines $\ell_1, \ell_2, \ell_3, m_1, m_2$. Then by multiple application of Lemma 12.1 (i) all eight lines must touch $\mathcal{Q}$. By Lemma 12.3 the existence of the incircles S_{32} and S_{23} imply the existence of a circle $\tilde{S}_{22}$ touching ℓ_2, ℓ_4, m_2, m_4. Further application of Lemma 12.3 to the circles S_{22} and $\tilde{S}_{22}$ imply the existence of S_{33}. □

Theorem 12.4 ensures that the construction can be carried on indefinitely and always yields an incircular net. After the choice of the first 4 lines the incircular net is (almost) uniquely determined. Thus, there are 8 degrees of freedom.

13 Associated points

The span of two conics $\mathcal{Q}_1, \mathcal{Q}_2$ in $\mathbb{C}\mathrm{P}^2$ is a pencil of conics. Its base points are the points common to all conics from the pencil and given by the intersection of any two of them. In $\mathbb{C}\mathrm{P}^2$ the intersection of two generic conics consists of four points. Vice versa, the family of all conics through 4 points in general position is a pencil of conics.

The span of three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ in $\mathbb{C}\mathrm{P}^3$ (not belonging to a common pencil) is a linear system of quadrics of dimension 2. Its base points are the points common to all conics from the pencil and given by the intersection of any three of them (not belonging to a common pencil). In $\mathbb{C}\mathrm{P}^3$ the intersection of three generic quadrics consists of eight points. On the other hand, the family of all conics through 7 points in general position already constitutes a linear system of quadrics of dimension 2.

Theorem 13.1 (associated points)**.** *Given eight distinct points which are the set of intersections of three quadrics in* $\mathbb{F}\mathrm{P}^3$*, all quadrics through any subset of seven of the points must pass through the eighth point. Such sets of points are called* ***associated points****.*

Proof (follows [BS2008]). Let $A_1, A_2, \ldots, A_8$ be the set of intersections of three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$. Note that no three of the eight points A_k can be collinear, since otherwise the set of intersection of the three quadrics would contain a whole line and not just eight points (see Lemma 8.4). For similar reasons no five of the eight points A_k can be coplanar. Indeed, five coplanar points no three of which are collinear determine a unique conic. The intersection of the three quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ would contain this conic and not just eight points.

Choose any subset of seven points $A_1, A_2, \ldots, A_7$. We show that any quadric $\mathcal{Q}$ through these seven points must belong to the family

$$\mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \mathcal{Q}_3.$$

As a consequence, the eighth intersection point A_8 will automatically lie on $\mathcal{Q}$. Suppose that, on the contrary, $\mathcal{Q}$ is linearly independent of $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$. Consider the family of quadrics

$$\mathcal{P} = \mathcal{Q}_1 \wedge \mathcal{Q}_2 \wedge \mathcal{Q}_3 \wedge \mathcal{Q}.$$

Due to the assumed linear independence, one could find a quadric in this family through any prescribed triple of points in $\mathbb{F}\mathrm{P}^3$. We show that this would lead to a contradiction.

First assume that no four points among $A_1, A_2, \ldots, A_7$ are coplanar. Choose three points B, C, D in the plane of A_1, A_2, A_3 so that the six points B, C, D, A_1, A_2, A_3 do not lie on a conic. Find a quadric $\mathcal{Q}'$ in the family $\mathcal{P}$ through B, C, D. This quadric must be reducible, one component being the plane of A_1, A_2, A_3 (indeed, otherwise $\mathcal{Q}'$ would cut this plane in a conic through A_1, A_2, A_3, B, C, D, which contradicts the choice of B, C, D). The other component of $\mathcal{Q}'$ must be a plane containing four points A_4, A_5, A_6, A_7, a contradiction.

The remaining case, when there are four coplanar points among $A_1, A_2, \ldots, A_7$, is dealt with analogously. Let A_1, A_2, A_3 and A_4 be coplanar. Denote the plane through these four points by Π. Take two points B, C in the plane Π so that the six points A_1, A_2, A_3, A_4, B, C do not lie on a conic, and take a point D not coplanar with A_5, A_6, A_7 (which is always possible, because the latter three points are not collinear). Then there exists a quadric $\mathcal{Q}'$ in the family $\mathcal{P}$ through B, C, D. Again, this quadric must be reducible, consisting of two planes, one of them being the plane Π. The other component of $\mathcal{Q}$ must be a plane containing A_5, A_6, A_7, D, a contradiction again (this time to the choice of D). $\square$

Theorem 13.2 (Miquel's theorem on quadrics)**.** *Let* $\mathcal{Q}$ *be a quadric in* $\mathbb{F}\mathrm{P}^3$ *of rank 3 or 4. Let* $x, x_1, x_2, x_3, x_{12}, x_{23}, x_{13}, x_{123} \in \mathcal{Q}$ *be eight points of a combinatorial cube (see Figure 31), such that five of its faces are coplanar and no two planes coincide. Then its sixth face is coplanar as well.*

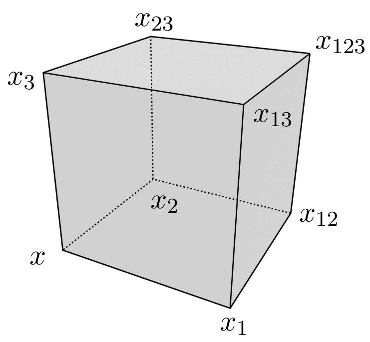


Figure 31. Combinatorial cube.

Proof. For $\{i, j, k\} = \{1, 2, 3\}$, $i < j$ define the six planes

$$\Pi^{ij} = x \wedge x_i \wedge x_j, \quad \Pi_k^{ij} = x_k \wedge x_{ik} \wedge x_{jk}.$$

The five plane $\Pi^{12}, \Pi^{23}, \Pi^{13}, \Pi_1^{23}, \Pi_2^{13}$ each contain one more of the eight point, and we need to show $x_{123} \in \Pi_3^{12}$.

Consider the two degenerate quadrics

$$\mathcal{Q}_1 = \Pi^{23} \cup \Pi_1^{23}, \quad \mathcal{Q}_2 = \Pi^{13} \cup \Pi_2^{13}.$$

Then, since $\mathcal{Q}$ does not contain any planes, the eight points are exactly the intersection

$$\mathcal{Q} \cap \mathcal{Q}_1 \cap \mathcal{Q}_2.$$

The degenerate quadric

$$\mathcal{Q}_3 = \Pi^{12} \cup \Pi_3^{12}$$

contains seven of the eight points, and therefore, by Theorem 13.1, also contains the eighth point x_{123}. This point must be contained in the plane Π_3^{12} since otherwise $x_{12} = x_{123}$. □

This implies the classical version of Miquel's six circle theorem.

Corollary 13.3. *Given four points x, x_1, x_2, x_{12} on a circle, and four circles passing through each adjacent pair of points, the alternate intersections of these four circles at $x_3, x_{13}, x_{23}, x_{123}$ then lie on a common circle.*

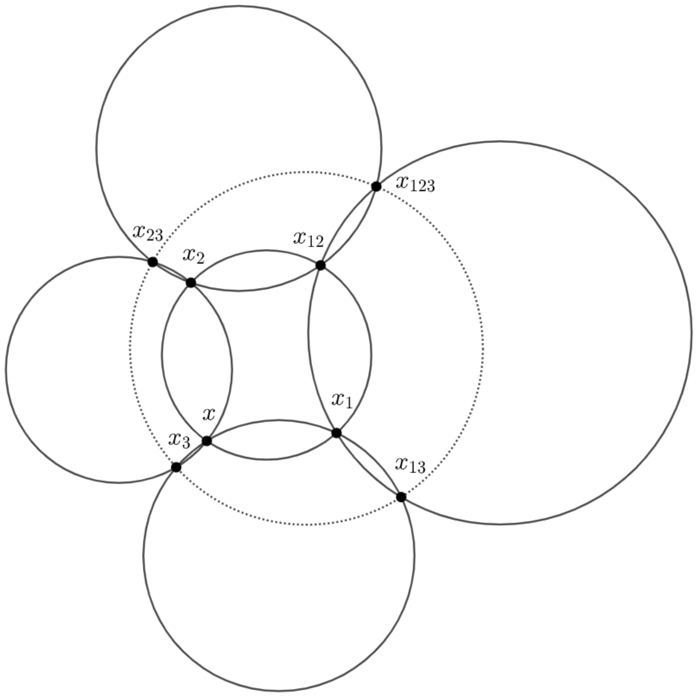


Figure 32. Miquel's six circle theorem.

