Unified signature cumulants and generalized Magnus expansions

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Cumulants in Stochastic Analysis

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Introduction

- Let $X = (X_t)_{0 \le t \le T}$ be a d-dimensional càdlàg semimartingale.
- The signature $\mathrm{Sig}(X)_{t,T}$ of X over the interval $[t, T]$ is given by the tensor series of iterated Stratonivich/Marcus integrals

$$\mathrm{Sig}(X)_{t,T} = 1 + X_{t,T} + \int_t^T X_{0,s} \circ \mathrm{d}X_s + \int_t^T \int_t^s (X_{0,u} \circ \mathrm{d}X_u) \circ \mathrm{d}X_s + \cdots .$$

- Define $\mathbf{X}_t = (0, X_t, 0 \ldots) \in T_0((\mathbb{R}^d)) = \prod_{i=1}^{\infty} (\mathbb{R}^d)^{\otimes i}$
- The signature $\mathrm{Sig}(\mathbf{X})_{t,\cdot}$ is the solution to the Marcus SDE

$$\mathrm{d}\mathbf{S} = \mathbf{S} \circ \mathrm{d}\mathbf{X}$$

with the initial condition

$$\mathrm{Sig}(\mathbf{X})_{t,t} = \mathbf{1} = (1, 0, \ldots) \in T_1((\mathbb{R}^d)).$$

Signature cumulants

- The (conditional) *expected signature* and the *signature cumulants*

$$\boldsymbol{\mu}_t(T) := \mathbb{E}_t(\mathrm{Sig}(\mathbf{X}_{t,T})), \quad \boldsymbol{\kappa}_t(T) := \log(\boldsymbol{\mu}_t(T))$$

  are fundamental characteristics on a process level.

- In analogy to moments and cumulants of a (scalar) random variable, signature cumulants simplify the of expression certain statistical properties.

- See (among many others) the following two references

  - Ilya Chevyrev and Terry Lyons. Characteristic functions of measures on geometric rough paths.
    Ann. Probab., 44(6):4049–4082, 2016

  - Patric Bonnier and Harald Oberhauser. Signature cumulants, ordered partitions, and independence of stochastic processes.
    Bernoulli, 26(4):2727–2757, 2020

Known expansions and our main result

- In case $\mathbf{X}$ is *deterministic* and has *finite variation* the log-signature $\boldsymbol{\kappa}_t(T) = \log \mathrm{Sig}(\mathbf{X})_{t,T}$ satisfies *Hausdorff's equation* and admits the *Magnus expansion*.
- On the other hand, in case $\mathbf{X}$ is one-dimensional (or in $\mathrm{Sym}_0((\mathbb{R}^d)))$ then $\boldsymbol{\kappa}_t(T) = \log \mathbb{E}_t \exp(\mathbf{X}_{t,T})$ is the sequence of "classical" cumulants and admits a *diamond expansion*.
- Our main result is a *fundamental equation* that *uniquely characterises* the signature cumulant $\boldsymbol{\kappa}(T)$, which is of the form

$$\boldsymbol{\kappa}_t = \mathbb{E}_t \left[\phi(\boldsymbol{\kappa}, \mathbf{X})_{t,T}\right], \quad 0 \leq t \leq T$$

and renders into an expansion after projecting to tensor levels

$$\boldsymbol{\kappa}_t^{(n)} = \mathbb{E}_t \left[\phi((\boldsymbol{\kappa}^{(1)}, ..., \boldsymbol{\kappa}^{(n-1)}), (\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}))_{t,T}\right], \quad n = 1, 2, \dots.$$

Hausdorff's equation

Theorem (Baker 1905, Hausdorff 1906)

Let $\mathfrak{g}$ be a Lie-algebra, $\mathcal{G} = \exp(\mathfrak{g})$ its Lie-group and Y the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = A(t)Y(t), \quad 0 \le t \le T, \quad Y(0) = Y_0 \in \mathcal{G},$$

with $A : [0, T] \to \mathfrak{g}$. Then under suitable conditions on A it holds

$$Y(t) = \exp(\Omega(t))Y_0 \in \mathcal{G}, \quad 0 \le t \le T,$$

and $\Omega : [0, T] \to \mathfrak{g}$ solves

$$\Omega(t) = \int_0^t H(\mathrm{ad}\,\Omega_u)(A(u))\mathrm{d}u, \quad 0 \le t \le T,$$

where $(\mathrm{ad}\, a \coloneqq [a, \cdot])$ and with Bernoulli numbers $(B_k)_{k\ge 0}$, $B_1 = \text{-}\frac{1}{2}$

$$H(z) = \frac{z}{1 - \exp(z)} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k.$$

