Cumulant operators for stochastic integrals

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Cumulants in Stochastic Analysis

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Cumulants and Thiele recursion formula

The cumulants $(\kappa_n^X)_{n\geq 1}$ of a random variable X are defined by

$$\log \mathbb{E}[e^{tX}] = \sum_{n=1}^{\infty} \kappa_n^X \frac{t^n}{n!}$$

and they satisfy the Thiele (1899) recursion formula

$$\mathbb{E}[X^n] = \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-l-1)!} \kappa_{n-l}^X \mathbb{E}[X^l],$$

which shows that

$$\begin{aligned}
\mathbb{E}[X^n] &= \sum_{k=0}^{n} \frac{n!}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{\kappa_{d_1}^X}{d_1!} \cdots \frac{\kappa_{d_k}^X}{d_k!} \\
&= \sum_{k=0}^{n} \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \kappa_{|P_1|}^X \cdots \kappa_{|P_k|}^X.
\end{aligned}$$

Gaussian cumulants

When $X \simeq \mathcal{N}(0, \sigma^2)$ is centered Gaussian we have

$$(\kappa_1^X, \kappa_2^X, \kappa_3^X, \kappa_4^X, \ldots) = (0, \sigma^2, 0, 0, \ldots).$$

This recovers the Wick theorem for the computation of Gaussian moments by counting the pair partitions of $\{1, \ldots, n\}$, cf. Isserlis (1918), as

$$\begin{aligned} \mathbb{E}[X^n] &= \sigma^n \sum_{k=1}^{n} \sum_{\substack{P_1 \cup \cdots \cup P_k = \{1, \ldots, n\} \\ |P_1|=2, \ldots, |P_k|=2}} \kappa_{|P_1|}^X \cdots \kappa_{|P_a|}^X \\ &= \begin{cases} \sigma^n (n-1)!!, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \end{aligned}$$

where the double factorial

$$(n-1)!! = \prod_{1 \leq 2k \leq n} (2k-1) = 2^{-n/2} \frac{n!}{(n/2)!}$$

counts the number of pair-partitions of $\{1, \ldots, n\}$ when n is even.

Moments of the Wiener integral

Let $f \in L^2([0, T])$. We have

$$\mathbb{E}\left[\left(\int_0^T f(t)dB_t\right)^n\right] = \begin{cases} \left(\int_0^T f^2(t)dt\right)^{n/2}(n-1)!!, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

When $u \in L^2(\Omega \times [0, T])$ is an adapted process, we have

$$\mathbb{E}\left[\left(\int_0^T u_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T |u_t|^2 dt\right].$$

Cumulant operators on the Wiener space.

Proposition 1

Nourdin and Peccati (2010) Let D and δ denote the gradient and divergence on the Wiener space, with $L = \delta D$ the Ornstein-Uhlenbeck operator. Let $\Gamma_0 F = F$, $\Gamma_1 F = \langle DF, -DL^{-1}F\rangle$, and

$$\Gamma_{n+1}F = \langle DF, -DL^{-1}\Gamma_n F\rangle, \qquad n \geq 0.$$

The cumulant of F of order $n+1$ is given by

$$\kappa_{n+1} = n!\mathbb{E}[\Gamma_n F], \qquad n \geq 0.$$

Cumulant operators for the Itô-Skorohod integral δ (ECP 2015, JTP 2015)

Let D denote the Malliavin gradient with adjoint of δ, i.e.

$$\mathbb{E}[F\delta(v)] = \mathbb{E}[\langle DF, v\rangle_H], \quad F \in \mathrm{Dom}(D), \quad v \in \mathrm{Dom}(\delta),$$

where $H := L^2(\mathbb{R}_+; \mathbb{R}^d)$.

Definition 1

Let $u \in D_{k,2}(H)$ be a (smooth) random process. The cumulant operator

$$\Gamma_k^u : D_{2,1} \longrightarrow L^2(\Omega), \qquad k \geq 2,$$

is defined by

$$\Gamma_k^u \mathbf{1} = \langle (Du)^{k-2} u, u\rangle_H + \langle D^* u, D((Du)^{k-2} u)\rangle_{H\otimes H}.$$

We extend Γ_k^u to all $F \in D_{2,1}$ by

$$\Gamma_k^u F := F\Gamma_k^u \mathbf{1} + \langle (Du)^{k-1} u, DF\rangle_H, \qquad k \geq 1.$$

The composition $(Du)^l$ and the adjoint D^* are defined in the sense of matrix powers with continuous indices.

A Thiele recursion for the Itô-Skorohod integral

Proposition 2

By repeated integration by parts we obtain the identity

$$\mathbb{E}[F\delta(u)^n] = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} \mathbb{E}\left[\delta(u)^k \Gamma_{n-k}^u F\right],$$

which can be seen as a stochastic version of the Thiele (1899) recursion formula for $\delta(u)$, under a measure with density F.

This yields the moment identity

$$\mathbb{E}[F\delta(u)^n] = \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\substack{d_1+\cdots+d_k=n \\ d_1\geq 1,\ldots,d_k\geq 1}} \frac{1}{d_1\cdots d_k} \mathbb{E}\left[\Gamma_{d_1}^u \cdots \Gamma_{d_k}^u F\right],$$

where the sum runs over the partitions $P_1, \ldots, P_k$ of $\{1, \ldots, n\}$ with cardinal $|P_i|$.

Moment identities

Proposition 3

For any $n \geq 1$ and $u \in D_{n+1,2}(H)$ we have the moment identity

$$\mathbb{E}[(\delta(u))^{n+1}] = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$$

$$\mathbb{E}\Bigg[(\delta(u))^{n-k} \left(\langle (Du)^{k-1}u, u\rangle_H + \textbf{trace}\,(Du)^{k+1}\right.$$

$$\left.\left. + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i}u, D\textbf{trace}\,(Du)^{i}\rangle_H \right)\right].$$

Note that $\textbf{Trace}\,(Du)^k = 0$ when u is an adapted process.

Corollary 2

Let $n \geq 1$ and $u \in D_{n+1,2}(H)$ such that $\|u\|_H$ is deterministic and

$$\textbf{trace}\,(Du)^{k+1} = 0, \qquad 1 \leq k \leq n.$$

Then

$$\mathbb{E}[(\delta(u))^{n+1}] = n\mathbb{E}\left[(\delta(u))^{n-1}\langle u, u\rangle_H\right] = n\mathbb{E}\|u\|_H^2 \left[(\delta(u))^{n-1}\right],$$

i.e. $\delta(u)$ has the same first $n+1$ moments as $\mathcal{N}(0, \|u\|_H^2)$.

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Edgeworth type expansions

Proposition 4

Assume that $u \in D_{k,2}(H)$ for all $k = 1, \ldots, n+2$ for some $n \geq 0$. Then for all $\Phi \in \mathcal{C}_b^{n+1}(\mathbb{R})$ and $F \in D_{2,1}$ we have

$$\begin{aligned}\mathbb{E}\left[F\delta(u)\Phi(\delta(u))\right] &= \sum_{k=0}^{n} \mathbb{E}\left[\Phi^{(k)}(\delta(u))\Gamma_{k+1}^{u} F\right] \\ &+\frac{1}{2}\mathbb{E}\left[F\Phi^{(n+1)}(\delta(u))\langle (Du)^{n-1}u, D\langle u,u\rangle\rangle\right] \\ &+\mathbb{E}\left[F\Phi^{(n+1)}(\delta(u))\langle (Du)^{n}u, \delta(Du)\rangle\right].\end{aligned}$$

When **trace** $(Du)^k = 0$ for all $k = 2, \ldots, n+1$ we have

$$\begin{aligned}\mathbb{E}\left[\delta(u)\Phi(\delta(u))\right] &= \mathbb{E}\left[\langle u,u\rangle \Phi'(\delta(u))\right] \\ &+\frac{1}{2}\sum_{k=2}^{n+1}\mathbb{E}\left[\langle (Du)^{k-2}u, D\langle u,u\rangle\rangle \Phi^{(k)}(\delta(u))\right] \\ &+\mathbb{E}\left[\Phi^{(n+1)}(\delta(u))\langle (Du)^{n}u, \delta(Du)\rangle\right], \qquad n \geq 0.\end{aligned}$$

This setting includes the particular case where u is an adapted process and $\delta(u)$ coincides with the Itô integral of u. If in addition $\langle u, u\rangle$ is deterministic then $\delta(u) \simeq \mathcal{N}(0, \langle u, u\rangle)$.