Proof. After stereographic projection to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ apply Theorem 13.2. $\square$

Part III
Laguerre Geometry

> Die vorliegende Arbeit behandelt die Geometrie einer algebraischen, siebengliedrigen Gruppe orientierter Berührungstransformationen in der Euklidischen Ebene. Die Gruppe ist dadurch definiert, daß ihre Abbildungen einerseits die orientierten geraden Linien, die man nach E. Study auch kurz *Speere* nennt, anderseits die orientierten Kreise untereinander vertauschen. (Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene (1910) – Wilhelm Blaschke)

Classically Laguerre geometry is the geometry of oriented lines and oriented circles in the Euclidean plane, and their oriented contact.

14 Models of Laguerre geometry

14.1 The Blaschke cylinder

A line in the Euclidean plane is given by

$$\left\{x \in \mathbb{R}^2 \;\middle|\; \nu \cdot x = d\right\} \subset \mathbb{R}^2$$

with some unit normal vector $\nu \in \mathbb{S}^1$ and signed distance to the origin $d \in \mathbb{R}$. The vector ν induces an orientation on the line by assigning it as a normal vector to every point on the line. Equivalently we can distinguish one of the two regions the line separates the Euclidean plane into, say

$$\left\{x \in \mathbb{R}^2 \;\middle|\; \nu \cdot x < d\right\} \subset \mathbb{R}^2,$$

and assign a direction on the line, such that this region always lies to the left of the line. Then the normal vectors always point into the other region.

The two points (ν, d) and $(-\nu, -d)$ determine the same line, but with opposite orientation. Thus, *oriented lines* in the Euclidean plane are in one-to-one correspondence with points (ν, d) on the *Blaschke cylinder*

$$\mathcal{Z} = \left\{(\nu, d) \in \mathbb{R}^3 \;\middle|\; |\nu| = 1\right\} = \mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3.$$

A circle in the Euclidean plane is given by

$$\left\{x \in \mathbb{R}^2 \;\middle|\; |x - c|^2 = r^2\right\} \subset \mathbb{R}^2$$

with some center $c \in \mathbb{R}^2$ and signed radius $r \in \mathbb{R}$. The sign of radius induces an orientation on the circle by assigning normal vectors that point outside the circle if $r > 0$ and inside if $r < 0$. Again this is equivalent to assigning a direction on the circle: left if $r > 0$ and right if $r < 0$.

Then the two tuples (c, r) and $(c, -r)$ describe the same circle, but with opposite orientation, where the special case of $r = 0$ is called a *null-circle* and is not oriented.

An oriented line and an oriented circle are said to be in *oriented contact* if the line is tangent to the circle and their direction, or equivalently their normal vectors, at the point of contact match.

Lemma 14.1. *An oriented line $(\nu, d) \in \mathcal{Z}$ and an oriented circle $(c, r) \in \mathbb{R}^3$ are in oriented contact if and only if*

$$c \cdot \nu + r = d. \tag{5}$$

Proof. Exercise. $\square$

Note that equation (5) is linear in (ν, d) and thus describes a plane.

Proposition 14.2.

(i) In the Blaschke cylinder model, the lines in oriented contact with the circle $(c, r) \in \mathbb{R}^3$ correspond to the planar section of the Blaschke cylinder

$$\{(\nu, r) \in \mathcal{Z} \mid c \cdot \nu + r - d = 0.\}$$

Vice versa, a planar section of the Blaschke cylinder with a plane non-parallel to the axis describes all oriented lines in oriented contact with a fixed oriented circle.

The planar section is a null-circle if and only if the plane contains the origin.

(ii) A planar section of the Blaschke cylinder with a plane parallel to the axis is either one generator and describes a family of parallel oriented lines or two generators and describes two families of parallel oriented lines.

Remark 14.1. For oriented lines *parallel* here means "parallel with matching orientation".

Proof.

(i) Follows from Lemma 14.1.

(ii) A generator of the Blaschke cylinder is given by $\big((\nu, d)\big)_{d \in \mathbb{R}}$ for some fixed $\nu \in \mathbb{S}^1$.

$\square$

Thus, in the *Blaschke cylinder model* of Laguerre geometry, oriented lines correspond to points on the Blaschke cylinder and oriented circles correspond to planes.

Corollary 14.3.

(i) Three oriented lines are in oriented contact with a unique oriented circle.

(ii) Three unoriented lines are in contact with four unoriented circles.

(iii) Four oriented lines are in oriented contact with an oriented circle if and only if their corresponding points on the Blaschke cylinder are coplanar.

14.2 The cyclographic model

In the Blaschke cylinder model oriented lines are the primary objects, while oriented circles are described as planes in the same space. Taking oriented circles as the primary objects gives rise to the *cyclographic model.*

An oriented circle with center $c \in \mathbb{R}^2$ and signed radius $r \in \mathbb{R}$ corresponds to a tuple $(c, r) \in \mathbb{R}^3$. We embed the Euclidean plane into the same space by identifying it with the $z = 0$ plane,

$$\mathbf{E} := \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; z = 0\right\} \cong \mathbb{R}^2,$$

which is the plane of null-circles and called the *base plane.*

For a point $(c, r) \in \mathbb{R}^3$ the cone

$$\mathcal{C}_{c,r} = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; (x - c_1)^2 + (y - c_2)^2 - (z - r)^2 = 0\right\}.$$

is called its *isotropic cone.* The intersection with the base plane $\mathcal{C}_{c,r} \cap \mathbf{E}$ gives the circle represented by the point (c, r).

A tangent plane of an isotropic cone is called an *isotropic plane.* It intersects the base plane in a $\frac{\pi}{4}$ angle. Note that equation (5) viewed as an equation in (c, r) describes an isotropic plane.

Proposition 14.4. *In the cyclographic model, the oriented circles in oriented contact to the oriented line $(\nu, d) \in \mathcal{Z}$ correspond to the points on the isotropic plane*

$$\mathcal{I}_{\nu,d} := \left\{(c, r) \in \mathbb{R}^3 \;\middle|\; \nu \cdot c + r - d = 0\right\}.$$

Vice versa, an isotropic plane describes all oriented circles in oriented contact with a fixed oriented line.

Proof. Follows from Lemma 14.1. □

The intersection of an isotropic plane with the base plane $\mathcal{I}_{\nu,d} \cap \mathbf{E}$ gives the line represented by the point (ν, d).

Thus, in the cyclographic model, points $(c, r) \in \mathbb{R}^3$ correspond to oriented circles and isotropic planes $\mathcal{I}_{\nu,d}$ with $(\nu, d) \in \mathcal{Z}$ correspond to oriented lines, while their oriented contact is described by the incidence

$$(c, r) \in \mathcal{I}_{\nu,d}.$$

14.3 The projective models and their duality

In the Blaschke cylinder model oriented lines correspond to points and oriented circles correspond to planes, while in the cyclographic model oriented circles correspond to points and oriented lines correspond to planes. In both models the oriented contact is given by the incidence of points and planes. Projectivizing both models by embedding their ambient space $\mathbb{R}^3$ as the affine part into $\mathbb{RP}^3$ reveals that they are related by duality.

Let $\langle \cdot, \cdot \rangle$ be the standard degenerate symmetric bilinear form of signature $(+ + - 0)$, i.e.,

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

for $x, y \in \mathbb{R}^4$, and

$$\mathcal{Z} := \left\{[x] \in \mathbb{RP}^3 \;\middle|\; \langle x, x \rangle = 0\right\} \subset \mathbb{RP}^3$$

the corresponding quadric in $\mathbb{RP}^3$, which we call the (projectivization of the) *Blaschke cylinder*. Projectively this is a cone with apex

$$q = [0, 0, 0, 1].$$

The Blaschke cylinder as described in Section 14.1 is recovered by introducing affine coordinates $x_3 = 1$. The only point missing in the affine picture is the apex of $\mathcal{Z}$, which may be indentified with the (non-orientable) line at infinity.

An oriented line in the Euclidean plane with unit normal vector $\nu \in \mathbb{S}^1$ and signed distance $d \in \mathbb{R}$ corresponds to the point

$$[\nu, 1, d] \in \mathcal{Z} \subset \mathbb{RP}^3,$$

Orientation reversion is given by the involution

$$\sigma : \mathbb{RP}^3 \to \mathbb{RP}^3, \quad [x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, -x_3, x_4].$$

It preserves $\mathcal{Z}$ and fixes the point

$$p = [0, 0, 1, 0]$$

and its polar plane.

An oriented circle in the Euclidean plane with center $c \in \mathbb{R}^2$ and signed radius $r \in \mathbb{R}$ corresponds to the plane

$$[c, r, -1]^\star \subset \mathbb{RP}^3.$$

It is a null-circle if and only if it contains the point p.

