# Realized cumulants for martingales

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FG6

## Definition (Realized variance)

For a partition $\Pi_{a,b}$ of $(a,b]$,

$$\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2$$

## Connecting high and low frequency distributions

$$E\left[\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2\right] = E[|M_{a,b}|^2]$$

A similar formula (Bartlett-type identity) appears in (Mykland, 1994).

## Definition (Aggregation property (Neuberger, 2012))

$$E[g(\mathbb{X}_{s,u}) \mid \mathcal{F}_s] = E[g(\mathbb{X}_{s,t}) \mid \mathcal{F}_s] + E[g(\mathbb{X}_{t,u}) \mid \mathcal{F}_s]$$

We have:

- $g(x) = x^2, \mathbb{X} = M$ satisfy the aggregation property,
- (Neuberger, 2012) $g(x,y) = x^3 + 3xy, \mathbb{X} = (M, M^{(2)})$ also satisfy the aggregation property (realized skewness), where
$$M_t^{(n)} := E[(M_T - M_t)^n \mid \mathcal{F}_t],$$
- (Bae–Lee, 2020) $g(x,y,z) = x^4 + 6x^2y + 3y^2 + 4xz, \mathbb{X} = (M, M^{(2)}, M^{(3)})$ also satisfies the aggregation property (realized kurtosis).

## Theorem (Fukasawa–Matsushita, 2020)

*The functions $g_n(x_1,\ldots,x_n) = B_{n+1}(x_1,\ldots,x_n,0)$ satisfy the aggregation property, with $\mathbb{X} = (X, X^{(2)},\ldots,X^{(n)})$. Here $B_n$ are the complete Bell polynomials*

$$B_n(x_1,\ldots,x_n) := \frac{\partial^n}{\partial z^n} \exp\left(\sum_{i=1}^n x_i \frac{z^i}{i!}\right)\Bigg|_{z=0}.$$

# Introduction

## Moments and cumulants

Similar relation holds (Faà di Bruno's formula, 1855):

$$E[X^n] = B_n(\kappa_1, \ldots, \kappa_n) = \sum_{\pi \in P(n)} \prod_{B \in \pi} \kappa_{|B|}.$$

In particular:

$$\{\{1\}, \ldots, \{n\}\} \Rightarrow B_n(x_1, 0, \ldots, 0) = x_1^n$$
$$\{1, \ldots, n\} \Rightarrow B_n(0, \ldots, 0, x_n) = x_n$$

## Lemma (Fukasawa–Matsushita, 2020)

*For stopping times $\tau \le \nu$,*

$$E[B_n(\mathbb{X}_\nu^{(n)}) \mid \mathcal{F}_\tau] = 0.$$

Consequences:

- Aggregation property, and
- $E[g_n(\mathbb{X}_{s,t}^{(n)}) \mid \mathcal{F}_s] = -E[X_{s,t}^{(n+1)} \mid \mathcal{F}_s]$,
- Realized cumulants:

$$X_\sigma^{(n+1)} = E\left[ \sum_{(\tau,\nu] \in \Pi_{\sigma,T}} g_n(\mathbb{X}_{\tau,\nu}^{(n)}) \middle| \mathcal{F}_\sigma \right].$$

## Theorem (Fukasawa–Matsushita 2020, Friz–Hager–T 2021, Lacoin–Rhodes–Vargas 2019, Friz–Gatheral–Radoičić 2020)

$$X_\sigma^{(n+1)} = E\left[ \sum_{s \in (\sigma,T]} g_n(\Delta \mathbb{X}_s^{(n)}) + \frac{1}{2} \sum_{j=1}^{n} \binom{n+1}{j} \langle X^{(n+1-j),c}, X^{(j),c} \rangle_{\sigma,T} \middle| \mathcal{F}_\sigma \right]$$

# Questions & comments

In the proof, the relation $E\left[e^{zX_T} \mid \mathcal{F}_t\right] = \mathrm{e}^{K_t(z)}$ is expressed in terms of Bell polynomials. We have, $\mathrm{e}^{zX_t + K_t(z)}$ is a martingale; this gives rise to the cumulant (diamond) recursion.

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- Similar formulas for non-commuting random variables? Free and boolean Appell sequences (Anshelevich 2004 & 2009).

  Free moments and cumulants (Speicher, 1997):

$$E[A_1 \cdots A_n] = \sum_{\pi \in NC(n)} \prod_{B \in \pi} \kappa(A_i : i \in B).$$