Magnus Expansion

Theorem (Magnus 1954)

Under suitable conditions on A, there is a converging expansion of Ω

$$\Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t), \quad 0 \leq t \leq T,$$

where the coefficients are recursively defined by

$$\Omega_1(t) = \int_0^t A(u)\mathrm{d}u,$$

$$\Omega_{n+1}(t) = \sum_{k=1}^{n} \frac{B_k}{k!} \sum_{\|\ell\|=n, |\ell|=k} \int_0^t \mathrm{ad}\,\Omega_{l_1} \cdots \mathrm{ad}\,\Omega_{l_k}(A(u))\mathrm{d}u.$$

where $\ell \in (\mathbb{N}_{\geq 1})^k$, $k \in \mathbb{N}_{\geq 1}$ with $|\ell| = k$ and $\|\ell\| = l_1 + \cdots + l_k$.

Example

The signature $\mathrm{Sig}(X)$ of an absolutely continuous path X is in the free Lie-Group. Magnus expansion for the log-signature.

Hausdorff's equation - Proof

The proof follows in two steps.

Lemma (Schur 1891, Poincaré 1899)

Let $\Omega : [0, T] \to \mathfrak{g}$ *be continuously differentiable then it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(\Omega(t)) = G(\operatorname{ad} \Omega(t))(\dot{\Omega}(t)) \exp(\Omega(t))$$

where

$$G(z) = \int_0^1 \exp(\tau z) \mathrm{d}\tau = \frac{\exp(z) - 1}{z} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}.$$

Lemma (Baker 1905, Hausdorff 1906)

Under suitable conditions on the eigenvalues of $\operatorname{ad} A$*, the operator* $G(\operatorname{ad} A)$ *is invertible and its inverse is given by* $H(\operatorname{ad} A)$*.*

Magnus Expansion - A few more references

- A. Iserles and S. P. Nørsett. On the solution of linear differential equations in lie groups.
  Philos. Trans. Roy. Soc. A, 357(1754):983–1019, 1999
- S. Blanes, F. Casas, J.A. Oteo, and J. Ros. The magnus expansion and some of its applications.
  Phys. Rep., 470(5-6):151–238, 2009
- Kevin Kamm, Stefano Pagliarani, and Andrea Pascucci. The stochastic magnus expansion, 2020

Diamond Expansion - Diamond Product

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space on which all semimartingales are continuous (e.g. filtration generated by BM).

Definition (Diamond Product (Elisa Alòs-Gatheral-Radoičić 2017))

Let X and Y be scalar martingales, such that $\mathbb{E}\langle X, Y\rangle_T < \infty$. Then define

$$\begin{aligned}(X \diamond Y)_t(T) :=& \mathbb{E}_t\big(\langle X, Y\rangle_{t,T}\big)\\ =& \mathbb{E}_t\big(\langle X, Y\rangle_T\big) - \langle X, Y\rangle_t \quad 0 \leq t \leq T.\end{aligned}$$

- Clearly also $(X \diamond Y)(T)$ is a continuous semimartingale.
- Observe also that $\diamond$ is commutative but not associative.

Diamond Expansion

Theorem (Friz-Gatheral-Radoičić 2020)

Let X_T be an $\mathcal{F}_T$ measurable random variable such that

$$\mathbb{E}(\exp(\lambda X_T)) < \infty$$

for all λ in a neighbour hood of zero. Then the recursion

$$\mathbb{K}^1(T) := (\mathbb{E}_t(X_T))_{0 \le t \le T}$$
$$\mathbb{K}^n(T) := \sum_{k=1}^{n} (\mathbb{K}^k \diamond \mathbb{K}^{n-k})(T),$$

is well defined for all $n \ge 1$. Further there exits an optimal $\rho = \rho_t(\omega)$, s.t.

$$\log \mathbb{E}_t(\exp(zX_T)) = \sum_{n=1}^{\infty} z^n \mathbb{K}_t^n(T), \quad z \in \mathbb{C}, |z| < \rho,$$

which identifies $n!\mathbb{K}^n(T)$ as the n-th conditional cumulant of X_T.