Random isometries

Remark 1

1) *When $\langle u, u \rangle$ is deterministic we find*

$$\begin{aligned}\mathbb{E}\left[\delta(u)\Phi(\delta(u))\right] &= \langle u, u\rangle \mathbb{E}\left[\Phi'(\delta(u))\right] \\ &+ \sum_{k=1}^{n} \mathbb{E}\left[\langle D^* u, D((Du)^{k-1}u)\rangle_{H\otimes H}\Phi^{(k)}(\delta(u))\right] \\ &+ \mathbb{E}\left[\Phi^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right], \qquad n \geq 0.\end{aligned}$$

2) *In the case of such (random) quasi-nilpotent isometry we get*

$$\begin{aligned}\mathbb{E}\left[\delta(u)\Phi(\delta(u))\right] &= \langle u, u\rangle \mathbb{E}\left[\Phi'(\delta(u))\right] \\ &+ \mathbb{E}\left[\Phi^{(n+1)}(\delta(u))\langle (Du)^n u, \delta(Du)\rangle\right] \\ &= \langle u, u\rangle \mathbb{E}\left[\Phi'(\delta(u))\right],\end{aligned}$$

which recovers the standard Gaussian integration by parts Üstünel and Zakai (1995) by induction on $n \geq 1$.

Stein approximation

Definition 3

Distances between probability measures.

1) The total variation distance between two real-valued random variables F and G is defined by

$$d_{TV}(F,G) := \sup_{A \in \mathcal{B}(\mathbb{R})} |P(F \in A) - P(G \in A)|.$$

2) The Wasserstein distance between the laws of F and G is defined by

$$d_W(F,G) := \sup_{h \in \mathrm{Lip}(1)} |\mathrm{E}[h(F)] - \mathrm{E}[h(G)]|,$$

where the supremum is taken over Lipschitz functions.

The integration by parts formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-x^2/2} dx$$

holds under the Gaussian measure, and the vanishing of the quantity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f'(x) - x f(x)) e^{-x^2/2} dx$$

can be used to characterize the Gaussian density.

Stein equation

1) When h is absolutely continuous with bounded derivative, the functional equation

$$f'(x) - xf(x) = h(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

admits a $\mathcal{C}_b^1(\mathbb{R})$ solution

$$f(x) = \mathrm{e}^{x^2/2} \int_{-\infty}^{x} (h(a) - \mathrm{E}[h(\mathcal{N})]) \mathrm{e}^{-a^2/2} \, \mathrm{d}a.$$

2) Let h be an absolutely continuous function with bounded derivative. The solution of the Stein equation

$$f'(x) - xf(x) = h(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

satisfies the bounds $\|f'\|_\infty \leq \|h\|_\infty$ and $\|f''\|_\infty \leq 2\|h\|_\infty$. By the Stein equation we can rewrite

$$\begin{aligned} \mathbb{E}[f_h'(F) - Ff_h(F)] &= \int_{-\infty}^{\infty} (f_h'(x) - xf_h(x)) \varphi(x) dx \\ &= \int_{-\infty}^{\infty} (h(x) - \mathbb{E}[h(\mathcal{N})]) \varphi(x) dx \\ &= \mathbb{E}[h(F)] - \mathbb{E}[h(\mathcal{N})]. \end{aligned}$$

Stein approximation

Proposition 5

Let $u \in D_{k,2}(H)$ for $k = 1, 2, 3$ We have

$$\begin{aligned} d_W(\delta(u), \mathcal{N}) &\leq \mathbb{E}\left[\left|1 - \langle u, u\rangle - \textbf{trace}\,(Du)^2\right|\right] \\ &+ \|u\|_2 \|D\langle u, u\rangle\|_2 + 2\mathbb{E}\left[|\langle (Du)u, \delta(Du)\rangle|\right]. \end{aligned}$$

By the relation

$$\Gamma_2^u \mathbf{1} = \langle u, u\rangle + \langle D^* u, Du\rangle_{H\otimes H} = \langle u, u\rangle + \textbf{trace}\,(Du)^2.$$

we also find

$$\begin{aligned} d_W(\delta(u), \mathcal{N}) &\leq \|1 - \langle u, u\rangle\|_2 + \|\textbf{trace}\,(Du)^2\|_2 \\ &+ \|u\|_2 \|D\langle u, u\rangle\|_2 + 2\mathbb{E}\left[|\langle (Du)u, \delta(Du)\rangle|\right]. \end{aligned}$$

(i) Quasi-nilpotent processes. When $\textbf{trace}\,(Du)^2 = 0$ we have

$$d_W(\delta(u), \mathcal{N}) \leq \mathbb{E}\left[|1 - \langle u, u\rangle|\right] + \|u\|_2 \|D\langle u, u\rangle\|_2 + 2\mathbb{E}\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

(ii) Random isometries. When $\langle u, u\rangle$ is deterministic we find

$$d_W(\delta(u), \mathcal{N}) \leq |1 - \langle u, u\rangle| + \|\textbf{trace}\,(Du)^2\|_2 + 2\mathbb{E}\left[|\langle (Du)u, \delta(Du)\rangle|\right].$$

Stein approximation

Corollary 4

Let $u \in D_{k,2}(H)$ for $k = 1, 2, 3$. We have

$$\begin{aligned} d_W(\delta(u), \mathcal{N}) \quad &\leq \quad |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\left[\|u\|_H^2 + \textbf{trace}\,(Du)^2\right]} \\ &\quad + \|u\|_2 \|D\langle u, u\rangle\|_2 + 2\mathbb{E}\left[|\langle (Du)u, \delta(Du)\rangle|\right]. \end{aligned}$$

Example. Consider the SDE

$$dX_t = \sigma(X_t) dW_t, \qquad X_0 = x_0,$$

where $\sigma \in \mathcal{C}_b^1(\mathbb{R})$. We have $X_t \in \mathrm{Dom}(D)$, $t \in [0, T]$, and

$$D_r X_s = \mathbf{1}_{[0,s]}(r)\sigma(X_r) e^{\int_r^s \sigma'(X_u) dW_u - \int_r^s |\sigma'(X_u)|^2 du/2}, \qquad 0 \leq r \leq s.$$

Since $X_T = \delta(\mathbf{1}_{[0,T]}\sigma(X))$, we get

$$\begin{aligned} d_W(X_T, \mathcal{N}) \quad &\leq \quad \mathbb{E}\left[|1 - \langle \sigma(X), \sigma(X)\rangle|\right] + \|\sigma(X)\|_2 \|D\langle \sigma(X), \sigma(X)\rangle\|_2 \\ &\quad + 2\mathbb{E}\left[|\langle (D\sigma(X))\sigma(X), \delta(D\sigma(X))\rangle|\right]. \end{aligned}$$

The last term can be bounded as

$$\mathbb{E}[|\langle (D\sigma(X))\sigma(X), \delta(D\sigma(X))\rangle|] \leq \frac{T^{5/2}}{\sqrt{15}} \|\sigma\|_\infty^3 \|\sigma'\|_\infty^2 e^{T\|\sigma'\|_\infty/2}.$$

Conditional Stein approximation (with Q. She, SPL 2017)

Proposition 6

Let $n \geq 1$ and assume that $u \in D_{k,2}(H)$ for all $k = 1, \ldots, n+2$ and $\langle u, (Du)^k u \rangle = 0$ for $k = 1, \ldots, n+1$. Then for all $f \in \mathcal{C}_b^{n+1}(\mathbb{R})$ we have

$$\mathbb{E}_{|u|}[\delta(u) f(\delta(u))] = \mathbb{E}_{|u|}\Big[\Big(\langle u, u \rangle + \textbf{trace}\,(Du)^2\Big)\, f'(\delta(u))\Big]$$

$$+ \sum_{k=2}^{n} \mathbb{E}_{|u|}\Big[\langle D^* u, D((Du)^{k-1} u)\rangle f^{(k)}(\delta(u))\Big] + \mathbb{E}_{|u|}\Big[\langle (Du)^n u, \delta(Du)\rangle f^{(n+1)}(\delta(u))\Big].$$

Proof.

