# On the use of Boolean cumulants in the study of free random variables


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Online meeting on

Cumulants in Stochastic Analysis

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**Summary:**

I am presenting arXiv:1907.10842. This is joint work with

Maxime Fevrier (University Paris-Saclay, France),
Mitja Mastnak (St. Mary's University, Halifax, Canada), and
Kamil Szpojankowski (Warsaw University of Technology, Poland).

The goal of the talk is to show how *Boolean* cumulants can be used in order to address operations with *freely independent* random variables, particularly in connection to the $*$-distribution of the product of two selfadjoint freely independent random variables, and in connection to the distribution of the anti-commutator of such random variables.

**Some free probabilistic terminology.**

Simplified look at free probability: one has $(*) + (**)$, with

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- For such $(\mathcal{A}, \varphi)$, refer to elements of $\mathcal{A}$ as *noncommutative random variables*. For $a \in \mathcal{A}$ and $n \in \mathbb{N}$, think of $\varphi(a^n)$ as *moment of order n* of $a$.

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- $a, b \in \mathcal{A}$ are declared to be *freely independent* when the subalgebras $\mathcal{M}$ and $\mathcal{N}$ are so, with $\mathcal{M} = \text{span}\{a^n \mid n \geq 0\}$ and $\mathcal{N} = \text{span}\{a^n \mid n \geq 0\}$.

**Free independence as a rule for calculation.**

$(\mathcal{A}, \varphi)$ noncomm. prob. space. Free independence of unital subalgebras $\mathcal{M}, \mathcal{N} \subseteq \mathcal{A}$ can be viewed as a rule for computing $\varphi$ of an alternating product of elements from $\mathcal{M}$ and $\mathcal{N}$.

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- For every $x \in \mathcal{M}$ and $y \in \mathcal{N}$ one has $\varphi(xy) = \varphi(x) \cdot \varphi(y)$.
- For every $x_1, x_2 \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{N}$ one has
$$\begin{aligned}\varphi(x_1 y_1 x_2 y_2) = \; & \varphi(x_1 x_2) \cdot \varphi(y_1)\varphi(y_2) \\ & + \varphi(x_1)\varphi(x_2) \cdot \varphi(y_1 y_2) \\ & - \varphi(x_1)\varphi(x_2) \cdot \varphi(y_1)\varphi(y_2).\end{aligned}$$

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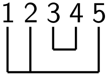
- **Question** (some kind of "free harmonic analysis"): given $a, b \in \mathcal{A}$, freely independent, compute the moments of $a + b$, of $ab$, of $ab + ba$ ...

## Non-crossing partitions and free cumulants.

**Important Idea** *(going back to Speicher, in the 1990's).*
*On a combinatorial level, calculations with freely independent elements are best done by using non-crossing partitions and free cumulants.*

**Notation.** $NC(n) :=$ the set of all *non-crossing partitions* of $\{1, 2, \ldots, n\}$. $NC(n)$ is partially ordered by *reverse refinement* ("$\pi \leq \sigma$" means that every block of $\pi$ is contained in a block of $\sigma$).

**Example** of $\pi \leq \sigma$ in $NC(5)$: $\pi =$  , $\sigma =$ 


$\pi$ of this example is an *interval partition* (its blocks have no *nestings*).

**Definition** *(free cumulant functionals).* $(\mathcal{A}, \varphi)$ noncomm. prob. space. There exists a sequence of multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, uniquely determined, such that for every $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has

$$\text{(M-FC)} \quad \varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n),$$

with $\kappa_\pi(a_1, \ldots, a_n) := \prod_{V \in \pi} \kappa_{|V|}(\,(a_1, \ldots, a_n) \mid V\,)$.

$\kappa_n$ is called the $n$th *free cumulant functional* of $(\mathcal{A}, \varphi)$. (M-FC) is a "moment–cumulant formula" (for free cumulants).

**Example:** for $n = 5$, the sum on right-hand side of (M-FC) has 42 terms (Catalan number!), and for $\pi =$  one gets the term $\kappa_\pi(a_1, \ldots, a_5) = \kappa_3(a_1, a_2, a_5)\kappa_2(a_3, a_4)$.