Diamond Expansion - References

- Many applications in particular for stochastic models in *mathematical finance* and *statistical physics/quantum field theory*.
(Concrete examples: Bessel processes, Levy's area formula, rough forward variance models, sine-gordon model)

- Elisa Alòs, Jim Gatheral, and Radoš Radoičić. Exponentiation of conditional expectations under stochastic volatility. *Quantitative Finance; SSRN (2017)*, 20(1):13–27, 2020 (1st version preprint Jun 2017)
- Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model, 2019 (1st version preprint Mar 2019)
- Peter K. Friz, Jim Gatheral, and Radoš Radoičić. Forests, cumulants, martingales, 2020 (1st version preprint Feb 2020)

Definitions - Tensor algebra

- Let $\mathcal{T} := T((\mathbb{R}^d)) = \prod_{i=0}^{\infty}(\mathbb{R}^d)^{\otimes i}$ be the *completed tensor algebra* over $\mathbb{R}^d$. An element $\mathbf{x} \in \mathcal{T}$ is a formal tensor series

$$\mathbf{x} = \sum_{n=0}^{\infty} \mathbf{x}^{(n)} = \sum_{w \in \mathcal{W}_d} \mathbf{x}^w e_w,$$

  where $\mathcal{W}_d$ is the set of words on $\{1, \ldots, d\}$ and for $w = i_1 \cdots i_n$, $|w| = n$ and $e_w = e_{i_1} \cdots e_{i_n}$, with basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$.

- The multiplication of two elements $\mathbf{x}, \mathbf{y} \in \mathcal{T}$ is given by

$$\mathbf{x}\mathbf{y} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbf{x}^{(k)} \mathbf{y}^{(n-k)}, \quad \text{where } \mathbf{x}^{(k)} \mathbf{y}^{(n-k)} \in (\mathbb{R}^d)^n.$$

- Lie-algebras and their (formal) Lie-groups

$$\mathcal{T}_0 := \{\mathbf{x} \in \mathcal{T} \mid \mathbf{x}_0 = 0\}, \qquad \mathcal{T}_1 := \{\mathbf{x} \in \mathcal{T} \mid \mathbf{x}_0 = 1\}$$

  with Lie bracket $[\mathbf{x}, \mathbf{y}] = \mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x} = (\operatorname{ad} \mathbf{x})\mathbf{y}$.

Definitions - Tensor algebra

- The exponential and logarithm

$$\exp: \mathcal{T}_0 \to \mathcal{T}_1, \quad \mathbf{x} \mapsto 1 + \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{x}^k,$$

$$\log: \mathcal{T}_1 \to \mathcal{T}_0, \quad (1+\mathbf{x}) \mapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\mathbf{x}^k.$$

- Denote by $\mathcal{T}^N$ the *truncated tensor algebra* with canonical projection

$$\pi^{(0,N)}: \mathcal{T} \to \mathcal{T}^N, \ (\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots) \mapsto (\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}).$$

- Let $\mathcal{S} = \mathrm{Sym}((\mathbb{R}^d))$ be the completed *symmetric algebra* with projection $\mathcal{T} \to \mathcal{S}, \mathbf{x} \mapsto \hat{\mathbf{x}}$. Then $\mathbf{x}, \mathbf{y} \in \mathcal{T}$ then $\hat{\mathbf{x}}\hat{\mathbf{y}} = \widehat{\mathbf{x}\mathbf{y}} = \hat{\mathbf{y}}\hat{\mathbf{x}} \in \mathcal{S}$

$$\hat{\mathbf{x}}^{i_1 \cdots i_n} = \frac{1}{n} \sum_{\sigma \in \mathbb{S}_n} \mathbf{x}^{i_{\sigma(1)} \cdots i_{\sigma(n)}}$$

for all $i_1 \cdots i_n \in \hat{\mathcal{W}}_d$ (non decreasing words).

Definitions - Tensor valued semimartingales

Definition (Tensor valued semimartingales)

$$\mathscr{S}(\mathcal{T}) = \left\{ \mathbf{X} : \Omega \times [0,T] \to \mathcal{T} \,\middle|\, X^w \text{ a càdlàg semimartingale}, \ w \in \mathcal{W}_d \right\}$$

Definition (Friz-H-Tapia 2021)