For F of the form $F = g\left(\int_0^T |u_t|^2 dt\right)$, $g \in \mathcal{C}_b^1(\mathbb{R})$ and $k \geq 1$, we have

$$\Gamma_k^u F = \mathbf{1}_{\{k=2\}} \langle u, u \rangle g\left(\int_0^T |u_t|^2 dt\right) + g'\left(\int_0^T |u_t|^2 dt\right) \int_0^T \langle D_t \int_0^T |u_s|^2 ds, (Du)^{k-1} u_t \rangle_{\mathbb{R}^d} dt$$

$$+ g\left(\int_0^T |u_t|^2 dt\right) \langle D^* u, D((Du)^{k-2} u)\rangle$$

$$= \mathbf{1}_{\{k=2\}} \langle u, u \rangle F + \mathbf{1}_{\{k \geq 2\}} \langle D^* u, D((Du)^{k-2} u)\rangle F.$$

□

Conditional Stein approximation

Proposition 7

Let $u \in \bigcap_{k=1}^{3} D_{k,2}(H)$*, such that* $\langle u, (Du)u \rangle = \langle u, (Du)^2 u \rangle = 0$*. We have*

$$d_{|u|}(\delta(u), \mathcal{N}_{g(\|u\|)}) \leq \frac{1}{\sqrt{g(\|u\|)}} \left| g(\|u\|) - \mathrm{Var}_{|u|}[\delta(u)] \right| + \frac{2}{g(\|u\|)} \mathbb{E}_{|u|}[|\langle (Du)u, \delta(Du) \rangle|] . \tag{1}$$

Theorem 5

Let $u \in \bigcap_{k \geq 1} D_{k,1}(H)$ *be an adapted process such that* $\langle u, (Du)^k u \rangle_H = 0$*,* $k \geq 1$*. We have*

$$\mathbb{E}\left[\exp\left(i \int_0^T u_t dB_t \right) \,\Big|\, \int_0^T |u_t|^2 dt \right] = \exp\left(-\frac{1}{2} \int_0^T |u_t|^2 dt \right).$$

See Yor (1980), Driver et al. (2016) for Lévy's stochastic area.

Conditionally Gaussian stochastic integrals

Corollary 6

Assume that $A^\dagger A^2 = 0$. *We have*

$$\mathbb{E}\left[\exp\left(i\int_0^T AB_t dB_t\right)\,\middle|\,(|AB_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\,dt\right).$$

The condition $A^\dagger A^2 = 0$ includes 2-nilpotent matrices.

Proposition 8

Let $u_s = AW_s$, $s \in [0,T]$. *The condition* $\langle u, (Du)u\rangle = 0$ *is satisfied if and only if the* $d \times d$ *matrix* A *is 2-nilpotent, and in this case we have*

$$d_{|u|}\left(\int_0^T AW_s dW_s, \mathcal{N}_{g(\|u\|)}\right) \leq \frac{1}{\sqrt{g(\|u\|)}}\left|g(\|u\|) - \int_0^T |AW_t|^2\,dt\right|.$$

Stein approximation - Standard Poisson processes (JTP 2019)

Let $(T_k)_{k\geq 1}$ denote the sequence of jump times of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$. The gradient operator D defined on random functionals

$$F \in \mathcal{S} := \left\{ F = f(T_1, \ldots, T_n) \ : \ f \in \mathcal{C}_b^1(\mathbb{R}^n) \right\},$$

as

$$D_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \frac{\partial f}{\partial x_k}(T_1, \ldots, T_n).$$

Let the operator $\widetilde{\nabla}$ be defined as

$$\widetilde{\nabla}_s u_t := D_s u_t - \dot{u}_t \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+,$$

where $\dot{u}_t$ denotes the time derivative of $t \longmapsto u_t$ with respect to t.

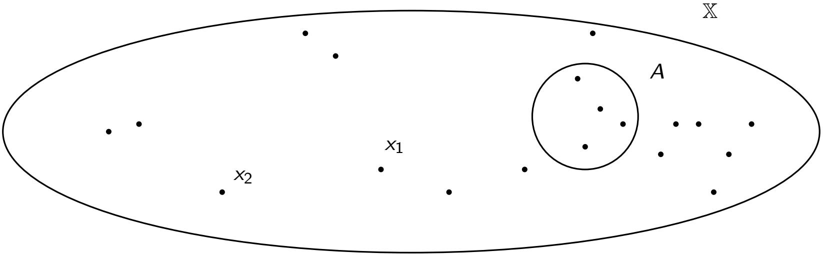
Proposition 9

Let $u \in \widetilde{D}_{2,1}(H)$ be adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We have

$$\begin{aligned} d(\delta(u), \mathcal{N}) \quad \leq \quad & |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\left[\|u\|_H^2\right]} + E\left[\left| \int_0^\infty u_s^3 ds + \left\langle u, D \int_0^\infty u_t^2 dt \right\rangle \right|\right] \\ & + 2E\left[|\langle (\widetilde{\nabla} u) u, \delta(\widetilde{\nabla}^* u) \rangle| \right]. \end{aligned}$$

Poisson point process

Let $\Omega^{\mathbb{X}} = \left\{ \omega = \sum_{k \geq 1} \delta_{x_k} \right\}$ denote the space of locally finite configurations on $\mathbb{X} = \mathbb{R}^d$.



Each element ω of $\Omega^{\mathbb{X}}$ is identified to $\omega = \sum_{x \in \omega} \delta_{x_i}$, where δ_x denotes the Dirac measure at $x \in \mathbb{X}$,

with $\omega(A) = \#\{k \ : \ x_k \in A\}$, and

$$\int_{\mathbb{X}} f(x)\omega(dx) = \sum_{x \in \omega} f(x).$$

For all compact disjoint subsets $A_1, \ldots, A_n$ of $\mathbb{X}$, $n \geq 1$, the mapping

$$\omega \mapsto (\omega(A_1), \ldots, \omega(A_n))$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective intensities $\lambda(A_1), \ldots, \lambda(A_n)$.

Stein approximation - Poisson point processes (ALEA 2018)

Let $d \geq 2$ and $0 < R < R' := 2R$, and consider the gradient operator D defined on random functionals $F \in \mathcal{S}$ as

$$D_y F := \sum_{n=1}^{\infty} \mathbf{1}_{\{\gamma(B(R))=n\}} \sum_{i=1}^{n} \langle \mathrm{G}_\eta(X_i, y), \nabla_{x_i}^{\mathbb{R}^d} f(X_1, \ldots, X_n) \rangle_{\mathbb{R}^d}, \qquad y \in B(R),$$

and the operator $\widetilde{\nabla}$ is defined on $u \in \mathcal{P}_0$ as $\widetilde{\nabla}_y u_x := D_y u_x + \langle \mathrm{G}_\eta(x, y), \nabla_x^{\mathbb{R}^d} u_x \rangle_{\mathbb{R}^d}$, $x, y \in B(R)$.

Proposition 10

For any random field $u \in \widetilde{D}_0^{1,\infty}$ we have

$$d_W(\delta(u), \mathcal{N}) \leq \mathbb{E}\left[\left|1 - \langle u, u \rangle - \langle \widetilde{\nabla}^* u, Du \rangle\right|\right] + \mathbb{E}\left[\left|\int_{B(R)} u_x^3 \, \lambda(dx) + \left\langle u, D \int_{B(R)} u_x^2 \, \lambda(dx) \right\rangle\right|\right]$$
$$+2\mathbb{E}\left[\left|\langle (\widetilde{\nabla} u) u, \delta(\widetilde{\nabla}^* u) \rangle\right|\right].$$

Given $\eta \in \mathcal{C}_0^\infty(B(R'))$ such that $\int_{B(R)} \eta(x) dx = 1$, $\mathrm{G}_\eta(x, y)$ is the kernel

$$\mathrm{G}_\eta(x, y) := \int_0^1 \frac{(x-y)}{s} \eta\left(y + \frac{x-y}{s}\right) \frac{ds}{s^d}, \qquad x, y \in B(R'),$$

see Acosta and Durán (2017), such that for $h \in \mathcal{C}_0^\infty(B(R))$ we have

$$h(y) = \int_{B(R')} \langle \mathrm{G}_\eta(x, y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dx), \qquad y \in B(R').$$

Moments of Poisson stochastic integrals

For Poisson stochastic integrals with respect to the Poisson random measure with intensity $\sigma(dx)$ on $\mathbb{X}$, the logarithmic generating function

$$\log \mathbb{E}\left[\exp\left(\int_{\mathbb{X}} h(x)\omega(dx)\right)\right] = \int_{\mathbb{X}} (e^{h(x)} - 1)\sigma(dx) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}} h^n(x)\sigma(dx),$$

shows that the cumulants of $\int_{\mathbb{X}} h(x)\omega(dx)$ are given by

$$\kappa_n^X = \int_{\mathbb{X}} h^n(x)\sigma(dx), \qquad n \geq 1,$$

and we have the moment identity

$$\mathbb{E}\left[\left(\int_{\mathbb{X}} h(x)\omega(dx)\right)^n\right] = \sum_{k=1}^{n} \sum_{P_1^n \cup \cdots \cup P_k^n = \{1,\ldots,n\}} \int_{\mathbb{X}} h^{|P_1^n|}(x)\sigma(dx) \cdots \int_{\mathbb{X}} h^{|P_k^n|}(x)\sigma(dx),$$

where the sum runs over all partitions $P_1^n, \ldots, P_k^n$ of $\{1, \ldots, n\}$, cf. Bassan and Bona (1990).