## Main theorem about free cumulants

**Definition.** $(\mathcal{A}, \varphi)$ noncomm prob space, $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ unital subalgebras. We say that $\mathcal{A}_1, \ldots, \mathcal{A}_s$ have *vanishing mixed free cumulants* to mean that whenever we pick:

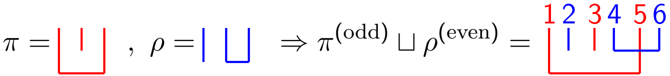
$$\left\{ \begin{array}{c} \text{an } n \geq 2, \text{ a non-constant colouring } c : \{1, \ldots, n\} \to \{1, \ldots, s\} \\ \text{and some } a_1 \in \mathcal{A}_{c(1)}, \ldots, a_n \in \mathcal{A}_{c(n)} \end{array} \right\}$$

it follows that $\kappa_n(a_1, \ldots, a_n) = 0$.

**Theorem** *(Speicher, 1994).* $(\mathcal{A}, \varphi)$ noncomm prob space, $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ unital subalgebras. Then $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are freely independent if and only if they have vanishing mixed free cumulants.

## Kreweras complement and the formula for $\kappa_n(ab)$.

Starting with $\pi, \rho \in NC(n)$, one can define a partition (with possible crossings!) $\pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})}$ of $\{1, \ldots, 2n\}$. E.g for $n = 3$:




**Definition.** For every $\pi \in NC(n)$, the set

$$\left\{\rho \in NC(n) \mid \pi^{(\mathrm{odd})} \sqcup \rho^{(\mathrm{even})} \in NC(2n)\right\}$$

has a maximal element $\rho_{\max}$ with respect to reverse refinement order. This $\rho_{\max}$ is called the *Kreweras complement* of $\pi$, and denoted as $\mathrm{Kr}(\pi)$. For example, $\mathrm{Kr}\Big($  $\Big) =$  .

**Formula for the free cumulant $\kappa_n(ab)$.**

$(\mathcal{A}, \varphi)$ noncomm prob space. For $a \in \mathcal{A}$ and $n \in \mathbb{N}$ write for short "$\kappa_n(a)$" instead of $\kappa_n(a, \ldots, a)$. Known since the 1990's: if $a, b \in \mathcal{A}$ are freely independent, then one has

$$(\diamondsuit) \qquad \kappa_n(ab) = \sum_{\pi \in NC(n)} \prod_{U \in \pi} \kappa_{|U|}(a) \cdot \prod_{V \in \mathrm{Kr}(\pi)} \kappa_{|V|}(b), \quad n \in \mathbb{N}.$$

Upon going to formal power series, one can use $(\diamondsuit)$ to derive the multiplicativity of the well-known $S$-transform of Voiculescu.

How is $(\diamondsuit)$ obtained? Strategy with 3 steps:
(1) Use formula for free cumulants with products as entries.
(2) Prune terms, based on vanishing of mixed free cumulants.
(3) Do a final combinatorial analysis of the terms that were left.

**Description of the 3-step strategy for getting ($\diamondsuit$).**

Step 1 is a formula of Krawczyk-Speicher (analogous to formula of Leonov-Shyriaev from classical cumulants):

$$\kappa_n(ab, ab, \ldots, ab) = \sum_{\sigma} \kappa_\sigma(a, b, a, b, \ldots, a, b),$$

with $\sigma$ running in $NC(2n)$ and subjected to the condition that

$$\sigma \vee \big\{ \{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\} \big\} = \big\{ \{1, 2, \ldots, 2n\} \big\}.$$

Step 2 forces $\sigma$ to separate $\{1, 3, \ldots, 2n-1\}$ from $\{2, 4, \ldots, 2n\}$.

Step 3 forces $\sigma$ to be of the form $\pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(\mathrm{even})}$ with $\pi \in NC(n)$, and the formula ($\diamondsuit$) is obtained.

**A problem: 3-step strategy fails to extend.**

It comes up naturally (e.g. when $\mathcal{A}$ has $*$-operation and $a, b$ are selfadjoint) that we need to keep track of a larger collection of free cumulants, with entries that are either $ab$ or $ba$.

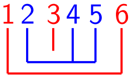
Instructive case study: $\kappa_3(ab, ab, ba) = ?$

Try the 3-step strategy. Steps 1 and 2 work fine, but the combinatorial analysis in Step 3 suffers from the absence of a suitable version of Kreweras complementation.

## Case study: 3-step strategy for $\kappa_3(ab, ab, ba)$?

Problem is in Step 3: when we fix a partition $\pi$ of the red dots, there is no control on the matching partitions of the blue dots which fulfil the conditions from Steps 1 and 2.