Dually the circle is represented by the point

$$[c, r, -1] \in (\mathbb{RP}^3)^*.$$

We identify $(\mathbb{RP}^3)^*$ with the cycligraphic model upon introducing affine coordinates $x_4 = 1$.

Orientation reversion acts on the dual space as

$$\sigma^* : (\mathbb{RP}^3)^* \to (\mathbb{RP}^3)^*, \quad [x_1, x_2, x_3, x_4] \mapsto [x_1, x_2, -x_3, x_4].$$

In particular it preserves the plane

$$\mathbf{E} = p^\star = \left\{ [x_1, x_2, x_3, x_4] \in (\mathbb{RP}^3)^* \;\middle|\; x_3 = 0 \right\},$$

which we identify with the base plane.

The Blaschke cylinder $\mathcal{Z}$ is a degenerate quadric of rank 3. Its dual quadric is given by

$$\mathcal{Z}^\star = \left\{ [x_1, x_2, x_3, x_4] \in (\mathbb{RP}^3)^* \;\middle|\; x_1^2 + x_2^2 - x_3^2 = 0,\ x_4 = 0 \right\},$$

which is a conic of signature $(++-)$ in the plane

$$q^\star = \left\{ [x_1, x_2, x_3, x_4] \in (\mathbb{RP}^3)^* \;\middle|\; x_4 = 0 \right\},$$

which becomes the plane at infinity in affine coordinates $x_4 = 1$.

The dual of a point on the Blaschke cylinder is a plane in $\mathbb{RP}^3$ that touches the dual quadric $\mathcal{Z}^\star$. Now the following proposition establishes the correspondence of the dual of the Blaschke cylinder model with the cyclographic model.

Proposition 14.5. *Upon introducing affine coordinates $x_4 = 1$ on the space $(\mathbb{RP}^3)^*$ containing the dual of the Blaschke cylinder $\mathcal{Z}^\star$ and the base plane $\mathbf{E}$ the following correspondence holds:*

(i) A plane in $(\mathbb{RP}^3)^*$ is an isotropic plane of the cyclographic model if and only if it touches $\mathcal{Z}^\star$.

(ii) A cone in $(\mathbb{RP}^3)^*$ is an isotropic cone of the cyclographic model if and only if it contains $\mathcal{Z}^\star$.

Proof. Exercises. □

14.4 Linear families of circles

Consider a line in the Blaschke cylinder model $L \subset \mathbb{RP}^3$ not going through the apex, $q \notin L$. Then its signature can assume the following three values:

(+−) The line L intersects the Blaschke cylinder in two points, which correspond to two oriented lines ℓ_1, ℓ_2 in the Euclidean plane. The planes through L correspond to all circles in oriented contact with both lines ℓ_1 and ℓ_2.

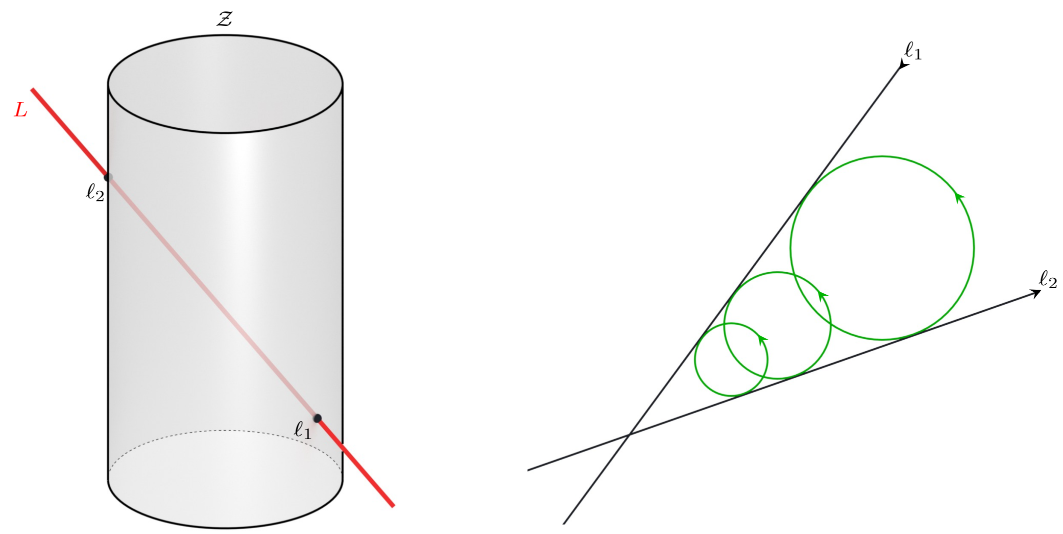


Figure 33. A line intersecting the Blaschke cylinder in two points corresponds to the circles in oriented contact with two oriented lines.

Dually, the line $L^\star$ makes an angle of less than $\frac{\pi}{4}$ with the base plane. Such a line is called *spacelike*.

(++) The line L does not intersect the Blaschke cylinder.

Dually, the line $L^\star$ makes an angle greater than $\frac{\pi}{4}$ with the base plane. Such a line is called *spacelike*, and L describes a nested family of circles.

(+0) The line L touches the Blaschke cylinder in one point ℓ. The planes through L correspond to circles that are in oriented contact with ℓ. Furthermore they all contain the point (null-circle) which corresponds to the plane through L and p. Such a family of oriented circles is called a *contact element*.

Dually, the line $L^\star$ makes an angle of $\frac{\pi}{4}$ with the base plane. Such a line is called *lightlike*.

15 Laguerre transformations

The transformation group of Laguerre geometry consists of all transformations that map oriented line to oriented lines, oriented circles to oriented circles, while preserving their oriented contact. It thus corresponds to all transformations $\mathbb{RP}^3$ that preserve the Blaschke cylinder and map planes to planes, i.e., the projective orthogonal group

$$\mathrm{PO}(2,1,1),$$

which is called the group of *Laguerre transformations*.

Remark 15.1. Mapping planes to planes means in particular that Laguerre transformations preserve parallelity of oriented lines.

Proposition 15.1.

(i) Every Laguerre transformation $[F] \in \mathrm{PO}(2,1,1)$ *in the Blaschke cylinder model is of the form*

$$[F] = \left[\begin{array}{c|c} A & 0 \\ \hline b^\intercal & c \end{array}\right]$$

with some $A \in \mathrm{O}(2,1)$, $b \in \mathbb{R}^3$, *and* $c \neq 0$.

(ii) Dually, every Laguerre transformation $[F] \in \mathrm{PO}(2,1,1)^\star$ *in the cyclographic model is of the form*

$$[F]^\star = \left[\begin{array}{c|c} A^{-\intercal} & -\frac{1}{c}A^{-\intercal}b \\ \hline 0 & \frac{1}{c} \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{b} \\ \hline 0 & \tilde{c} \end{array}\right]$$

with some $\tilde{A} \in \mathrm{O}(2,1)$, $\tilde{b} \in \mathbb{R}^3$, *and* $\tilde{c} \neq 0$.

Thus in affine coordinates $x_4 = 1$ *of the cyclographic model it takes the form*

$$x = (x_1, x_2, x_3) \mapsto \lambda \tilde{A} x + \tilde{b}$$

with some $\tilde{A} \in \mathrm{O}(2,1)$, $\tilde{b} \in \mathbb{R}^3$, *and* $\lambda \neq 0$.

Proof. Analogous to the corresponding statement in Section 9.5. □

Corollary 15.2. *Let* (ℓ_1, ℓ_2, ℓ_3) *and* $(\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3)$ *each be a triple of oriented lines in the Euclidean plane such that no two in each triple are parallel. Then there exists a unique Laguerre transformation with*

$$\ell_i \mapsto \tilde{\ell}_i, \qquad i = 1, 2, 3.$$

To better understand the group of Laguerre transformations we first establish that it contains the group of similarity transformations.

Proposition 15.3. *A Laguerre transformation* $f \in \mathrm{PO}(2,1,1)$ *is a similarity transformation if and only if it fixes the point*

$$p = [0, 0, 1, 0].$$

Proof. Dually, and in affine coordinates $x_4 = 1$ the condition on the transformation

$$x = (x_1, x_2, x_3) \mapsto \lambda A x + b$$

translates to

$$a_{31}x_1 + a_{32}x_2 + b_3 = 0$$

for all $x_1, x_2 \in \mathbb{R}$. This yields $b_3 = a_{13} = a_{23} = 0$, and since $A \in \mathrm{O}(2,1)$ this further implies $a_{31} = a_{32} = 0$. Thus, we obtain

$$A = \left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right), \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3, \qquad \lambda \neq 0$$

with $R \in \mathrm{O}(2)$, which describes a similarity transformation. □

As an example of Laguerre transformations which are not similarity transformations we introduce the following two families of transformations:

Laguerre offset Consider the family of Laguerre transformations

$$S_t = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix} \end{array}\right], \qquad t \in \mathbb{R}.$$

It acts on a line by

$$S_t \begin{bmatrix} \nu \\ 1 \\ d \end{bmatrix} = \begin{bmatrix} \nu \\ 1 \\ d+t \end{bmatrix}$$

and thus maps every line to a parallel line at distance t.