Nonlinear Mecke Identity (PMS 2012)

Given $\mathfrak{z}_n = (z_1, \ldots, z_n) \in \mathbb{X}^n$, we use the shorthand notation $\varepsilon^+_{\mathfrak{z}_n}$ for the operator

$$(\varepsilon^+_{\mathfrak{z}_n} F)(\omega) = F(\omega \cup \{z_1, \ldots, z_n\}), \qquad \omega \in \Omega,$$

where F is a random variable on $\Omega^{\mathbb{X}}$. We also use the finite difference operator

$$\mathbf{D}_x F(\omega) = F(\omega \cup \{x\}) - F(\omega).$$

Mecke identity:

$$\mathbb{E}\left[\sum_{x \in \omega} u(x, \omega)\right] = \mathbb{E}\left[\int_{\mathbb{X}} \varepsilon^+_x u(x) \sigma(dx)\right].$$

where ε^+_x is the addition operator $\varepsilon^+_x F(\omega) = F(\omega \cup \{x\})$.

Proposition 11

For $n \geq 1$ we have

$$\mathbb{E}\left[\left(\sum_{x \in \omega} u(x, \omega)\right)^n\right] = \sum_{k=1}^{n} \sum_{P_1 \cup \cdots \cup P_k = \{1, \ldots, n\}} \mathbb{E}\left[\int_{\mathbb{X}^k} \varepsilon^+_{x_1, \ldots, x_k} (u^{|P_1|}(x_1) \cdots u^{|P_k|}(x_k)) \sigma(dx_1) \cdots \sigma(dx_k)\right],$$

where $\varepsilon^+_{x_1, \ldots, x_k}$ is the addition operator $\varepsilon^+_{x_1, \ldots, x_k} F(\omega) = F(\omega \cup \{x_1, \ldots, x_k\})$.

Case $p = 2$

By the Slivnyak-Mecke formula, we have

$$\left\langle \sum_{x_1 \in \xi} u_1(x_1, \xi) \sum_{x_2 \in \xi} u_2(x_2, \xi) \right\rangle = \left\langle \sum_{x_1 \in \xi} \left(\sum_{x_2 \in \xi} u_2(x_2, \xi) \right) u_i(x_1, \xi) \right\rangle$$
$$= \mathbb{E}\left[\int_{\mathbb{X}} \epsilon^+_{x_1} \left(\sum_{x_2 \in \xi} u_2(x_2, \xi) u_1(x_1, \xi) \right) \mu(\mathrm{d}x_1) \right],$$

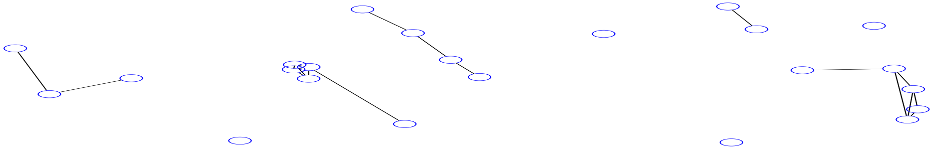
with

$$\epsilon^+_{x_1} \sum_{x_2 \in \xi} u_2(x_2, \xi) = \sum_{x_2 \in \xi} \epsilon^+_{x_1} u_2(x_2, \xi) + \epsilon^+_{x_1} u_2(x_1, \xi).$$

Hence, applying again the Slivnyak-Mecke formula we have

$$\left\langle \sum_{x_1 \in \xi} u_1(x_1, \xi) \sum_{x_2 \in \xi} u_2(x_2, \xi) \right\rangle$$
$$= \sum_{\pi_1 = \{1\}} \mathbb{E}\left[\int_{\mathbb{X}} \sum_{x_2 \in \xi} \epsilon^+_{x_1} (u_2(x_2, \xi) u_1(x_1, \xi)) \mu(\mathrm{d}x_1) \right] + \sum_{\pi_1 = \{1\}} \mathbb{E}\left[\int_{\mathbb{X}} \epsilon^+_{x_1} (u_2(x_1, \xi) u_1(x_1, \xi)) \mu(\mathrm{d}x_1) \right]$$
$$= \sum_{\pi_1 = \{1\}} \mathbb{E}\left[\int_{\mathbb{X}^2} \epsilon^+_{x_1} (u_2(x_2, \xi) u_1(x_1, \xi)) \mu(\mathrm{d}x_1) \right] + \sum_{\pi_1 = \{1\}} \mathbb{E}\left[\int_{\mathbb{X}} \epsilon^+_{x_1} (u_2(x_1, \xi) u_1(x_1, \xi)) \mu(\mathrm{d}x_1) \right].$$

Random-connection model (JAP 2019)



Two point process vertices $x \neq y$ are connected ($x \leftrightarrow y$) with probability $H(x,y)$, independently of $\omega(dx)$:

$$\mathbb{E}\left[\epsilon^+_{\mathfrak{x}_r}\epsilon^+_{\mathfrak{y}_r}\prod_{i=1}^{r}\prod_{j=1}^{i-1}\mathbb{1}_{\{x_i \leftrightarrow y_j\}}(\omega)\,\Big|\,\omega\right] = \prod_{i=1}^{r}\prod_{j=1}^{i-1}H(x_i,y_j),$$

for distinct $x_1,\ldots,x_r,y_1,\ldots,y_{r-1} \in \mathbb{X}$.

Given $x,y \in \mathbb{X}$, the number of $(r+1)$-hop sequences $z_1,\ldots,z_r \in \omega$ of vertices connecting x to y in the random graph is the multiparameter stochastic integral

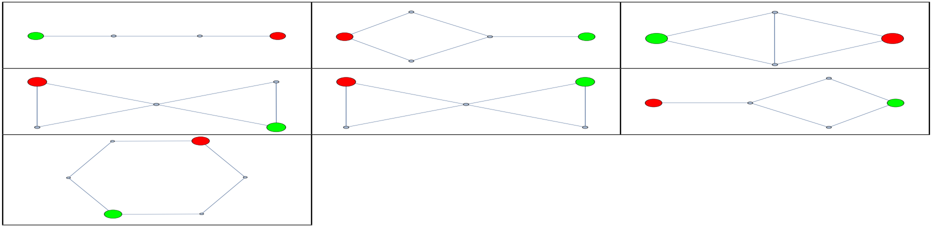
$$N^{x,y}_{r+1} = \int_{\mathbb{X}^r} u(z_1,\ldots,z_r)\omega(dz_1)\cdots\omega(dz_r)$$

of the r-process

$$u(z_1,\ldots,z_r,\omega) := \mathbb{1}_{\{z_i \neq z_j,\ 1\leq i<j\leq r\}}\mathbb{1}_{\{z_1,\ldots,z_r\in\omega\}}\prod_{i=0}^{r}\mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\omega)$$

which vanishes on the diagonals in $\mathbb{X}^r$, with $z_0 := x$ and $z_{r+1} := y$.

Computing the second moment of a 3-hop count requires to identify and count the 7 possible multigraphs that can connect x to y via two 3-hop paths with possible common nodes as in the next figure, in which every path in each multigraph is followed from the green node x to the red node y. Common nodes break independence and have to be dealt with separately.



We develop a systematic approach to the computation of the moments of the $r+1$-hop count $N_{r+1}^{x,y}$ using non-flat partitions, via the combinatorial identity

$$\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^n\right] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^n u(z^\rho_{\pi_k})\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

when the process $u(z_1,\ldots,z_r;\omega)=0$ vanishes on diagonals whenever, excluding the count of connections of a node to itself.