Let $\mathbf{X}, \mathbf{Y} \in \mathscr{S}(\mathcal{T})$ then their *outer bracket* is given by

$$[\![\mathbf{X}, \mathbf{Y}]\!]_t = \sum_{w,v \in \mathcal{W}_d} \left(\langle \mathbf{X}^{w,c}, \mathbf{Y}^{v,c} \rangle_t + \sum_{0<u\le t} \Delta \mathbf{X}_t^w \Delta \mathbf{Y}_t^v \right) e_w \otimes e_v$$

as an element in $\mathcal{T} \otimes \mathcal{T}$ and the *inner bracket* is given by

$$\langle \mathbf{X}^c, \mathbf{Y}^c \rangle_t = \mathrm{m}([\![\mathbf{X}^c, \mathbf{Y}^c]\!]_t) = \sum_{w \in \mathcal{W}_d} \left(\sum_{w=uv} \langle \mathbf{X}^{uc}, \mathbf{Y}^{vc} \rangle_t \right) e_w \in \mathcal{T},$$

where m is the linear multiplication map, i.e. $\mathrm{m}(e_w \otimes e_v) = e_{wv}$.

A generalized signature

Definition (Friz-Shekhar 2017, Friz-H-Tapia 2021)

Let $\mathbf{X} = (0, \mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots) \in \mathscr{S}(\mathcal{T}_0)$, then the signature $\mathrm{Sig}(\mathbf{X})_{0,\cdot}$ is the unique solution to the Marcus SDE

$$\mathbf{S}_t = \mathbf{S}_t \circ \mathrm{d}\mathbf{X}_t,$$

for all $0 \leq t \leq T$ with initial condition $\mathbf{S}_0 = \mathbf{1} \in \mathcal{T}_1$, or equivalently of

$$\mathbf{S}_t = \mathbf{1} + \int_{(0,t]} \mathbf{S}_{u\text{-}} \mathrm{d}\mathbf{X}_u + \frac{1}{2}\int_0^t \mathbf{S}_{u\text{-}} \mathrm{d}\langle \mathbf{X}^c \rangle_u + \sum_{0<u\leq t} \mathbf{S}_{u\text{-}}(\exp(\Delta \mathbf{X}_u) - 1 - \Delta \mathbf{X}_u).$$

- $\mathrm{Sig}(\mathbf{X})_{s,t} := \mathrm{Sig}(\mathbf{X})_{0,s}^{-1}\mathrm{Sig}(\mathbf{X})_{0,t}$ and $\mathrm{Sig}(\mathbf{X})_{s,s+\cdot}$ also solves the SDE.
- In case $\mathbf{X} = (0, X, 0, \dots)$ with $X \in \mathscr{S}(\mathbb{R}^d)$ then $\mathrm{Sig}(\mathbf{X})_{0,T} \in \exp(\mathfrak{g})$ is the "usual" signature.

The Signature - Semimartingale integrability

Definition (Friz-H-Tapia 2021 - $\mathscr{H}$-spaces)

Let $q \in [1, \infty)$ and $\mathbf{X} \in \mathscr{S}((\mathbb{R}^d)^{\otimes n})$ define

$$\|\mathbf{X}\|_{\mathscr{H}^q} := \inf_{\mathbf{X}-\mathbf{X}_0=\mathbf{M}+\mathbf{A}} \left\| \left|[\mathbf{M}]_T\right|^{1/2} + |\mathbf{A}|_{1-\mathrm{var};[0,T]} \right\|_{\mathcal{L}^q}.$$

Further let $\mathbf{X} \in \mathscr{S}(\mathcal{T}^N)$, then define

$$\|\!|\mathbf{X}|\!\|_{\mathscr{H}^{q,N}} := \sum_{n=1}^{N} \left(\|\mathbf{X}^{(n)}\|_{\mathscr{H}^{qN/n}} \right)^{1/n}.$$

Define also the following spaces

$$\mathscr{H}^{q,N} := \left\{ \mathbf{X} \in \mathscr{S}(\mathcal{T}_0^N) \;\middle|\; \|\!|\mathbf{X}|\!\|_{\mathscr{H}^{q,N}} < \infty \right\} \quad \text{and}$$

$$\mathscr{H}^{\infty-}(\mathcal{T}) := \left\{ \mathbf{X} \in \mathscr{S}(\mathcal{T}) \;\middle|\; \|\mathbf{X}^{(n)}\|_{\mathscr{H}^q} < \infty,\ n \in \mathbb{N}_{\geq 1},\ q \in [1, \infty) \right\}.$$

The Signature - Semimartingale integrability

Theorem (Friz-H-Tapia 2021)