Note that S_t preserves the line $p \wedge q$ and maps p to an arbitrary point on this line (except q).

Laguerre boost Consider the family of Laguerre transformations

$$T_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh t & \sinh t & 0 \\ 0 & \sinh t & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad t \in \mathbb{R}.$$

It preserves the line $p \wedge [0,1,1,0]$ and maps p to an arbitrary point on this line inside the Blaschke cylinder.

Up to similarity transformations a Laguerre transformation is a either a Laguerre offset or a Laguerre boost.

Proposition 15.4. *Let $f \in \mathrm{PO}(2,1,1)$ be a Laguerre transformation. Then there exist two similarity transformations $\Phi, \Psi \in \mathrm{PO}(2,1,1)$ such that either*

$$f = \Phi \circ S_t = S_t \circ \Psi$$

for some $t \in \mathbb{R}$, or

$$f = \Phi \circ T_t \circ \Psi$$

for some $t \in \mathbb{R}$.

Proof. Consider the line $L = p \wedge f(p)$.

- If L contains the point q, let $t \in \mathbb{R}$ such that $S_t(p) = f(p)$. Then $\Phi = S_t^{-1} \circ f$ fixes the point p and thus is a similarity transformation.

- If L does not contain the point q it is a line of signature $(+-)$ and intersects the Blaschke cylinder $\mathcal{B}$ in two points. Let Ψ be a similarity transformation that maps $[0,1,1,0]$ to one of the intersection points. Then it maps the line $\tilde{L} = p \wedge [0,1,1,0]$ to the line $L = p \wedge q$, and thus $\tilde{p} = \Psi^{-1} \circ f(p) \in \tilde{L}$. Let $t \in \mathbb{R}$ such that $T_t(p) = \tilde{p}$. Then $\Phi = T_t^{-1} \circ \Psi^{-1} \circ f$ fixes the point p and thus is a similarity transformation.

□

15.1 Tangent distance

For two oriented circles $x = (c, r), \tilde{x} = (\tilde{c}, \tilde{r}) \in \mathbb{R}^3$ represented in the cyclographic model its squared distance in the $\|\cdot\|_{2,1}$-norm is

$$\|x - \tilde{x}\|_{2,1}^2 = \langle x - \tilde{x}, x - \tilde{x} \rangle = |c - \tilde{c}|^2 - (r - \tilde{r})^2,$$

which is called the *(squared) tangent distance* of the two circles (c, r) and $(\tilde{c}, \tilde{r})$.

Proposition 15.5. *Let $x = (c, r), \tilde{x} = (\tilde{c}, \tilde{r}) \in \mathbb{R}$ be two points in the cyclographic model. Then the following statements are equivalent.*

(i) The two corresponding oriented circle possess a common oriented tangent line.

(ii) The line $x \wedge \tilde{x}$ is spacelike.

(iii) The tangent distance $\|x - \tilde{x}\|_{2,1}^2$ is non-negative.

In this case the tangent distance is equal to the Euclidean distance between the two touching points of a common oriented tangent line of the two corresponding circles.

Proof. Exercise. □

Euclidean transformations preserve Euclidean distances, while similarity transformations preserve their ratios. Similarly, in the cyclographic model, a Laguerre transformations

$$f : x = (x_1, x_2, x_3) \mapsto \lambda A x + b$$

with some $A \in \mathrm{O}(2,1)$, $b \in \mathbb{R}^3$, $\lambda \neq 0$ preserves the ratios of tangent distances

$$\|f(x) - f(\tilde{x})\|_{2,1}^2 = \lambda^2 \|x - \tilde{x}\|_{2,1}^2 ,$$

Remark 15.2. This property characterizes Laguerre transformations in the cyclographic model. They are exactly the transformations preserving ratios of the tangent distance.

16 Conics and hypercycles

[Bla1910] W. Blaschke.
Untersuchungen über die Geometrie der Speere in der Euklidischen Ebene,
Separatdruck aus “Monatshefte f. Mathematik u. Physik”, XXI, Hamburg (1910).

[BST2018] A.I. Bobenko, W.K. Schief and J. Techter.
Checkerboard incircular nets. Laguerre geometry and parametrisation,
Geometriae Dedicata **204:1** (2020), 97-129

The oriented tangent lines of an oriented curve in the plane yield a curve on the Blaschke cylinder. Conversely, a curve on the Blaschke cylinder, corresponds to a one-parameter family of oriented lines in the plane, which envelopes a “point curve”.

As a simple example we have come across curves on the Blaschke cylinder which are given by planar sections and correspond to circles. We will not study which curves on the Blaschke cylinder correspond to conics (more precisely ellipses and hypebolas).

By means of a rotation and a translation (which constitute special Laguerre transformations) an ellipse or a hyperbola may be brought into the form

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\} \tag{6}$$

with some $a, b \neq 0$. The case $a > 0, b > 0$ corresponds to an ellipse and the case $ab < 0$ to a hyperbola.

Proposition 16.1. *The curve on the Blaschke cylinder $\mathcal{Z}$ corresponding to the tangent lines (with both orientations) of the conic C is given by the intersection of $\mathcal{Z}$ with the cone*

$$\mathcal{C} = \left\{ [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 \;\middle|\; a x_1^2 + b x_2^2 - x_4^2 = 0 \right\}. \tag{7}$$

Proof. The tangent line to C at a point $(x_0, y_0) \in C$ is given by

$$\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x x_0}{a} + \frac{y y_0}{b} = 1 \right\},$$

and its two lifts to the Blaschke cylinder by

$$\left[\frac{x_0}{a}, \frac{y_0}{b}, \pm\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}, 1 \right] = \left[\frac{x_0 d}{a}, \frac{y_0 d}{b}, \pm 1, d \right] \in \mathcal{Z}$$

where

$$d = \frac{1}{\sqrt{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}}$$

□

In particular, we found that the curve on the Blaschke cylinder corresponding to an ellipse or hyperbola is given by the intersection with a quadric.

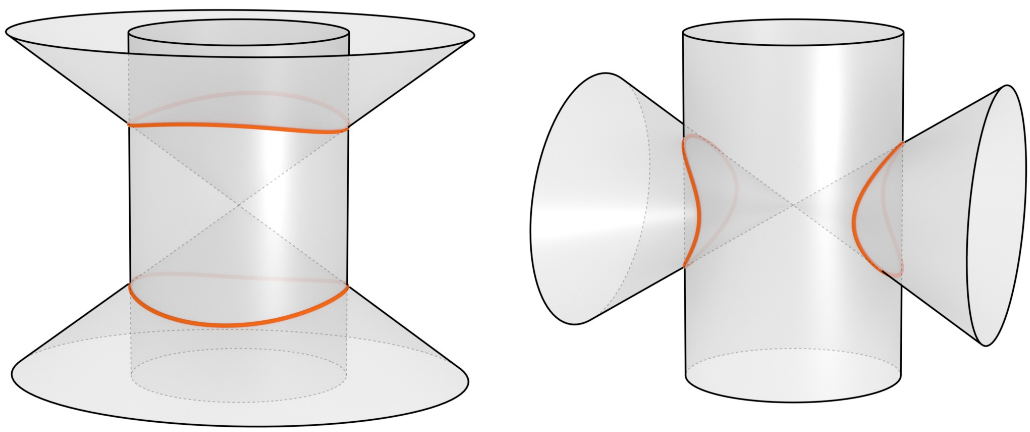


Figure 34. Hypercycle base curves corresponding to an ellipse and hyperbola respectively.

Generally, the intersection curve of the Blaschke cylinder $\mathcal{Z}$ with another quadric $\mathcal{Q}$ is called a *hypercycle base curve*. The envelope of the corresponding lines in the plane is called a *hypercycle*.

The hypercycle base curve is the base curve of the pencil of quadrics spanned by $\mathcal{Z}$ and $\mathcal{Q}$. The intersection of any quadric from this pencil with the Blaschke cylinder yields the same curve $\mathcal{Z} \cap \mathcal{Q}$.