The above sum is over *non-flat* partitions ρ in the set $\Pi[n\times r]$ of partitions of $\{1,\ldots,n\}\times\{1,\ldots,r\}$ (Bogdan et al. (2017)).

Variance of 3-hop counts

We recover Theorem II.2 of Kartun-Giles and Kim (2018) for the variance of 3-hop counts by a shorter argument.

Corollary 7

The variance of the 3*-hop count between* $x \in \mathbb{X}$ *and* $y \in Y$ *is given by*

$$\begin{aligned} \mathrm{Var}\left[N_3^{x,y}\right] &= 2\lambda^3\left(\frac{\pi^3}{8\beta^3}\right)^{d/2} e^{-\beta\|x-y\|^2/2} + \lambda^2\left(\frac{\pi^2}{3\beta^2}\right)^{d/2} e^{-\beta\|x-y\|^2/3} \\ &\quad + 2\lambda^3\left(\frac{\pi^3}{12\beta^3}\right)^{d/2} e^{-3\beta\|x-y\|^2/4} + \lambda^2\left(\frac{\pi^2}{8\beta^2}\right)^{d/2} e^{-\beta\|x-y\|^2}. \end{aligned}$$

Random stopping sets (ESAIM 2021)

We consider (possibly random) sets A based on a Poisson point process, and let $N(\mathrm{A})(\omega)$ denote the cardinality of $\omega \cap \mathrm{A}(\omega)$. We first consider the factorial moment $\mathbb{E}[N(\mathrm{A})_{(n)}]$.

Proposition 12

(Breton and Privault (2014)) Let A be a random measurable subset of X. For all $n \geq 1$ and sufficiently integrable random variable F, we have

$$\mathbb{E}\Big[F\ N(\mathrm{A})_{(n)}\Big] = \mathbb{E}\left[\int_{X^n} \varepsilon^+_{\mathfrak{x}_n}(F\mathbf{1}_{\mathrm{A}^n}(x_1,\ldots,x_n))\,\hat{\sigma}^n(dx_1,\ldots,dx_n)\right].$$

Corollary 8

For A a random set and F a bounded random variable, we have

$$\mathbb{E}\Big[F(1+t)^{N(\mathrm{A})}\Big] = \mathbb{E}[F] + \sum_{k=1}^{\infty} \frac{t^k}{k!}\mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(F\mathbf{1}_{\mathrm{A}^k}(x_1,\ldots,x_k))(\omega)\,\hat{\sigma}^k(dx_1,\ldots,dx_k)\right], \quad t \in (-2,0).$$

Random stopping sets

This corollary allows us to recover the distribution of the discrete random variable $N(\mathrm{A})$.

Corollary 9

For A *a random set and* F *a bounded random variable, we have*

$$\mathbb{E}\left[F\mathbf{1}_{\{N(\mathrm{A})=n\}}\right] = \frac{1}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mathbb{E}\left[\int_{X^{k+n}} \varepsilon^{+}_{\mathfrak{x}_{k+n}}(F\mathbf{1}_{\mathrm{A}^{k+n}}(x_1,\ldots,x_{k+n}))(\omega)\,\hat{\sigma}^{k+n}(dx_1,\ldots,dx_{k+n})\right],$$

$n \geq 0$.

Given K in the collection $\mathcal{K}(X)$ of compact subsets of X, let

$$\mathcal{F}_K := \sigma(\omega(U) \ : \ U \subset K, \ \sigma(U) < \infty)$$

denote the sigma-algebra generated by $\omega \mapsto \omega(U)$, with $U \subset K$ and $\sigma(U) < \infty$.

Definition 10

(Zuyev (1999), Molchanov (2005)).

1) A random compact set S is called a stopping set if

$$\{\omega \in \Omega^X \ : \ \mathrm{S}(\omega) \subset K\} \in \mathcal{F}_K \quad \text{for all } K \in \mathcal{K}(X).$$

2) Given S a stopping set, we consider the stopped sigma-algebra

$$\mathcal{F}_{\mathrm{S}} := \sigma(B \in \mathcal{F} \ : \ B \cap \{\omega \in \Omega^X \ : \ \mathrm{S}(\omega) \subset K\} \in \mathcal{F}_K, \ K \in \mathcal{K}(X)).$$

In addition to the stopping set property, we will need the following two conditions.

Definition 11

1. A stopping set S is said to be non-increasing if

$$\mathrm{S}(\omega \cup \{x\}) \subset \mathrm{S}(\omega), \qquad \omega \in \Omega^X, \quad x \in X.$$

2. A stopping set S is said to be stable if

$$x \in \mathrm{S}(\omega) \Longrightarrow x \in \mathrm{S}(\omega \cup \{x\}), \qquad \omega \in \Omega^X, \quad x \in X.$$

The above monotonicity and stability conditions are satisfied by common examples of stopping sets, starting with deterministic compact subsets of X.

Proposition 13

The complement $\overline{\mathrm{S}}$ of a stable and non-increasing stopping set S fulfills the condition

$$\varepsilon^+_{\mathfrak{x}_n}(\mathbf{1}_{\overline{\mathrm{S}}}(x_1)\cdots\mathbf{1}_{\overline{\mathrm{S}}}(x_n)) = \mathbf{1}_{\overline{\mathrm{S}}}(x_1)\cdots\mathbf{1}_{\overline{\mathrm{S}}}(x_n), \quad x_1,\ldots,x_n \in X, \quad n \geq 1.$$

From Proposition 13 we find the following consequences of Corollaries 8 and 9, starting with the next factorial moment identity.

Proposition 14

Let $\overline{\mathrm{S}}$ be the complement of a stable, non-increasing stopping set S. For all $n \geq 1$, we have

$$\mathbb{E}\Big[F \; N(\overline{\mathrm{S}})_{(n)}\Big] = \mathbb{E}\bigg[\int_{\overline{\mathrm{S}}^n} \varepsilon_x^+ F \; \hat{\sigma}^n(d\mathfrak{x}_n)\bigg],$$

for F a bounded random variable.

From Proposition 14 we recover the distribution of $N(\overline{\mathrm{S}})$.

Corollary 12

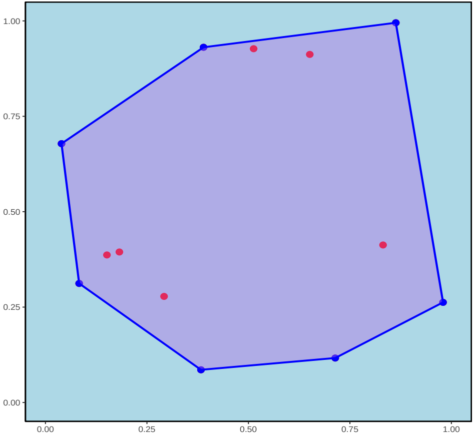
Let $\overline{\mathrm{S}}$ be the complement of a stable and non-increasing stopping set S. We have

$$\mathbb{P}(N(\overline{\mathrm{S}}) = n \mid \mathcal{F}_{\mathrm{S}}) = \frac{\mathrm{e}^{-(\sigma(\overline{\mathrm{S}}))}}{n!}(\sigma(\overline{\mathrm{S}}))^n, \qquad n \in \mathbb{N}.$$

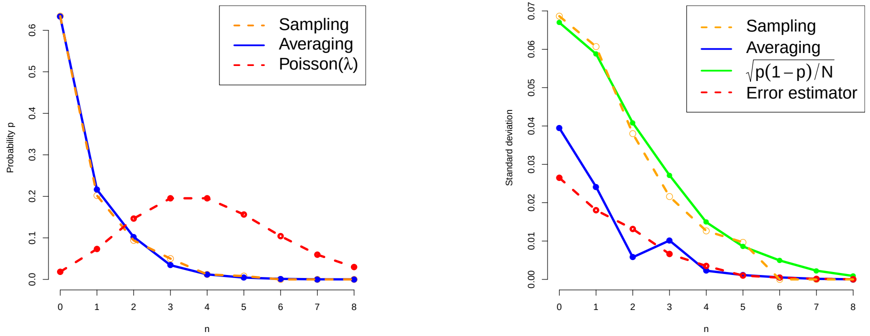
We estimate the distribution $\mathbb{P}(N(\overline{\mathrm{S}}) = n)$ of the number of Poisson vertices inside the complement $\overline{\mathrm{S}}$ of a stopping set S using both the standard sampling estimator $\mathbf{1}_{\{N(\overline{\mathrm{S}})=n\}}$ and the alternative estimator

$$\mathbb{P}(N(\overline{\mathrm{S}}) = n \mid \mathcal{F}_{\mathrm{S}}) = \frac{(\sigma(\overline{\mathrm{S}}))^n}{n!} \mathrm{e}^{-\sigma(\overline{\mathrm{S}})}. \tag{2}$$

obtained from Corollary 12. The closed complement $\mathrm{S} = \overline{\mathrm{A}}$ of the (open) convex hull $\overline{\mathrm{S}} = \mathrm{A}$ of a Poisson point process in a convex domain X of finite intensity measure in $\mathbb{R}^d$ is a stable and non-increasing stopping set, see Figure 1.1.