For instance $\pi =$  has 3 possibilities of a matching $\rho$:



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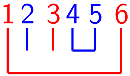

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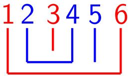

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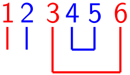
But $\pi =$  has no matching $\rho$'s at all, because the join condition from Step 1 can never be satisfied:


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- This alternative approach uses *Boolean cumulants*.
- It turns out there is a tractable formula for Boolean cumulants with freely independent entries.
- As a consequence, one can do "free harmonic analysis with Boolean cumulants": for $a, b \in \mathcal{A}$ freely independent, have formulas for Boolean cumulants of $a + b, ab, ab + ba$.

**What are the Boolean cumulants.**

**Notation.** $\mathrm{Int}(n) :=$ set of all *interval partitions* of $\{1, 2, \ldots, n\}$. In reference to the partial order $\leq$ by reverse refinement: $\mathrm{Int}(n)$ is a *sublattice* of $(NC(n), \leq)$.

**Definition** *(Boolean cumulant functionals).* $(\mathcal{A}, \varphi)$ noncomm. prob. space. There exists a sequence of multilinear functionals $(\beta_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, uniquely determined, such that for every $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{A}$ one has

$$\text{(M-BC)} \quad \varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \beta_\pi(a_1, \ldots, a_n),$$

$\beta_n$ is called the $n$th *Boolean cumulant functional* of $(\mathcal{A}, \varphi)$. (M-BC) is a moment–cumulant formula, for Boolean cumulants.

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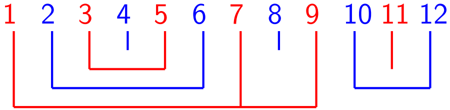
  (This was known, in relation to "Boolean Bercovici-Pata bijection".)
- Boolean cumulants of $ab+ba$: competitive formula, can give new examples of explicit calculations. (Will show such an example.)

**Alternative approach to Kreweras, via VNRP.**

Go directly to the partition of interest in $NC(2n)$, of the form $\pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(\mathrm{even})}$ with $\pi \in NC(n)$. Observe:
– always two outer blocks (one red, one blue);
– the two colours alternate in any vertical cross-section.

An example for $n = 6$:




The next definition isolates the property of vertical alternance (or "no-repeat" for colours).

**The vertical no-repeat property.**

Isolates a key-property of partitions "$\pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(\mathrm{even})}$".
The next definition refers to some (easily checked) facts about the blocks of a partition $\sigma \in NC(m)$:

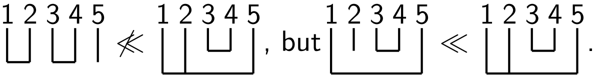
– One has a well-defined *nesting* relation between blocks.
– A block can be "outer" (not nested in anything else), otherwise it is said to be "inner".
– For every inner block $V$ there is a special "parent-block" $\mathrm{Parent}(V)$ that $V$ is immediately nested into.

**Definition** *(VNRP).* Let $\sigma \in NC(m)$ and consider a colouring $c : \{1, \ldots, m\} \to \{1, \ldots, s\}$ which is constant along the blocks of $\sigma$. We say that $\sigma$ and $c$ have the *vertical no-repeat property* (VNRP) to mean that $c(\,\mathrm{Parent}(V)\,) \neq c(V)$, for every inner block $V$ of $\sigma$.

**The combinatorics result that we will build on.**

We use a partial order relation "$\ll$" on $NC(n)$, coarser than reverse refinement, and known from previous work on how to use Boolean cumulants in free probability (e.g. Belinschi-N 2009, "evolution towards $\boxplus$-infinite divisibility").

**Definition** *(The partial order $\ll$).* $m \in \mathbb{N}$ and $\pi, \sigma \in NC(m)$. We write "$\pi \ll \sigma$ to mean that $\pi \leq \sigma$ (reverse refinement) and that, in addition: for every block $W \in \sigma$ there exists a block $V \in \pi$ such that $\min(W), \max(W) \in V$.

E.g. 


Non-trivial: for fixed $\pi \in NC(n)$, $\ll$ gives a structure of *Boolean lattice* on $\{\sigma \in NC(m) \mid \sigma \gg \pi\}$.

Our combinatorics result then goes as follows (next slide).