We make the following assumptions of *non-degeneracy* and assume that the hypercycle base curve contains at least 8 points in general position such that it can uniquely be identified with the pencil $\mathcal{Z} \wedge \mathcal{Q}$.

We call a hypercycle *generic* if[4]

- the corresponding hypercycle base curve does not contain the apex $q = [0, 0, 0, 1]$ of the Blaschke cylinder,
- and the corresponding pencil of quadrics is generic in the sense that it contains 4 (distinct, possibly imaginary) degenerate quadrics.[5]

In particular, a generic hypercycle base curve is not contained in a plane or a pair of planes.

Proposition 16.2. *For a generic hypercycle the following three statements are equivalent*

(i) The hypercycle is a conic (doubly covered with opposite orientation).

(ii) The hypercycle consists of two components that coincide up to their orientation.

(iii) The hypercycle base curve is given by the intersection of the Laguerre quadric with a cone with apex p.

(iv) The pencil associated with the hypercycle is diagonal up to an isometry (Laguerre transformation fixing $\boldsymbol{p}$).

Proof. The hypercycle base curve is invariant under the involution σ reversing the orientation, if and only if p is the apex of a cone intersecting the Blaschke cylinder in the hypercycle base curve.

The apex p is the dual point of the base plane in the cyclographic model. The dual of a cone with apex p is therefore a conic contained in the base plane.

The pencil associated with an ellipse or hyperbola (centered and rotated as in (6)) is diagonal. Vice versa, if a generic pencil is diagonal one easily sees that it must contain a cone of the form (7). □

A Laguerre transformation separates the two copies of a conic and one obtains a hypercycle which consists of two possibly intersecting pieces. Vice versa, a hypercycle is Laguerre equivalent to a conic if and only if it is diagonalizable by a Laguerre transformation. Thus, we investigate the question of the diagonalisability of generic pencils of quadrics.

[4]The first condition is as in [BST2018], the second condition is more restrictive.

[5]This second condition already implies the first.

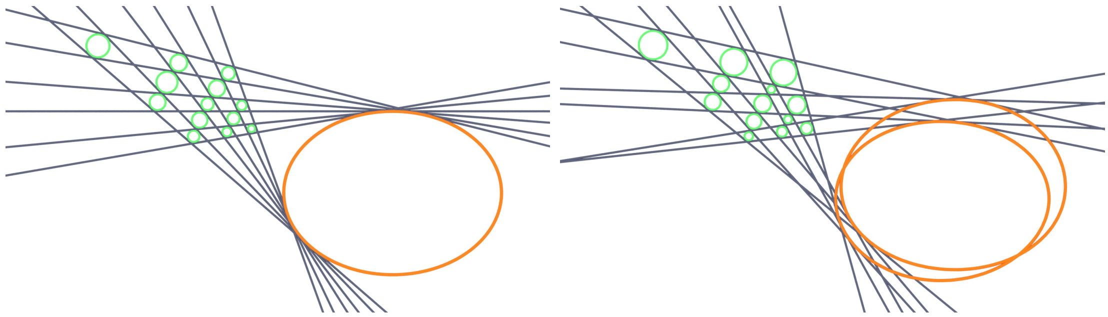


Figure 35. Applying a Laguerre transformation to a conic yields a hypercycle.

16.1 Classification of hypercylces

We will now examine under what circumstances generic hypercycles may be regarded as Laguerre transforms of conics.

Consider a pencil of quadrics spanned by the Blaschke cylinder $\mathcal{Z}$ and another quadric $\mathcal{Q}$. The pencil is diagonalizable by a Laguerre transformation if and only if the quadric $\mathcal{Q}$ is diagonalizable by a Laguerre transformation.

Proposition 16.3. *Let $\mathcal{Q} \subset \mathbb{RP}^3$ be a quadric that does not contain the apex $q = [0,0,0,1]$ of the Blaschke cylinder. Then its representative matrix can be brought into the block diagonal form*

$$Q = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix},$$

by a Laguerre transformation.

Proof. $p \notin \mathcal{Q} = [\tilde{Q}]$ is equivalent to $\tilde{Q}_{4,4} \neq 0$. In the normalization $\tilde{Q}_{4,4} = 1$ the representative matrix is of the form

$$\tilde{Q} = \begin{pmatrix} \tilde{S} & a \\ a^\intercal & 1 \end{pmatrix},$$

and is brought into block diagonal form by a Laguerre transformation with the matrix

$$A = \begin{pmatrix} I & 0 \\ -a^\intercal & 1 \end{pmatrix}.$$

$\square$

A Laguerre transformation of the form

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

with $B \in O(2,1)$ acts on the block diagonal matrix Q by

$$A^\intercal Q A = A = \begin{pmatrix} B^\intercal S B & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the study of the pencil of quadrics $\mathcal{Z} \wedge \mathcal{Q} \subset \mathbb{RP}^3$ up to Laguerre transformations may be reduced to the study of the pencil of conics $\hat{\mathcal{Z}} \wedge \hat{\mathcal{Q}} \subset \mathbb{RP}^2$ up to transformations from $\mathrm{PO}(2,1)$, where

$$\hat{\mathcal{Z}} = \left\{[x_1, x_2, x_3] \in \mathbb{RP}^2 \;\middle|\; x_1^2 + x_2^1 - x_3^2 = 0\right\}, \qquad \hat{\mathcal{Q}} = \left\{[x] \in \mathbb{RP}^2 \;\middle|\; x^\intercal S x = 0\right\},$$

We call $\hat{\mathcal{Z}}$ the "Blaschke circle" and denote its representative matrix by $Z = \mathrm{diag}(1, 1, -1)$.

Proposition 16.4. *The pencil $\mathcal{Z} \wedge \mathcal{Q} \subset \mathbb{RP}^3$ is generic if and only if the corresponding pencil $\hat{\mathcal{Z}} \wedge \hat{\mathcal{Q}} \subset \mathbb{RP}^2$ is of type Ia, Ib, or Ic.*

Proof. are as stated in Table The degenerate qudarics of the pencil $\mathcal{Z} \wedge \mathcal{Q} \subset \mathbb{RP}^3$ are given by the roots of the equation

$$\lambda \det(Z + \lambda S) = 0.$$

The root $\lambda = 0$ corresponds to the Blaschke cylinder. The remaining degenerate quadrics correspond to the degenerate conics of the pencil $\hat{\mathcal{Z}} \wedge \hat{\mathcal{Q}} \subset \mathbb{RP}^2$. □

Proposition 16.5. *The pencil $\mathcal{Z} \wedge \mathcal{Q} \subset \mathbb{RP}^3$ is diagonalizable by a Laguerre transformation if and only if the corresponding pencil $\hat{\mathcal{Z}} \wedge \hat{\mathcal{Q}} \subset \mathbb{RP}^2$ is of type Ia or Ic.*

Proof. The $\mathcal{Z} \wedge \mathcal{Q} \subset \mathbb{RP}^3$ is diagonalizable by a Laguerre transformation if and only if $\hat{\mathcal{Q}} \subset \mathbb{RP}^2$ is diagonalizabel by a transformation from $\mathrm{PO}(2,1)$.

By Proposition 10.12 a pencil of type Ib is not diagonalizable. Pencils of type Ia and Ic are diagonalizable by a projective transformation, yet we still have to show that they are diagonalizable by a projective orthogonal transformation from $\mathrm{PO}(2,1)$.

Type Ia) In this case, we may apply a projective transformation which transforms the Blaschke circle into an ellipse and maps the four base points to the four vertices of a square. An appropriate subsequent affine transformation then maps the ellipse to the Blaschke circle without affecting the symmetry of the rectangle. The composition of these two transformations constitutes an $\mathrm{PO}(2,1)$ transformations since it leaves the Blaschke circle invariant. This compound transformation results in a symmetric distribution of the base points and, hence, the transformed pencil is symmetric. A degenerate conic in the this pencil is given by

$$x_2^2 - a^2 x_3^2 = 0$$

with some $0 < a < 1$.