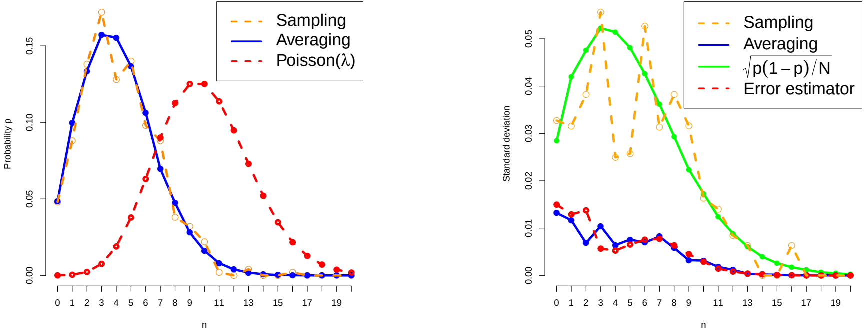


In the sequel, we consider a Poisson point process with flat intensity $\lambda > 0$ on the unit square $X = [0,1]^2$. Using the estimator $\mathbf{1}_{\{N(\overline{\mathrm{S}})=n\}}$ ("Sampling") and the alternative estimator (2) ("Averaging"), the following simulations provide estimates for the distribution $\mathbb{P}(N(\overline{\mathrm{S}}) = n)$ of the count of points strictly inside the convex hull $\overline{\mathrm{S}}$ complement of S, generated by the Poisson point process on $X = [0,1]^2$, see in Figures 1.2a and 1.3a.



(a) Probability distribution (b) Standard error

Figure 1.2: Distribution and standard error for the inside of the Poisson convex hull, $\lambda = 4$.



(a) Probability distribution

(b) Standard error

Figure 1.3: Distribution and standard error for the inside of the Poisson convex hull, $\lambda = 10$.

Consider the stopping set given by the Voronoi flower S based on a typical cell containing the point $(1/2, 1/2)$ in the unit square $X = [0, 1] \times [0, 1]$, up to a translation of the Poisson point process with flat intensity $\lambda > 0$. In case the window $X = [0, 1] \times [0, 1]$ does not contain any cell around the point $(1/2, 1/2)$ we let $\mathrm{S} = [0, 1] \times [0, 1]$, which is the case in particular when $\omega = \emptyset$.

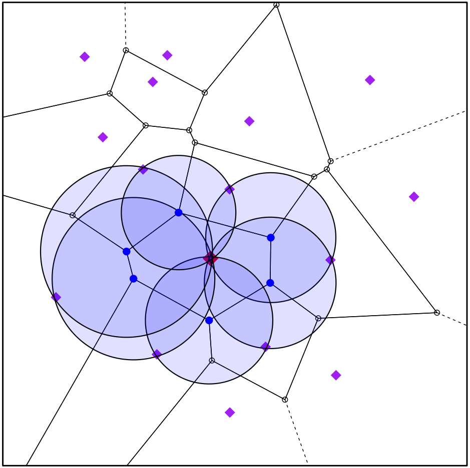


Figure 1.4: Voronoi flower S.

The next simulations provide estimates for the distribution $\mathbb{P}(N(\overline{\mathrm{S}}) = n)$ of the count of points in the complement $\overline{\mathrm{S}}$ of the Voronoi flower S around the point $(1/2, 1/2)$, generated by a Poisson point process with flat intensity $\lambda > 0$ on the unit square $X = [0,1]^2$.

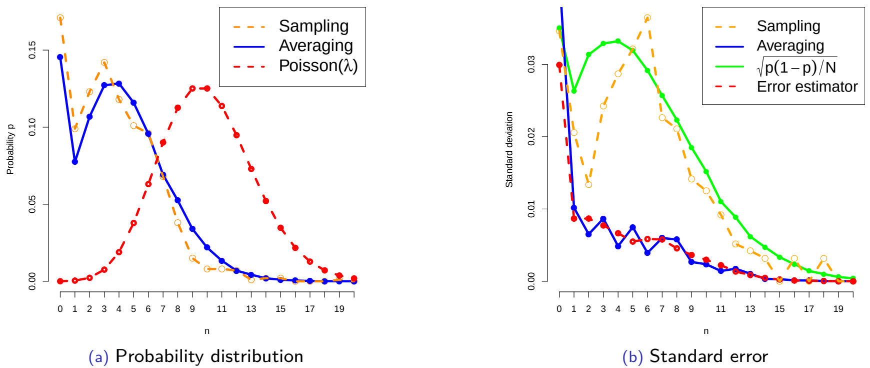


Figure 1.5: Distribution and standard error for the Voronoi flower complement, $\lambda = 10$.

We check that the estimator (2) ("Averaging") is more accurate, as it has a lower variance than standard sampling when estimating the count of points in the complement $\overline{\mathrm{S}}$ of the Voronoi flower S for two different values of the Poisson intensity parameter λ.

Moments of shot noise processes (BICY 2020)

We work in the multiple source model of Brigham and Destexhe (2015), based on a Poisson point process $\xi(dx)$ on the space

$$\Omega := \left\{ \xi = \{x_i\}_{i \in I} \subset \mathbb{X} \ : \ \#(A \cap \xi) < \infty \text{ for all compact } A \in \mathcal{B}(\mathbb{X}) \right\}$$

of locally finite configurations with the intensity measure $\mu(dt, d\theta)$ on $\mathbb{X} = \mathbb{R} \times S$, where $S = [0, N]$. We consider N shot noise conductance processes

$$Q_k(t, \xi) = \int_{(-\infty, t] \times S} g_k(t - s, \theta) \xi(ds, d\theta) = \sum_{(s_j, \theta_j) \in \xi} g_k(t - s_j, \theta_j), \quad k = 1, \ldots, N,$$

which represent the inputs of N conductance synapses. The shot noise kernels $g_k(u, \theta)$ represent the impulse response functions and are such that $g_k(u, \theta) = 0$ for $u < 0$, where $g_k(t - s, \theta)$ represents the leak conductance, the s_j's are the presynaptic events, and the θ_j's are modeling possible synaptic inhomogeneities. The system response is modeled by the membrane potential $Y_N(t, \xi)$ satisfying the unit-less shot noise SDE

$$\tau \frac{dY_N}{dt}(t, \xi) = -Y_N(t, \xi) + \sum_{k=1}^{N} (w_k - Y_N(t, \xi)) Q_k(t, \xi), \tag{3}$$

where $\tau > 0$ is the membrane time constant and $w_k \in \mathbb{R}$, $k = 1, \ldots, N$, represent the (renormalized) leak potentials.

Proposition 15

The solution of (3) *is given by the filtered shot noise process*

$$Y_N(t,\xi) = \frac{1}{\tau}\sum_{k=1}^{N} w_k \int_{-\infty}^{t} Q_k(z,\xi)\mathrm{e}^{-\int_z^t Q_0(u,\xi)du} dz, \qquad t \in \mathbb{R},$$

where

$$Q_0(u,\xi) := \frac{1}{\tau} + \frac{1}{\tau}\sum_{k=1}^{N} Q_k(u,\xi).$$

The next proposition gives a general formula for the computation of the joint moments of $Y_N(t_1,\xi),\ldots,Y_N(t_n,\xi)$ in the multiple source model.

