Cumulants for Lévy-type processes by Michael Anshelevich

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3 Ideas featured in M. Anshelevich's talk:

- In a theory, typically non-commutative, where it is a priori not clear how to define independence or convolutions, one can start with a plausible candidate for "Lévy processes" and use them to define cumulants and, as a consequence, independence and convolutions.
- Higher variations may play a role of cumulants for processes which do not have finite moments.
- For unbounded random variables, questions which are trivial in the commutative case are open, and seem to be difficult, in the non-commutative case.

Here: I would like to come back on the move to noncommutativity, on the various approaches to cumulants and on higher variations and discuss the problems that have been raised from a purely algebraic point of view, motivated by the use of (generalized) symmetric functions in *Higher variations for free Lévy processes* by M. Anshelevich and Z. Wang, 2020.

4 approaches to cumulants

- Cumulants as generators ("moments are exponentials of cumulants")
- Cumulants using transforms (Fourier-type)
- Cumulants from partitions
- Cumulants from higher variations

Is there possibly a common algebraic framework? Where to look for it?

The log/exp approach –groups and Hopf algebras

- Exponential generating series (classical case) $c(t) = \log(\mathbf{E}(\exp^{tX})) = \log(m(t))$
- Lie group interpretation: cumulants are of Lie-type, momenta of group-type
- One step beyond: Hopf algebras: momenta and cumulants viewed as formal variables, $\Delta(m(t)) = m(t) \otimes m(t)$, $\Delta(c(t)) = c(t) \otimes 1 + 1 \otimes c(t)$ (j.w. K. Ebrahimi-Fard, N. Tapia, L. Zambotti).
- This group-theoretic approach also holds for other cumulants (free, Boolean...) but one has to use nonstandard Lie algebras/Lie groups exponential and log maps (shuffle algebra approach, j.w. K. Ebrahimi-Fard).

Question: how to generalize these ideas? Is it possible to account for the higher variations approach? Tentative path...

Symmetric functions and beyond

- The Hopf algebra describing the momenta/cumulants relations is isomorphic to the Hopf algebra of symmetric functions (e.g. cumulants $c_n/n!$ corresponds to the power sum $p_n(X) = \sum_i x_i^n$, where X is a countable alphabet).
- Symmetric functions are closely related to commutative RB algebras (Rota Waring identity analysis of the Spitzer identity).
- Rota-Baxter algebras: $R(x)R(y) = R(xR(y) + R(x)y + \theta xy)$. In the free commutative Rota-Baxter algebra over x, $Rx^{[n]} := R(Rx^{[n-1]}x)$, $\Delta(Rx^{[n]}) := \sum_{i=0}^n Rx^{[i]} \otimes Rx^{[n-i]}$ defines a Hopf algebra structure isomorphic to the Hopf algebra of symmetric functions (on the algebra generated by the $Rx^{[n]}$).
- Interest of RB algebras: encode simultaneously integration (weight 0), for example rough paths-type iterated integrals AND summation of series: (f_n) in an associative algebra, $R(f)_n := \sum_{i=0}^{n-1} f_i$, weight -1: Possible approach to higher variation calculus.

Question: how to generalize these ideas? Is it possible to account for the higher variations approach? Tentative path...

Noncommutativity?

- Already pointed out: symmetric functions in noncommutative variables appear already in the study of higher variations (Anshelevich/Wang).
- Many noncommutative Hopf algebraic variants: NCSF, WQSym, FQSym...
- They are relevant to the study of noncommutative RB algebras
- One can develop the study of Lie algebras/Groups correspondences and analogs (for example) of the calculus with the lattice of partitions inside noncommutative RB algebras (jw K. Ebrahimi-Fard, J. Gracia-Bondia, D. Manchon).
- Not quite clear how these last observations should be formulated when dealing for example with generalizations of free or Boolean cumulants.

Hope: new developments at the interface Hopf algebras/generalized cumulant-momenta relations stimulated by the higher variations approach and more generally the various advanced notions of cumulants discussed in this conference.