Let $q \in [1, \infty)$ and $N \in \mathbb{N}_{\geq 1}$ then there exists constants $c, C > 0$ deeding on q, N and d such that for all $\mathbf{X} \in \mathscr{S}(\mathcal{T}_0^N)$ we have

$$c \|\mathbf{X}\|_{\mathscr{H}^{q,N}} \leq \|\mathrm{Sig}(\mathbf{X})\|_{\mathscr{H}^{q,N}} \leq C \|\mathbf{X}\|_{\mathscr{H}^{q,N}}.$$

- In particular, if $\mathbf{X} \in \mathscr{H}^{\infty-}$, then $\mathrm{Sig}(\mathbf{X}) \in \mathscr{H}^{\infty-}$ and therefore

$$\boldsymbol{\mu}(T) = (\mathbb{E}_t(\mathrm{Sig}(\mathbf{X})_{t,T}))_{0 \leq t \leq T} \in \mathscr{S}(\mathcal{T}_1)$$
$$\boldsymbol{\kappa}(T) = \log(\boldsymbol{\mu}(T)) \in \mathscr{S}(\mathcal{T}_0).$$

- In case $\mathbf{X} = (0, M, 0, \dots)$ for a martingale $M \in \mathcal{M}(\mathbb{R}^d)$ then the above implies the known estimate (from Friz-Victoir's ('06), Chevyrev-F's ('19) BDG for enhanced martingales)

$$\max_{i=1,\dots,N} \left\| \sup_{0 \leq t \leq T} \left| \mathrm{Sig}(M)_{0,t}^{(n)} \right| \right\|_{\mathcal{L}^{qN/n}}^{1/n} \leq C \left\| \left| [M]_T \right|^{1/2} \right\|_{\mathcal{L}^{qN}}.$$

Signature cumulants - Functional equation

- Recall the definition of the classic functions G and H

$$G(\operatorname{ad}\mathbf{x}) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\operatorname{ad}\mathbf{x})^k \qquad \text{and} \qquad H(\operatorname{ad}\mathbf{x}) = \sum_{k=0}^{\infty} \frac{B_k}{k!}(\operatorname{ad}\mathbf{x})^k,$$

where $(\operatorname{ad}\mathbf{x})\mathbf{y} = \mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x}$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{T}_0$.

- Further define $Q(\operatorname{ad}\mathbf{x}) : \mathcal{T}_0 \otimes \mathcal{T}_0 \to \mathcal{T}_0$ by

$$Q(\operatorname{ad}\mathbf{x}) = \sum_{m,n=0}^{\infty} 2\frac{(\operatorname{ad}\mathbf{x})^n \odot (\operatorname{ad}\mathbf{x})^m}{(n+1)!(m)!(n+m+2)}$$

where for linear $f, g : \mathcal{T}_0 \to \mathcal{T}_0$

$$(f \odot g)(\mathbf{y} \otimes \mathbf{z}) = f(\mathbf{y})g(\mathbf{z})$$

for all $\mathbf{y}, \mathbf{z} \in \mathcal{T}_0$ extended by linearity.

Signature cumulant - Functional equation

Theorem (Friz-H-Tapia 2021)

Let $\mathbf{X} \in \mathscr{H}^{\infty-}$*, then the signature cumulant* $\boldsymbol{\kappa} = \boldsymbol{\kappa}(T)$ *is the unique solution to the following functional equation:*

$$
\begin{aligned}
\boldsymbol{\kappa}_t = \mathbb{E}_t \Big\{ & \int_{(t,T]} H(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\left\langle \mathbf{X}^c \right\rangle_u) \\
& + \frac{1}{2}\int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}}) \circ Q(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\left[\!\left[\boldsymbol{\kappa}^c, \boldsymbol{\kappa}^c\right]\!\right]_u) \\
& + \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}}) \circ (\mathrm{Id} \odot G(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}}))(\mathrm{d}\left[\!\left[\mathbf{X}^c, \boldsymbol{\kappa}^c\right]\!\right]_u) \\
& + \sum_{t<u\leq T} \Big(H(\operatorname{ad}\boldsymbol{\kappa}_{u\text{-}})\Big(\exp(\Delta\mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u\text{-}}) - 1 - \Delta\mathbf{X}_u \Big) - \Delta\boldsymbol{\kappa}_u \Big) \Big\},
\end{aligned}
$$

for all $0 \leq t \leq T$.