**The combinatorics result that we will build on.**

**Theorem 1** (combinatorics). *Let $m$ be in $\mathbb{N}$, let $c : \{1, \dots, m\} \to \{1, \dots, s\}$ be a colouring, and let $NC(m; c) := \{\sigma \in NC(m) \mid \ c$ is constant on every block of $\sigma\}$. For every $\sigma \in NC(m; c)$ there exists a $\tau \in NC(m; c)$, uniquely determined, such that $\sigma \ll \tau$ and such that $\tau$ has the VNRP property.*

**Remark.** This gives in particular a description of the partitions $\pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(even)}$ as *maximal elements* with respect to $\ll$, under suitable hypotheses: if $m = 2n$, $c : \{1, \dots, 2n\} \to \{1, 2\}$ colours according to parity, and $\sigma \in NC(2n; c)$ has exactly two outer blocks, then $\tau$ of Theorem 1 is of the form $\pi^{(\mathrm{odd})} \sqcup \mathrm{Kr}(\pi)^{(even)}$, with $\pi \in NC(n)$.

## Free independence in terms of Boolean cumulants.

Next theorem is obtained when we:
– use the explicit formula for $\beta_n$'s in terms of $\kappa_n$'s;
– invoke the vanishing of mixed $\kappa_n$'s; and (last but not least)
– group terms by using $\ll$ and Theorem 1.

**Theorem 2.** $(\mathcal{A}, \varphi)$ *noncomm prob space with Boolean cumulants* $(\beta_n)_{n=1}^{\infty}$. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ *be unital subalgebras. TFAE:*

*(1)* $\mathcal{A}_1, \ldots, \mathcal{A}_s$ *are freely independent.*

*(2) For every* $n \in \mathbb{N}$*, every colouring* $c : \{1, \ldots, n\} \to \{1, \ldots, s\}$*, and every* $a_1 \in \mathcal{A}_{c(1)}, \ldots, a_n \in \mathcal{A}_{c(n)}$*, one has*

$$\beta_n(a_1, \ldots, a_n) = \sum_{\pi} \beta_\pi(a_1, \ldots, a_n),$$

*where on the r.h.s we sum over* $\pi \in NC(n)$ *which: (i) respects the colouring* $c$*; (ii) has VNRP; (iii) has a unique outer block.*

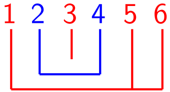
An example of how this works...(next slide)

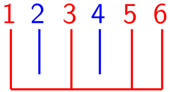
**Example of how the formula stated in Theorem 2 is working.**

Suppose $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are freely independent and we have $a_1, a_3, a_5, a_6 \in \mathcal{A}_1$ and $a_2, a_4 \in \mathcal{A}_2$. Then

$$\beta_6(a_1, a_2, a_3, a_4, a_5, a_6) = \beta_3(a_1, a_5, a_6)\beta_1(a_3) \cdot \beta_2(a_2, a_4) + \beta_5(a_1, a_3, a_5, a_6) \cdot \beta_1(a_2)\beta_1(a_4).$$

That is, the sum on the rhs of the preceding slide has 2 terms,

corresponding to 

and to 

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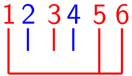
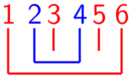
[But no terms given by 

, or by 

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**Boolean cumulants of a free anti-commutator.**

Building on Theorem 2, one can start doing free harmonic analysis with Boolean cumulants: with $a, b \in \mathcal{A}$ freely independent, seek formulas for the Boolean cumulants of $a + b$, of $ab$, of $ab + ba$,...

Interesting that one can give a *structural description* for the set of partitions that appear in the calculations for $\beta_n(ab + ba)$. Refer to these partitions as "ac-friendly".

**Notation.** For $\sigma \in NC(2n)$, denote

$$\mathrm{OuterMax}(\sigma) := \{\max(W) \mid \ W \text{ is an outer block of } \sigma\}.$$

**Definition.** Say that $\sigma \in NC(2n)$ is *ac-friendly* when it satisfies:

(acf1) $\mathrm{OuterMax}(\sigma) \subseteq \{1, 3, \ldots, 2n-1\} \cup \{2n\}$.

(acf2) For every $j \in \{1, 3, \ldots, 2n-1\} \setminus \mathrm{OuterMax}(\sigma)$, one has $\mathrm{depth}_\sigma(j) \neq \mathrm{depth}_\sigma(j+1)$, where "$\mathrm{depth}_\sigma(j)$" stands for the depth of the block of $\sigma$ which contains the number $j$.