Type Ic) In this case, there exists a degenerate conic consisting of two intersecting real lines which do not intersect the Blaschke circle. A suitable combination of a projective and an affine transformation sends the vertex of this degenerate conic to infinity and maps the intermediate ellipse back to the Blaschke circle. The degenerate conic may therefore be transformed into $(x_2 - a)(x_2 + b) = 0$, where $a, b > 1$. Application of a further transformation of the form

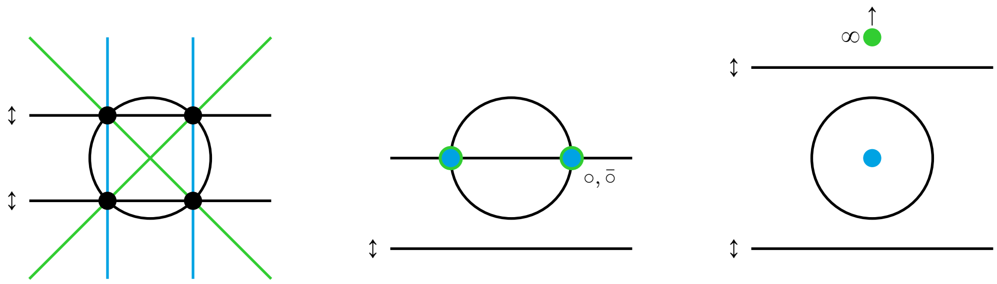
$$B = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}.$$

with $C \in \mathrm{O}(1,1)$ leads to $a = b$ so that the degenerate conic simplifies to

$$x_2^2 - a^2 x_3^2 = 0,$$

which is symmetric. □

In the process of the proof we have derived normal forms for the pencils of type Ia and Ic. Similarly one arrives at a normal form for the non-diagonalizable type Ib.



Ia Ib Ic

Type	base points	# real	Deg. conics	Roots	Normal form (one conic)
Ia	$1, 1, 1, 1$	4	$\times, \times, \times$	$1, 1, 1$	$x_2^2 - a^2 x_3^2 = 0, \quad 0 < a < 1$
Ib	$1, 1, (1, \bar{1})$	2	$\times, \circ, \bar{\circ}$	$1, (1, \bar{1})$	$bx_2^2 - x_2 x_3 = 0, \quad \lvert b \rvert < 1$
Ic	$(1, \bar{1}), (1, \bar{1})$	0	$\times, \bullet, \bullet$	$1, 1, 1$	$x_2^2 - a^2 x_3^2 = 0, \quad a > 1$

Figure 36. The classification of real generic pencils of conics containing the Blaschke circle $\hat{\mathcal{Z}}$ up to transformations from $\mathrm{PO}(2, 1)$. The normal forms of one conic in the pencil are given. They still depend on one parameter.

Lifting these normal forms to $\mathbb{R}\mathrm{P}^3$ we obtain the following normal forms (and different classes) of generic hypercycle base curves.

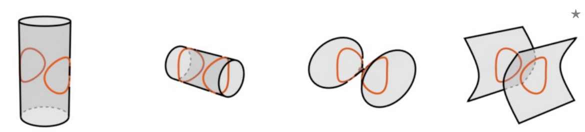
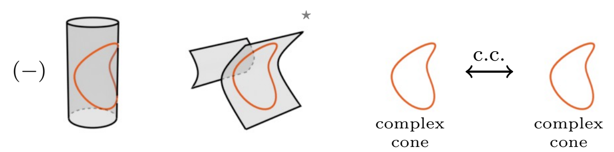
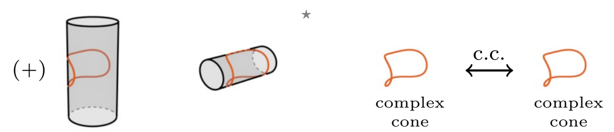
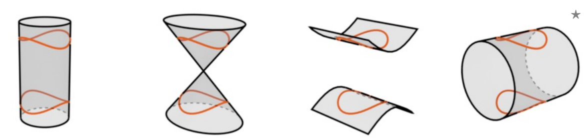
Type	Normal form (one quadric)	Type and multiplicity of degenerate quadrics
Ia	$x_2^2 - a^2 x_3^2 + x_4^2, \quad 0 < a < 1$	
$\mathrm{Ib}_\pm$	$bx_2^2 - x_2 x_3 \pm x_4^2 = 0, \quad \lvert b \rvert < 1$	 
Ic	$x_2^2 - a^2 x_3^2 + x_4^2 = 0, \quad a > 1$	

Figure 37. Classification of generic hypercycle base curves.

Proposition 16.6. *A generic hypercycle can be transformed into an ellipse or hyperbola by a Laguerre transformation if its hypercycle base curve consists of two separate components.*

Furthermore the two components of the hypercycle base curve are either both null-homotopic on the Blaschke cylinder or both wrap around the Blaschke cylinder once. In the first case (type Ia) the hypercycles can be transformed into a hyperbola and the the second case (type Ic) into an ellipse.

17 Checkerboard incircular nets

[BST2018] A.I. Bobenko, W.K. Schief and J. Techter.
Checkerboard incircular nets. Laguerre geometry and parametrisation, *Geometriae Dedicata* **204:1** (2020), 97-129

[BLPT2020] A.I. Bobenko, C.O.R. Lutz, H. Pottmann and J. Techter.
Non-Euclidean Laguerre geometry and incircular nets, in preparation.

Two families $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ of oriented lines in the Euclidean plane are called a *checkerboard incircular net* if for every $i, j \in \mathbb{Z}$ with even $i + j$ the four lines $\ell_i, \ell_{i+1}, m_j, m_{j+1}$ touch a common oriented circle.

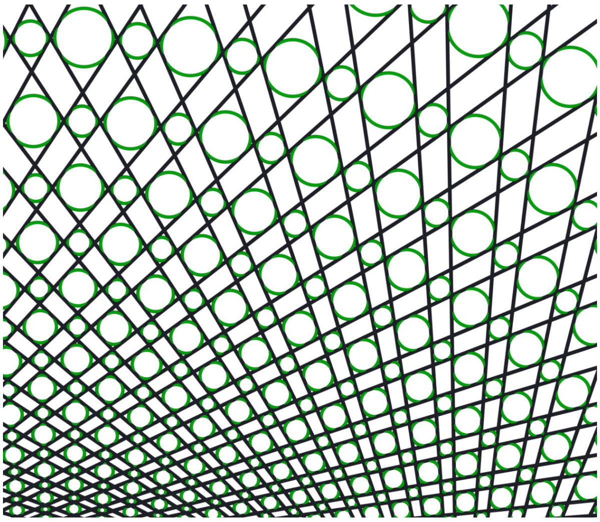


Figure 38. An example of a checkerboard incircular net.

In the limit in which all incircles of the quadrilaterals $\ell_{2i}, \ell_{2i+1}, m_{2j}, m_{2j+1}$ of a checkerboard incircular net collapse to a point, the pairs of lines ℓ_{2i}, ℓ_{2i+1} as well as the pairs of lines m_{2j}, m_{2j+1} coincide respectively up to their orientation. Such a pair of oriented lines may be regarded as a non-oriented line. In this limit a checkerboard incircular net becomes an "ordinary" incircular net.

Proposition 17.1. *Let $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ be a checkerboard incircular net with $\ell_{2i} = \ell_{2i+1}$ for $i \in \mathbb{Z}$ and $m_{2j} = m_{2j+1}$ for $j \in \mathbb{Z}$. Then*

$$\tilde{\ell}_i := \ell_{2i}, \qquad \tilde{m}_j := m_{2j}$$

defines an incircular net. In particular it satisfies the regularity condition (R).

Proof. Exercises. □

17.1 Miquel's theorem in Laguerre geometry

Theorem 13.2 yields a Laguerre geometric version of Miquel's theorem.

Theorem 17.2 (Miquel's theorem in Laguerre geometry). *Let $\ell_1, \ell_2, \ell_3, \ell_4, m_1, m_2, m_3, m_4$ be eight oriented lines in Euclidean plane. If the five quadrilaterals $(\ell_1, \ell_2, m_1, m_2)$, $(\ell_1, \ell_2, m_3, m_4)$, $(\ell_3, \ell_4, m_1, m_2)$, $(\ell_3, \ell_4, m_3, m_4)$, $(\ell_2, \ell_3, m_2, m_3)$ are circumscribed (each quadruple of lines touches a common oriented circle), then so is the quadrilateral $(\ell_1, \ell_4, m_1, m_4)$ (cf. Figure 39).*

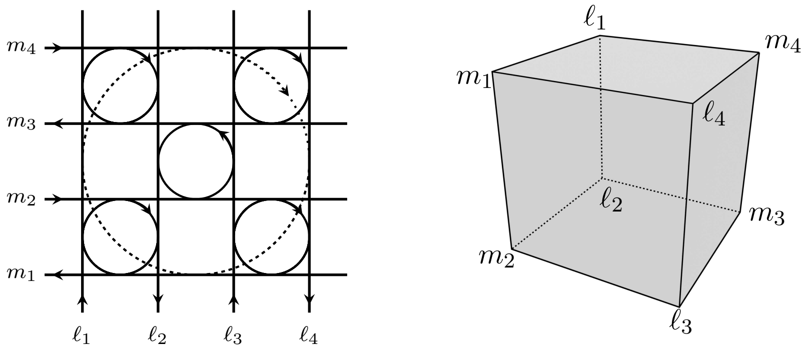


Figure 39. Combinatorial pictures on Miquel's theorem in Laguerre geometry. *Left:* The eight oriented lines and six incircles in the plane. *Right:* The eight corresponding points on the Blaschke cylinder and how to associate them with the vertices of a cube.