Signature cumulants - Expansion

Corollary (Friz-H-Tapia 2021)

Let $\mathbf{X} \in \mathscr{H}^{1,N}$ for some $N \in \mathbb{N}_{\geq 1}$, then we have

$$\boldsymbol{\kappa}_t^{(1)}(T) = \mathbb{E}_t\left(\mathbf{X}_{t,T}^{(1)}\right)$$

and for $n > 1$

$$\begin{aligned}\boldsymbol{\kappa}_t^{(n)} &= \mathbb{E}_t\left(\mathbf{X}_{t,T}^{(n)}\right) \\ &\quad + \sum_{|\ell|\geq 2,\ ||\ell||=n} \mathbb{E}_t\Bigg\{\frac{B_{k-1}}{(k-1)!}\int_{(t,T]} \operatorname{ad}\boldsymbol{\kappa}_{u-}^{(l_2)}\cdots\operatorname{ad}\boldsymbol{\kappa}_{u-}^{(l_k)}\left(\mathrm{d}\mathbf{X}_u^{(l_1)}\right) \\ &\qquad + \frac{B_{k-2}}{(k-2)!}\int_t^T \operatorname{ad}\boldsymbol{\kappa}_{u-}^{(l_3)}\cdots\operatorname{ad}\boldsymbol{\kappa}_{u-}^{(l_k)}\left(\mathrm{d}\left\langle\mathbf{X}^{(l_1)c},\mathbf{X}^{(l_2)c}\right\rangle_u\right) + \cdots\Bigg\}\end{aligned}$$

with $\ell = (l_1, \ldots, l_k)$, $l_i \geq 1$, $|\ell| = k$, $||\ell|| = l_1 + \cdots + l_k$.

Signature cumulants - Expansion - Overview

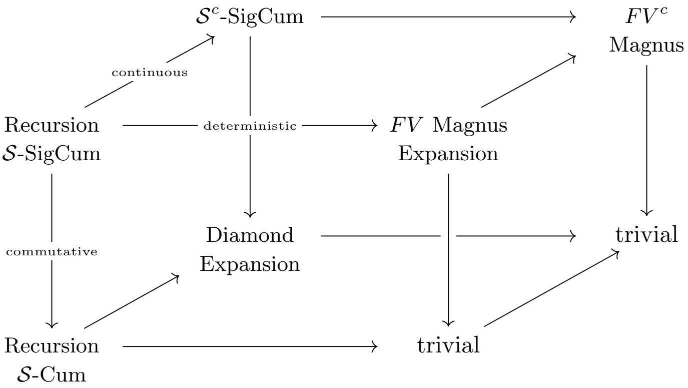

Figure: Signature cumulant expansion and implications [FHT 2021]

Signature cumulants - Functional equation

Theorem (Friz-H-Tapia 2021)

Let $\mathbf{X} \in \mathscr{H}^{\infty-}$*, then the signature cumulant* $\boldsymbol{\kappa} = \boldsymbol{\kappa}(T)$ *is the unique solution to the following functional equation:*

$$\begin{aligned}\boldsymbol{\kappa}_t = \mathbb{E}_t\Bigg\{ \int_{(t,T]} H(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\,\langle \mathbf{X}^c\rangle_u) \\ + \frac{1}{2}\int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}}) \circ Q(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\,[\![\boldsymbol{\kappa}^c, \boldsymbol{\kappa}^c]\!]_u) \\ + \int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}}) \circ (\mathrm{Id} \odot G(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}}))(\mathrm{d}\,[\![\mathbf{X}^c, \boldsymbol{\kappa}^c]\!]_u) \\ + \sum_{t<u\leq T} \Big(H(\mathrm{ad}\,\boldsymbol{\kappa}_{u\text{-}})\Big(\exp(\Delta\mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u\text{-}}) - 1 - \Delta\mathbf{X}_u\Big) - \Delta\boldsymbol{\kappa}_u \Big)\Bigg\},\end{aligned}$$

for all $0 \leq t \leq T$.

Consequences - Hausdorff's equation

Theorem (Hausdorff 1906, Magnus 1954, (Friz-H-Tapia 2021))

Let $\mathbf{X}$ *be a deterministic càdlàg path of bounded variation. The log-signature* $\Omega_t(T) = \log \mathrm{Sig}(\mathbf{X})_{t,T}$ *satisfies*

$$\Omega_t = \int_t^T H(\mathrm{ad}\,\Omega_{u\text{-}})(\mathrm{d}\mathbf{X}_u^c) + \sum_{t \leq u \leq T} \int_0^1 \psi\Big(\exp(\theta\,\mathrm{ad}\,\Delta\mathbf{X}_u) \circ \exp(\mathrm{ad}\,\Omega_u)\Big)(\Delta\mathbf{X}_u)\mathrm{d}\theta$$

for all $0 \leq t \leq T$ *where* $\psi(z) = H(\log z) = \frac{\log z}{z-1}$.