Proposition 16

We have the joint moment identity

$$\langle Y_N(t_1,\xi)\cdots Y_N(t_n,\xi)\rangle = \frac{1}{\tau^n}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_n} m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n)dz_1\cdots dz_n, \tag{4}$$

where

$$m_{n,N}(z_1,\ldots,z_n;t_1,\ldots,t_n) := \left\langle \prod_{k=1}^{n}\left(\mathrm{e}^{-\int_{z_l}^{t_l} Q_0(u,\xi)du}\int_{(-\infty,z_k]\times S} f^{(w)}(z_k-u,\theta)\xi(du,d\theta)\right)\right\rangle.$$

The function $m_{n,N}(z_1, \ldots, z_n; t_1, \ldots, t_n)$ can be evaluated as a sum over the set $\Pi[n]$ of partitions $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ with cardinality $k = |\pi| = 1, \ldots, n$, as

$$m_{n,N}(z_1, \ldots, z_n; t_1, \ldots, t_n)$$

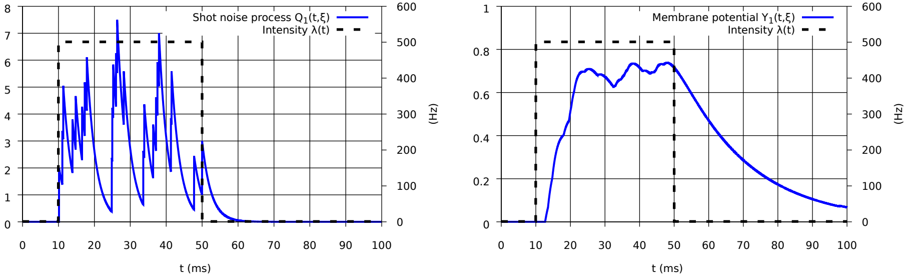
$$= \left\langle \mathrm{e}^{-\sum_{l=1}^{n} \int_{z_l}^{t_l} Q_0(u,\xi) du} \right\rangle \sum_{\pi \in \Pi[n]} \prod_{j=1}^{|\pi|} \int_{(-\infty, \hat{z}_{\pi_j}] \times S} \prod_{l=1}^{n} \mathrm{e}^{-\frac{1}{\tau} \int_{z_l}^{t_l} f(u-y,\eta) du} \prod_{i \in \pi_j} f^{(w)}(z_i - y, \eta) \mu(\mathrm{d}y, \mathrm{d}\eta$$

$(z_1, \ldots, z_n) \in (-\infty, t_1] \times \cdots \times (-\infty, t_n]$, with $\hat{z}_{\pi_j} = \min_{i \in \pi_j} z_i$, where, by the Lévy-Khintchine formula

$$\left\langle \mathrm{e}^{-\sum_{l=1}^{n} \int_{z_l}^{t_l} Q_0(u,\xi) du} \right\rangle$$

$$= \mathrm{e}^{-\frac{1}{\tau} \sum_{l=1}^{n} (t_l - z_l)} \exp\left(\int_{(-\infty, \max(t_1, \ldots, t_n)] \times S} (\mathrm{e}^{-\frac{1}{\tau} \sum_{l=1}^{n} \int_{z_l}^{t_l} f(u-s,\theta) du} - 1) \mu(ds, d\theta) \right).$$

Figure 1.6 presents random simulations of the shot noise process $Q_1(t,\xi)$ and membrane potential $Y_1(t,\xi)$ in the unit-less single source model with the parameters of Brigham and Destexhe (2015), plotted together with the intensity $\lambda(t) := \lambda \mathbf{1}_{[t_a,t_b]}(t)$.



(a) Shot noise process $Q_1(t,\xi)$.

(b) Membrane potential $Y_1(t,\xi)$.

Figure 1.6: Filtered shot noise processes and intensity function $\lambda(t) = \lambda \mathbf{1}_{[t_a,t_b]}(t)$.

Figure 1.7 presents numerical simulations of first moment and standard deviation in the unit-less single source model of Figure 1.6, together with the mean obtained by Monte Carlo simulations, and is consistent with Figure 1 in Brigham and Destexhe (2015).

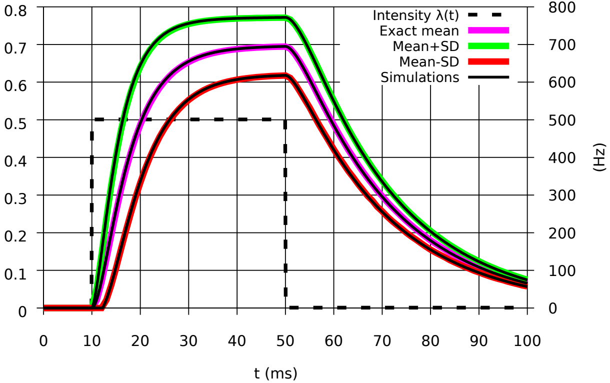


Figure 1.7: Mean and standard deviation of $Y_1(t,\xi)$ with $\lambda(t) = \lambda \mathbf{1}_{[t_a,t_b]}(t)$.

When $n = 3$ there are 5 partitions of $\{1, 2, 3\}$ which can be listed as

$\pi_1 =$ '123',

$\pi_1|\pi_2 =$ '12|3'; '1|23'; '13|2',

$\pi_1|\pi_2|\pi_3 =$ '1|2|3',

hence by (4) the third joint moment $\langle Y_N(t_1, \xi) Y_N(t_2, \xi) Y_N(t_3, \xi)\rangle$ can be computed from

$$\left\langle \mathrm{e}^{-\sum_{l=1}^{3}\int_{z_l}^{t} Q_0(u,\xi)du} \right\rangle \Bigg(\int_{(-\infty,\min(z_1,z_2,z_3)]\times S} \mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{3}\int_{z_l}^{t} f(u-y,\eta)du} \prod_{i=1}^{3} f^{(w)}(z_i - y, \eta)\mu(\mathrm{d}y, \mathrm{d}\eta)$$

$$+ 3\int_{(-\infty,\min(z_1,z_2)]\times S} \mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{3}\int_{z_l}^{t} f(u-y,\eta)du} f^{(w)}(z_1 - y, \eta) f^{(w)}(z_2 - y, \eta)\mu(\mathrm{d}y, \mathrm{d}\eta)$$

$$\times \int_{(-\infty,z_3]\times S} \mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{3}\int_{z_l}^{t} f(u-y,\eta)du} f^{(w)}(z_3 - y, \eta)\mu(\mathrm{d}y, \mathrm{d}\eta)$$

$$+ \prod_{j=1}^{3}\int_{(-\infty,z_j]\times S} \mathrm{e}^{-\frac{1}{\tau}\sum_{l=1}^{3}\int_{z_l}^{t} f(u-y,\eta)du} f^{(w)}(z_j - y, \eta)\mu(\mathrm{d}y, \mathrm{d}\eta)\Bigg). \tag{5}$$

For $n = 4$ there are 15 partitions of $\{1, 2, 3, 4\}$, which can be listed as

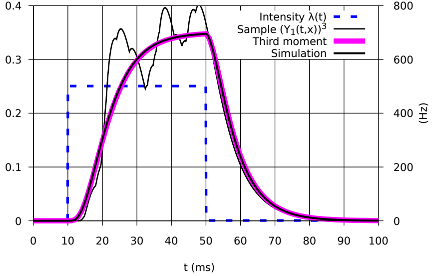
$\pi_1 =$ '1234',

$\pi_1|\pi_2 =$ '12|34'; '13|24'; '14|23'; '1|234'; '2|134'; '3|124'; '4|123',

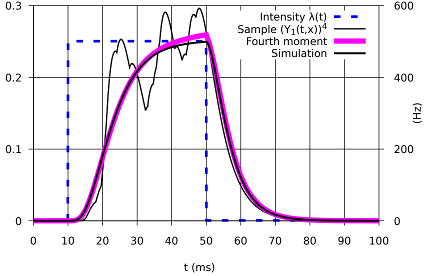
$\pi_1|\pi_2|\pi_3 =$ '1|2|34'; '1|3|24'; '2|3|14'; '2|4|12'; '1|4|23'; '3|4|12',

$\pi_1|\pi_2|\pi_3|\pi_4 =$ '1|2|3|4'.

The next Figure 1.8 presents numerical estimates of the exact third and fourth moment expressions for the membrane potential $Y_1(t, \xi)$ in the unit-less single source model of Figures 1.6-1.7.



(a) Third moments of $Y_1(t, \xi)$.



(b) Fourth moments of $Y_1(t, \xi)$.