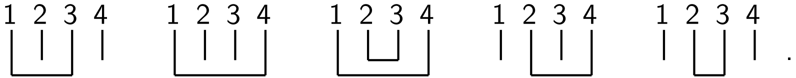
**Recall:** (acf1) $\mathrm{OuterMax}(\sigma) \subseteq \{1, 3, \ldots, 2n-1\} \cup \{2n\}$, and

(acf2) For every $j \in \{1, 3, \ldots, 2n-1\} \setminus \mathrm{OuterMax}(\sigma)$, one has $\mathrm{depth}_\sigma(j) \neq \mathrm{depth}_\sigma(j+1)$.

**Notation.**

$NC_{\mathrm{ac-friendly}}(2n) := \big\{ \sigma \in NC(2n) \mid \sigma$ satisfies (acf1) and (acf2)$\big\}$.

**Example.** $NC_{\mathrm{ac-friendly}}(4)$ has 5 partitions, shown below.




**Remark.** Have explicit formula for generating series:

$$\sum_{n=1}^{\infty} |\, NC_{\mathrm{ac-friendly}}(2n) \,| z^n = \frac{1}{2} - \sqrt{(1-8z)\frac{1-2z-\sqrt{1-8z}}{8z}}.$$

**Formula for Boolean cumulants of a free anti-commutator.**

Take for simplicity the case when $a, b$ are identically distributed.

**Theorem 3.** *$(\mathcal{A}, \varphi)$ noncomm prob space and $a, b \in \mathcal{A}$ freely independent, with $\beta_n(a) = \beta_n(b) =: \lambda_n$, $n \geq 1$. Then*

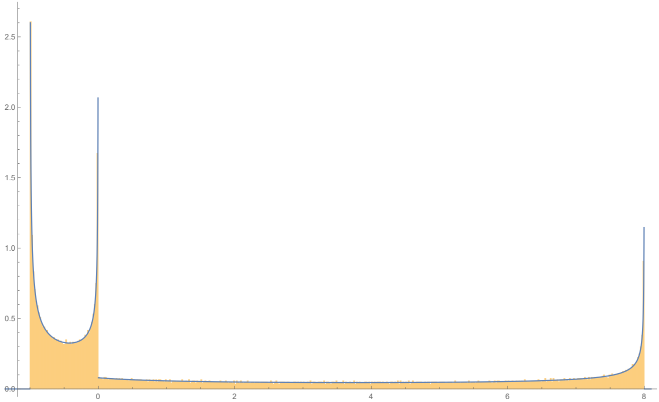
$$\beta_n(ab + ba) = 2 \cdot \sum_{\sigma \in NC_{\mathrm{ac-friendly}}(2n)} \prod_{V \in \sigma} \lambda_{|V|}, \quad n \geq 1.$$

**Example.** Suppose the common distribution of $a$ and $b$ is the shifted Bernoulli distribution $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$. Then $\varphi(a^n) = \varphi(b^n) = 2^{n-1}$ for all $n \geq 1$, and in Theorem 3 we get $\lambda_n = 1$ for all $n$. It thus follows that

$$\beta_n(ab + ba) = 2 \cdot |\, NC_{\mathrm{ac-friendly}}(2n)\, |, \quad \forall\, n \geq 1.$$

In this example it is possible to compute explicitly the moment generating series of $ab + ba$, and find out that it is given by a density $f(x)$, with $-1 \leq x \leq 8$. Have explicit (though not nice) formula for $f(x)$, with lots of radicals. Next slide shows picture.

Density $f(x)$ for the distribution of $ab + ba$, with $a, b$ as above




**Remark.** For same $a, b$, it is easily found that the distribution of the *commutator* $i(ab - ba)$ is the arcsine distribution on $[-2, 2]$. (Free anti-commutators can be reduced to free commutators when $a, b$ have symmetric distributions, but this is not the case here.)

**Some References.**

Paper presented today:

M. Fevrier, M. Mastnak, A. Nica, K. Szpojankowski.
Using Boolean cumulants to study multiplication and anti-commutators of free random variables, arXiv:1907.10842.
*Transactions of the American Math Society*, 373(2020), 7167–7205.

Other recent work related to free anti-commutator:

- W. Ejsmont, F. Lehner.
  Sums of commutators in free probability, arXiv:2002.06051.
- D. Perales.
  On the anti-commutator of two free random variables, arXiv:2101.09444.

**Thank you for your attention!**