Consequences - Diamond expansion with jumps

- Let $\mathbf{X}_T$ be an $\mathcal{T}_0$-valued $\mathcal{F}_T$-measurable random variable in $\mathcal{L}^{\infty-}$. We associate the martingale $\mathbf{X} = (\mathbb{E}_t(\mathbf{X}_T))_{0 \le t \le T}$. Then note that

$$\mathrm{Sig}(\hat{\mathbf{X}})_{t,T} = \widehat{\mathrm{Sig}(\mathbf{X})}_{t,T} = \exp(\hat{\mathbf{X}}_T - \hat{\mathbf{X}}_t).$$

- In case $\mathbf{X} = (0, X, 0, \dots)$ we have

$$\hat{\boldsymbol{\mu}}_t(T) = \widehat{\boldsymbol{\mu}_t(T)} = \mathbf{1} + \sum_{n=1}^{\infty} \mathbb{E}_t\left((\hat{X}_T - \hat{X}_t)^{\otimes n} \right),$$

i.e. the symmetrized expected signature is the tensor sequence of multivariate moments.

Definition (Tensorfied diamond product)

For semimartingales $\hat{\mathbf{X}}, \hat{\mathbf{Y}} \in \mathscr{S}(\mathcal{S})$ define

$$(\hat{\mathbf{X}} \diamond \hat{\mathbf{Y}})_t(T) := \mathbb{E}_t\left(\langle \hat{\mathbf{X}}^c, \hat{\mathbf{Y}}^c \rangle_T \right) - \langle \hat{\mathbf{X}}^c, \hat{\mathbf{Y}}^c \rangle_t, \quad 0 \le t \le T$$

In particular $(\hat{\mathbf{X}} \diamond \hat{\mathbf{Y}})(T) \in \mathscr{S}(\mathcal{S})$.

Signature cumulants - Functional equation

Theorem (Friz-H-Tapia 2021)

Let $\mathbf{X} \in \mathscr{H}^{\infty-}$, then the signature cumulant $\boldsymbol{\kappa} = \boldsymbol{\kappa}(T)$ is the unique solution to the following functional equation:

$$\begin{aligned}\boldsymbol{\kappa}_t = \mathbb{E}_t\bigg\{ \int_{(t,T]} H(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\left\langle \mathbf{X}^c \right\rangle_u) \\ + \frac{1}{2}\int_t^T H(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}}) \circ Q(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}})(\mathrm{d}\, [\![\boldsymbol{\kappa}^c, \boldsymbol{\kappa}^c]\!]_u) \\ + \int_t^T H(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}}) \circ (\mathrm{Id} \odot G(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}}))(\mathrm{d}\, [\![\mathbf{X}^c, \boldsymbol{\kappa}^c]\!]_u) \\ + \sum_{t<u\leq T} \Big(H(\mathrm{ad}\, \boldsymbol{\kappa}_{u\text{-}})\Big(\exp(\Delta \mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u\text{-}}) - 1 - \Delta\mathbf{X}_u\Big) - \Delta\boldsymbol{\kappa}_u \Big)\bigg\},\end{aligned}$$

for all $0 \leq t \leq T$.

Consequences - Diamond expansion with jumps

Theorem (Friz-H-Tapia 2021)

Let $\hat{\mathbf{X}}_T$ be an $\mathcal{F}_T$-measurable random variable with values in $\mathcal{S}_0$, componentwise in $\mathcal{L}^{\infty -}$. Then

$$\mathbb{K}_t(T) := \log \mathbb{E}_t \exp(\hat{\mathbf{X}}_T)$$

satisfy the following functional equation, for all $0 \le t \le T$,

$$\mathbb{K}_t(T) = \mathbb{E}_t(\hat{\mathbf{X}}_T) + \frac{1}{2}(\mathbb{K} \diamond \mathbb{K})_t(T) + \mathbb{J}_t(T)$$

with jump component,

$$\mathbb{J}_t(T) = \mathbb{E}_t\bigg(\sum_{t<u\le T} \big(\exp(\Delta\mathbb{K}_u) - 1 - \Delta\mathbb{K}_u \big) \bigg)$$
$$= \mathbb{E}_t\bigg(\sum_{t<u\le T} \big(\tfrac{1}{2!}(\Delta\mathbb{K}_u)^2 + \tfrac{1}{3!}(\Delta\mathbb{K}_u)^3 + \dots \big) \bigg).$$