Proof. The eight oriented lines correspond to eight points on the Blaschke cylinder. Associate them with the vertices of a combinatorial cube (see Figure 39). Coplanarity of the bottom and side faces corresponds to the assumed circumscribility. By Theorem 13.2 the top face is planar as well. □

Corollary 17.3. *Let $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ be a checkerboard incircular net. Then for every $i, j, k \in \mathbb{Z}$ with even $i + j$ the quadrilateral $(\ell_i, m_j, \ell_{i+2k+1}, m_{j+2k+1})$ is circumscribed.*

17.2 Tangency to a hypercycle

The lines of an incircular nets are tangent to a conic. It turns out that the lines of a checkerboard incircular net are tangent to a hypercycle.

Lemma 17.4. *Let p_1, p_2 be two points which belong to all members of a pencil of quadrics $\mathcal{Q}_\lambda$. Then, there exists a unique quadric $\mathcal{Q}_{\lambda_{12}}$ from the pencil which contains the whole line $L_{12} = p_1 \wedge p_2$.*

If the line $L_{34} = p_3 \wedge p_4$ associated with another pair of base points p_3, p_4 intersects the line L_{12} then the two quadrics $\mathcal{Q}_{\lambda_{12}}$ and $\mathcal{Q}_{\lambda_{34}}$ coincide.

Proof. Let q_1, q_2 be two quadratic forms generating the pencil with the quadratic form $q_\lambda = q_1 + \lambda q_2$. The points $p_1 = [v_1]$, $p_2 = [v_2]$ belong to all quadrics of the pencil if and only if $q_1(v_1) = q_1(v_2) = q_2(v_1) = q_2(v_2) = 0$. The line $L_{12} = p_1 \wedge p_2$ belongs to the quadric determined by $q_{\lambda_{12}}$ if and only if $q_{\lambda_{12}}(v_1, v_2) = 0$ so that $t_{12} = -\frac{q_1(v_1,v_2)}{q_2(v_1,v_2)}$. Vanishing of the denominator is the case when the line lies on the quadric determined by q_2.

Moreover, if the line $L_{34} = p_3 \wedge p_4$ passing through another pair of common points p_3, p_4 intersects the line L_{12} then the point of intersection and p_3, p_4 belong to the quadric $\mathcal{Q}_{\lambda_{12}}$. Accordingly, the line L_{34} is contained in $\mathcal{Q}_{\lambda_{12}}$ so that $\mathcal{Q}_{\lambda_{12}} = \mathcal{Q}_{\lambda_{34}}$. □

Theorem 17.5. *All lines of a generic checkerboard incircular net are in oriented contact with a common hypercycle.*

Moreover, the corresponding pencil of quadrics, which contains the hypercycle base curve, contains two unique hyperboloids $\mathcal{H}, \widetilde{\mathcal{H}}$ distinguished in the following way: Let $(\ell_i)_{i\in\mathbb{Z}}$, $(m_j)_{j\in\mathbb{Z}}$ be the points on the Blaschke cylinder $\mathcal{Z} \subset \mathbb{RP}^3$ corresponding to the oriented lines of the checkerboard incircular net. Consider the lines

$$L_i := \ell_i \wedge \ell_{i+1}, \qquad M_j := m_j \wedge m_{j+1}.$$

Then, all lines L_{2k}, M_{2l} lie on a common hyperboloid $\mathcal{H} \subset \mathbb{RP}^3$. Similarly, all lines L_{2k+1}, M_{2l+1} lie on a common hyperboloid $\widetilde{\mathcal{H}} \subset \mathbb{RP}^3$.

Proof. Due to the inscribability property of checkerboard incircular nets every line L_{2k} intersects every line M_{2l}, and vice versa. Thus, all lines L_{2k}, M_{2l} generically lie on a common hyperboloid $\mathcal{H}$. Similarly, all lines L_{2k+1}, M_{2l+1} lie on a common hyperboloid $\widetilde{\mathcal{H}}$. We now show that both hyperboloids $\mathcal{H}$, $\widetilde{\mathcal{H}}$ intersect the Blaschke cylinder $\mathcal{Z}$ in the same curve, that is, they belong to the same pencil of quadrics. Indeed, according to Lemma 17.4, for each line L_{2k+1}, there exists a unique quadric in the pencil spanned by $\mathcal{Z}$ and $\mathcal{H}$ containing L_{2k+1}. Same for each line M_{2l+1}. Since the lines L_{2k+1} and M_{2l+1} pairwise intersect, again according to Lemma 17.4, the corresponding quadrics coincide with each other and eventually with $\widetilde{\mathcal{H}}$. Thus, all points ℓ_i, m_j lie on the intersection $\mathcal{Z} \cap \mathcal{H} = \mathcal{Z} \cap \widetilde{\mathcal{H}}$. □

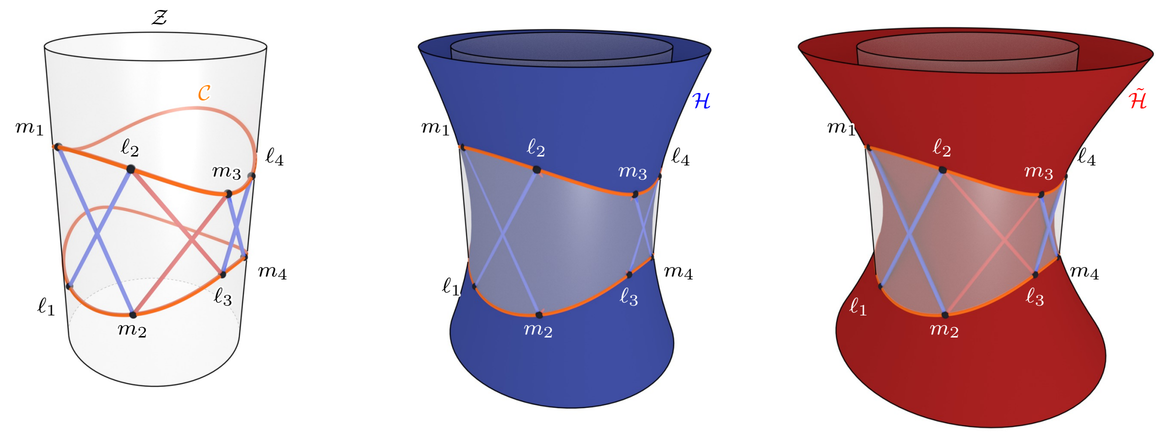


Figure 40. Construction of checkerboard IC-nets in the Blaschke cylinder model. The lines L_{2k+1}, M_{2k+1} (red) and L_{2k}, M_{2k} (blue) are generators of the quadrics $\mathcal{H}$ and $\tilde{\mathcal{H}}$ respectively.

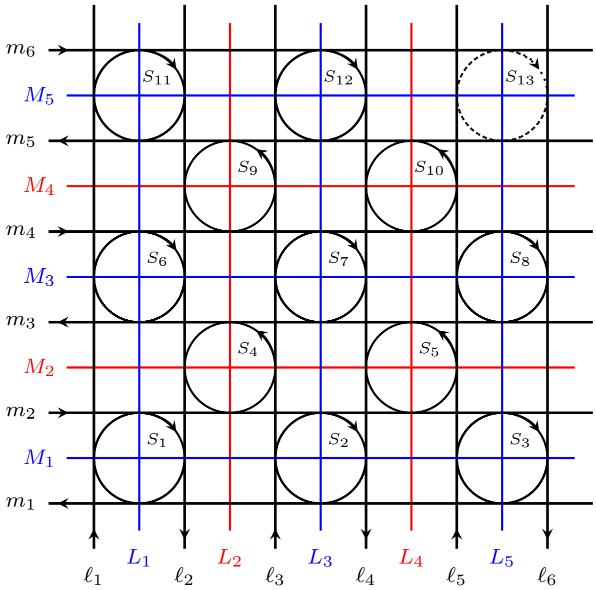


Figure 41. Combinatorial picture of the lines of a checkerboard incircular net and the corresponding 13th-circle incidence theorem.

17.3 Construction of checkerboard incircular nets

The elementary construction of a checkerboard incircular net from a small patch (line by line, while ensuring the incircle constraint) is guaranteed to work due to the following incidence theorem (see Figure 41). This construction has 12 real degrees of freedom.