Figure 1.8: Third and fourth moments of $Y_1(t, \xi)$.

Proposition 17

The Gram-Charlier expansion of the continuous probability density function $\phi_X(x)$ of a random variable X is given by

$$\phi_X(x) = \frac{1}{\sqrt{\kappa_2}}\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right) + \frac{1}{\sqrt{\kappa_2}}\sum_{n=3}^{\infty} c_n H_n\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right)\varphi\left(\frac{x-\kappa_1}{\sqrt{\kappa_2}}\right), \qquad (6)$$

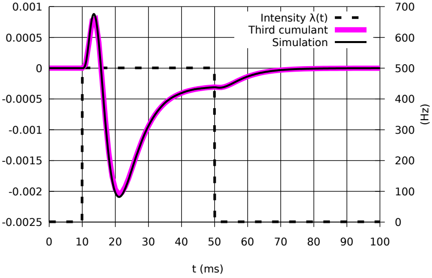
where the sequence $(c_n)_{n\geq 3}$ is given from the cumulants $(\kappa_n)_{n\geq 1}$ of X as

$$c_n = \frac{1}{\kappa_2^{n/2}} \sum_{m=1}^{[n/3]} \sum_{\substack{l_1+\cdots+l_m=n \\ l_1,\ldots,l_m\geq 3}} \frac{\kappa_{l_1}\cdots\kappa_{l_m}}{m!\,l_1!\cdots l_m!}, \qquad n \geq 3.$$

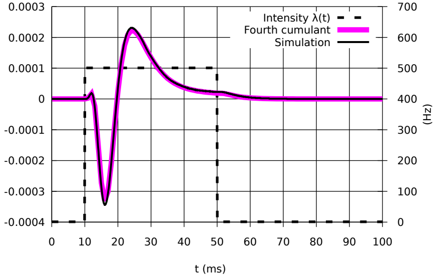
In particular, the coefficients c_3 and c_4 can be expressed from the skweness $\kappa_3/\kappa_2^{3/2}$ and the excess kurtosis κ_4/κ_2^2 as

$$c_3 = \frac{\kappa_3}{3!\kappa_2^{3/2}} \quad \text{and} \quad c_4 = \frac{\kappa_4}{4!\kappa_2^2}, \quad \text{with} \quad c_6 = \frac{\kappa_3^2}{2(3!)^2\kappa_2^3},$$

Figure 1.9 presents time-dependent numerical estimates of the third and fourth cumulant for the membrane potential $Y_1(t, \xi)$, based on exact moment expressions in the unit-less single source model of Figures 1.6 and 1.7.



(a) Third cumulants of $Y_1(t, \xi)$.



(b) Fourth cumulants of $Y_1(t, \xi)$.

Figure 1.9: Third and fourth cumulants of $Y_1(t, \xi)$.

Figure 1.10 presents numerical estimates of skewness $\langle\langle X^3\rangle\rangle/(\langle\langle X^2\rangle\rangle)^{3/2}$ and excess kurtosis $\langle\langle X^4\rangle\rangle/(\langle\langle X^2\rangle\rangle)^2$ obtained from exact moment expressions in the single source model. Negative skewness and positive excess kurtosis are observed starting after $t = 20ms$.

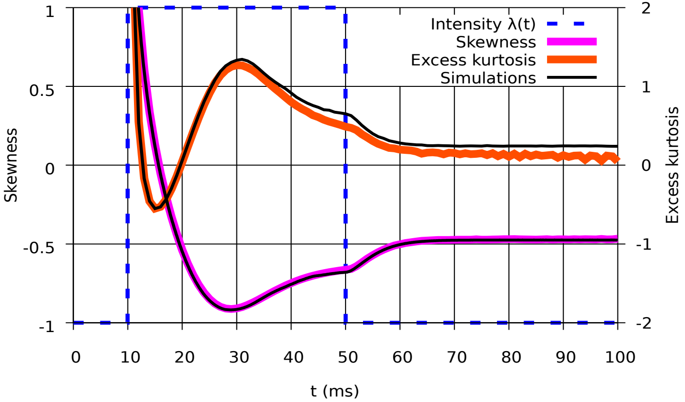
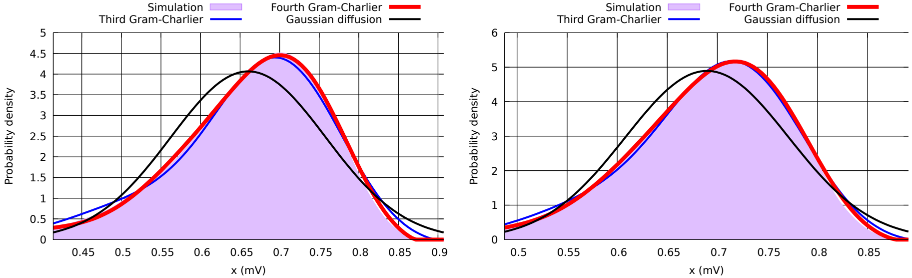


Figure 1.10: Skewness and excess kurtosis of $Y_1(t,\xi)$.

Figure 1.11 presents second, third and fourth-order Gram-Charlier expansions (6) based on exact moment expressions computed at different times, for the probability density function of the membrane potential $Y_1(t, \xi)$ in the single source model of Figures 1.6 and 1.7.



(a) t=30 ms.

(b) t=40 ms.

Figure 1.11: Gram-Charlier density expansions *vs* Monte Carlo density estimation.

The purple areas correspond to probability density estimates obtained by Monte Carlo simulations of the numerical solution of (3). The second-order expansions correspond to the Gaussian diffusion approximation obtained from first and second-order moments.

Actual probability density estimates obtained by simulation show significant differences from their Gaussian diffusion approximations.

Figure 1.12 presents time-dependent fourth-order Gram-Charlier expansions (6), based on exact moment formulas at different times, for the probability density function of $Y_1(t,\xi)$ in the single source model of Figure 1.6 and 1.7.

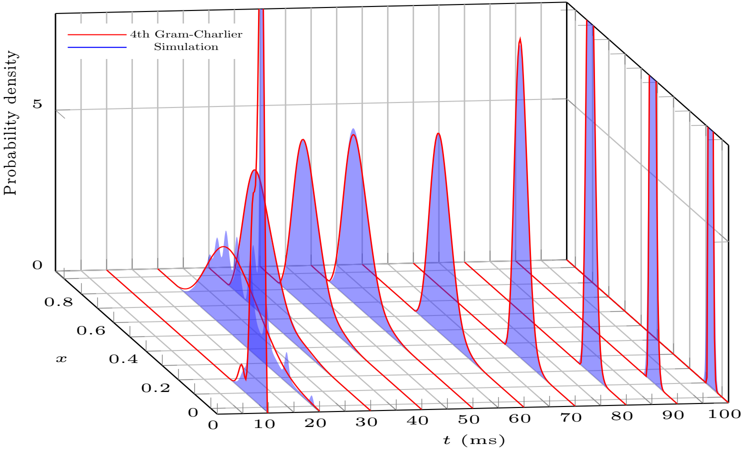


Figure 1.12: Fourth-order Gram-Charlier expansions vs simulated densities.

Fourth-order Gram-Charlier expansions appear to give the best fit to the actual probability densities, which have negative skewness and positive excess kurtosis.

Figure 1.13 shows the discrepancies over time between second and fourth-order Gram-Charlier expansions for the probability density function of $Y_1(t, \xi)$ in the unit-less single source model.

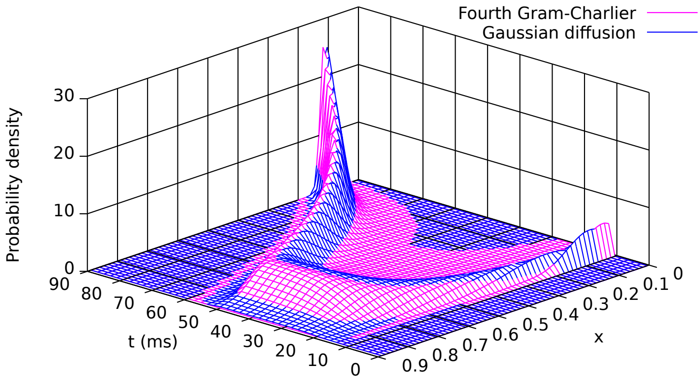


Figure 1.13: Fourth-order Gram-Charlier expansion *vs* diffusion approximation.

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