Example - Time dependent Brownian motion

Theorem (Friz-H-Tapia 2021, extends Fawcett 2002)

For a d-dimensional, time dependent Brownian motion $X = \int_0^\cdot \sigma(t)\mathrm{d}B_t$ *the signature cumulant* $\boldsymbol{\kappa}(T)$ *is the unique solution to*

$$\boldsymbol{\kappa}_t(T) = \int_t^T H(\operatorname{ad}\boldsymbol{\kappa}_u)(a(u))\mathrm{d}u, \quad a = \sigma.\sigma^T.$$

Further we have the following expansion for the tensor levels

$$\boldsymbol{\kappa}_t^{(1)}(T) = 0, \quad \boldsymbol{\kappa}_t^{(2)}(T) = \tfrac{1}{2}\int_t^T a(u)\mathrm{d}u,$$

and the general term is given by $\boldsymbol{\kappa}_t^{(2n-1)}(T) \equiv 0$ *and*

$$\boldsymbol{\kappa}_t^{(2n)}(T) = \sum_{\|\ell\|=n-1} \frac{B_k}{k!}\int_t^T \operatorname{ad}\boldsymbol{\kappa}_u^{(2\cdot l_1)} \cdots \operatorname{ad}\boldsymbol{\kappa}_u^{(2\cdot l_k)}(a(u))\,\mathrm{d}u.$$

In particular if B is a standard BM then $\boldsymbol{\kappa}_t(T) = (T-t)(\sum_{i=1}^d e_{ii})$.

Example - Inhomogeneous Lévy Process

Let X be a d-dimensional time-inhomogeneous Lévy processes of the form

$$\mathrm{d}X_t = b(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \mathrm{d}\big(1_{|x|\le 1} * (\mu^X - \nu)\big)_t + \mathrm{d}\big(1_{|x|>1} * \mu^X\big)_t,$$

with μ^X a Poisson random measure with intensity $\nu(\mathrm{d}t, \mathrm{d}x) = K_t(dx)dt$.

Corollary (Friz-H-Tapia 2021, extents Friz-Shekar 2017)

Assume that K has uniformly bounded moments of all orders. Then the signature cumulant $\boldsymbol{\kappa}_t(T) = \log(\mathbb{E}_t \mathrm{Sig}(X)_{t,T})$ is the unique solution to the integral equation

$$\boldsymbol{\kappa}_t(T) = \int_t^T H(\mathrm{ad}\,\boldsymbol{\kappa}_u)(\mathfrak{y}(u))\,\mathrm{d}u, \quad 0 < t \le T,$$

where $a = \sigma\sigma^T \in \mathbb{R}^d \otimes \mathbb{R}^d \subset \mathcal{T}_0$ and

$$\mathfrak{y}(t) := b(t) + \frac{1}{2}a(t) + \int_{\mathbb{R}^d} (\exp(x) - 1 - x\mathbf{1}_{|x|\le 1}) K_t(\mathrm{d}x) \in \mathcal{T}_0.$$

Thank You !

References I

Elisa Alòs, Jim Gatheral, and Radoš Radoičić.
Exponentiation of conditional expectations under stochastic volatility.
Quantitative Finance; SSRN (2017), 20(1):13–27, 2020.

S. Blanes, F. Casas, J.A. Oteo, and J. Ros.
The magnus expansion and some of its applications.
Phys. Rep., 470(5-6):151–238, 2009.

Patric Bonnier and Harald Oberhauser.
Signature cumulants, ordered partitions, and independence of stochastic processes.
Bernoulli, 26(4):2727–2757, 2020.

Ilya Chevyrev and Terry Lyons.
Characteristic functions of measures on geometric rough paths.
Ann. Probab., 44(6):4049–4082, 2016.

Peter K. Friz, Jim Gatheral, and Radoš Radoičić.
Forests, cumulants, martingales, 2020.

References II

A. Iserles and S. P. Nørsett.
On the solution of linear differential equations in lie groups.
Philos. Trans. Roy. Soc. A, 357(1754):983–1019, 1999.

Kevin Kamm, Stefano Pagliarani, and Andrea Pascucci.
The stochastic magnus expansion, 2020.

Hubert Lacoin, Rémi Rhodes, and Vincent Vargas.
A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model, 2019.