Theorem 17.6. *Let $\ell_1, \cdots, \ell_6$, $m_1, \cdots, m_6$ be 12 oriented lines in the Euclidean plane which are in oriented contact with 12 oriented circles $S_1, \cdots, S_{12}$, in a checkerboard manner, as shown in Figure 41. In particular, the lines ℓ_1, ℓ_2, m_1, m_2 are in oriented contact with the circle S_1, the lines ℓ_3, ℓ_4, m_1, m_2 are in oriented contact with the circle S_2 etc. Then, the 13th checkerboard quadrilateral also has an inscribed circle, i.e., the lines ℓ_5, ℓ_6, m_5, m_6 have a common circle S_{13} in oriented contact.*

Though possible in principle, the elementary construction from, e.g., 6 lines as initial data, is not stable, and thus impractical for the construction of large checkerboard incircular nets. Yet, by Theorem 17.5, we find that a checkerboard incircular net can equivalently be prescribed by

- choosing a hypercycle (8 degrees of freedom),
- choosing two hyperboloids $\mathcal{H}, \tilde{\mathcal{H}}$ in the pencil of quadrics corresponding to the hypercycle base curve (2 degrees of freedom),
- and choosing two initial lines tangent to the hypercycle, one from each of the ℓ- and m-family (2 degrees of freedom).

Then further lines of, say, the ℓ-family are obtained by alternately going along a chosen family of rulings of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ from one point of the base curve to the next (see Figure 40). Similarly for the m-family of lines, while using the respective other families of rulings of the two hyperboloids. The intersection of two rulings from the two different families of the same hyperboloid implies the coplanarity of the four intersection points with the base curve, which, in turn, corresponds to the existence of an incircle.

17.3.1 Parametrization of checkerboard incircular nets tangent to an ellipse

By parametrizing the hypercycle base curve in terms of Jacobi elliptic functions one obtains explicit formulas for the corresponding checkerboard incircular nets. We demonstrate this in the case of checkerboard incircular nets tangent to an ellipse.

We have seen that the hypercycle base curve corresponding to an ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = 1, \qquad a > b > 0$$

is given by the intersection of the Blaschke cylinder

$$\mathcal{Z} = \left\{ [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 \;\middle|\; x_1^2 + x_2^2 - x_3^2 = 0 \right\}$$

with the cone

$$\mathcal{C} = \left\{ [x_1, x_2, x_3, x_4] \in \mathbb{RP}^3 \;\middle|\; a x_1^2 + b x_2^2 - x_4^2 = 0 \right\}.$$

The curve $\mathcal{Z} \cap \mathcal{C}$ consists of two components which can be parametrized in terms of Jacobi elliptic functions by

$$v_\pm(u) = \left[\operatorname{cn}(u,k),\ \operatorname{sn}(u,k),\ \pm 1,\ \sqrt{a}\operatorname{dn}(u,k) \right],$$

for $u \in \mathbb{R}$, where the modulus k is given by

$$k = \sqrt{1 - \frac{b}{a}}.$$

Note that the parametrization of the two components satisfy

$$v_\pm(u) = \sigma\left(v_\mp(u) \right),$$

i.e., points on the different components with the same argument u represent the same line with opposite orientation.

This parametrization features the following remarkable property:

Proposition 17.7.

(i) *Let $u, \tilde{u}, s \in \mathbb{R}$. Then the four points $v_+(u), v_-(u+s), v_-(\tilde{u}), v_+(\tilde{u}+s)$ are coplanar.*

(ii) *Let $s \in \mathbb{R}$. Then the lines $v_+(u) \wedge v_-(u+s)$ with $u \in \mathbb{R}$ constitute one family of rulings of a common hyperboloid in the pencil $\mathcal{Z} \wedge \mathcal{C}$. The second family of rulings of this hyperboloid is given by the lines $v_+(u) \wedge v_-(u-s)$ with $u \in \mathbb{R}$.*

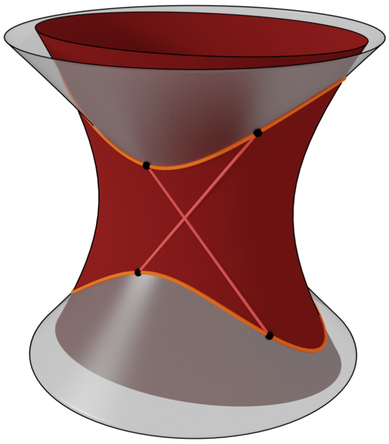


Figure 42. Parametrization of the hypercycle base curve by Jacobi elliptic functions.

This allows to parametrize a confocal checkerboard incircular net tangent to a given ellipse in the following way:

Theorem 17.8. *Let* $a > b > 0$, $s, \tilde{s} \in \mathbb{R}$, *and* $u_0^\ell, u_0^m \in \mathbb{R}$. *Then the two families of lines* $(\ell_i)_{i\in\mathbb{Z}}$ *and* $(m_j)_{j\in\mathbb{Z}}$ *given by*

$$\begin{aligned}
\ell_{2k} &= v_+(u_0^\ell + k(s+\tilde{s})) \\
\ell_{2k+1} &= v_-(u_0^\ell + k(s+\tilde{s}) + s) \\
m_{2l} &= v_-(u_0^m + l(s+\tilde{s})) \\
m_{2l+1} &= v_+(u_0^m + l(s+\tilde{s}) + s)
\end{aligned}$$

constitute a checkerboard incircular net tangent to an ellipse.

The choice of

- a and b determines the ellipse,
- s and $\tilde{s}$ determines two hyperboloids in the pencil of quadrics, and further allows to distinguish the two families of rulings on each of them,
- u_0^{v}, u_0^{h} determines one initial line tangent to the ellipse in each of the two families of lines.

Note that for $s = 0$ the corresponding hyperboloid degenerates to the cone $\mathcal{C}$. In this case,

$$\ell_{2k} = \ell_{2k+1}, \qquad m_{2l} = m_{2l+1},$$

and thus, $(\ell_i)_{i\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ constitutes an "ordinary" incircular net.

Periodicity can be achieved by setting

$$s + \tilde{s} = \frac{4\mathsf{K}(k)}{N}.$$

To achieve "embeddedness" of the confocal checkerboard incircular nets one has to additionally demand that the two different families of lines agree (up to their orientation), e.g.,

$$\ell_i = \sigma(m_i),$$

which is obtained by setting

$$u_0^\ell = u_0^m.$$

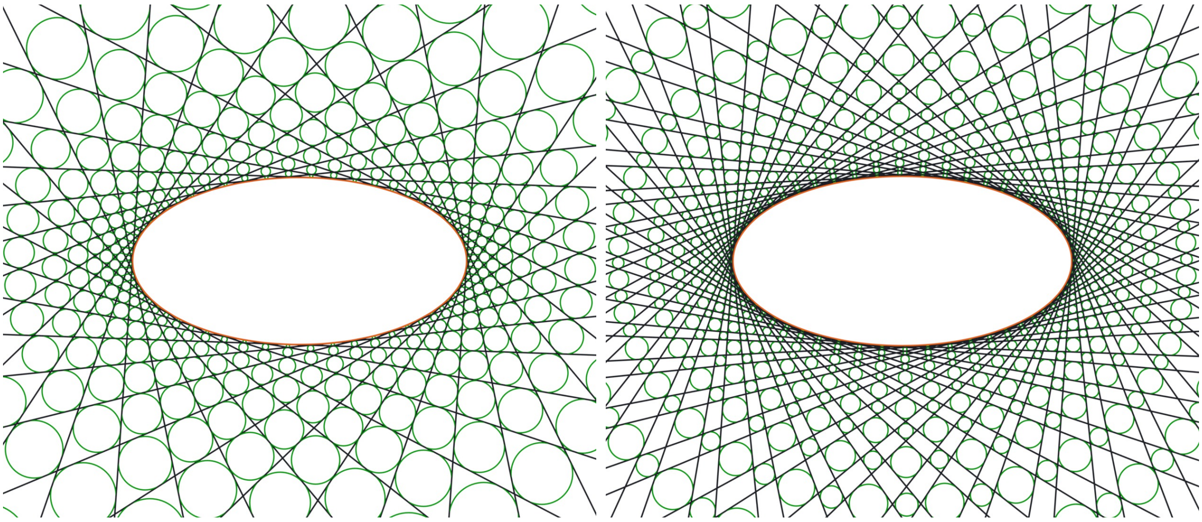


Figure 43. Periodic checkerboard incircular nets tangent to an ellipse with $a = 4$, $b = 1$, $N = 32$. *Left:* $s = 0$, corresponding to the degenerate case of an incircular net. *Right:* $s = 0.1$.

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