Moment Cumulant Formulae for Multifaced Universal Products

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Motivation

Classical stochastic independence

$$X, Y \text{ independent} \iff \mathbb{E}\big(f(X)g(Y)\big) = \mathbb{E}\big(f(X)\big)\mathbb{E}\big(g(Y)\big)$$
$$\iff \mathbb{P}_{(X,Y)} = \mathbb{P}_X \otimes \mathbb{P}_Y$$
$$\iff \text{mixed cumulants vanish}$$

Classical moment cumulant formula

$$m_n = \sum_{\pi \in P(n)} \prod_{\beta \in \pi} \kappa_{|\beta|}$$

"mixed moments" in noncommutative situations

- a_i hermitian random matrices, independent *entries* $\sim N(0,1)$

$$\mathrm{tr}(a_1a_2a_1a_2) = ?, \quad \mathrm{tr}\left(\frac{a_1+\cdots+a_k}{\sqrt{k}}\right)^n = ?$$

- T_1, T_2 rooted trees, $T_1 \diamond T_2$: "glued together at root"

$$\langle e_0, (A_{T_1\diamond T_2})^n e_0\rangle = \langle e_0, (A_{T_1}\otimes p_0 + p_0 \otimes A_{T_2})^n e_0\rangle = ?$$

- T_1, T_2 rooted trees, $T_1 \triangleright T_2$: "glue T_2 to every vertex"

$$\langle e_0, (A_{T_1\triangleright T_2})^n e_0\rangle = \langle e_0, (A_{T_1}\otimes p_0 + \mathrm{id} \otimes A_{T_2})^n e_0\rangle = ?$$

- λ, ρ left/right regular representation of $\mathbb{F}_n$

$$\Big\langle \delta_e, p\big(\lambda(g_i), \rho(g_i)\big)\delta_e\Big\rangle = ?$$

Motivation

Noncommutative independences

In all situations ∃

- notion of "independence"
- special kinds of cumulants
- combinatorial moment cumulant and mixed moment formulae

Last example: independence for pairs $(\lambda(g_i), \rho(g_i)) \rightsquigarrow$ *2-faced*

Example: mixed moments for free independence

$A_1, A_2 \subset \mathcal{A}$ free wrt $\varphi \in \mathcal{A}'$ if for $a_1, \ldots, a_n$ alternating form A_1, A_2:

$$\varphi(a_k) = 0 \implies \varphi(a_1 b_1 \ldots a_n b_n) = 0$$

Then, e.g., $\varphi(abab) = \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2$

Overview

Overview

Non-commutative probability: $*$-algebraic setting

Definition (**non-commutative probability space**)

pair $(\mathcal{A}, \Phi)$ with

- unital $*$-algebra $\mathcal{A}$
- state Φ on $\mathcal{A}$

Definition (**random variable**)

$*$-homomorphism $j\colon B \to \mathcal{A}$ (B is $*$-algebra)

- $\widetilde{B}$ unitization of B, $\widetilde{j}$ unital extension of j
- $\Phi \circ \widetilde{j}$ is called *distribution* of j
- selfadjoint $a \in \mathcal{A} \rightsquigarrow j_a\colon \mathbb{C}[x]_0 \to \mathcal{A}, x \mapsto a$
- distribution of $j_a \leftrightsquigarrow$ collection of *moments* $\big(\Phi(a^k)\big)_{k\in\mathbb{N}}$
- $*$-subalgebra $B \rightsquigarrow$ embedding $\iota\colon B \hookrightarrow \mathcal{A}$

Augmented algebras and unitization

Already appears for Boolean or monotone independence:

Take care with units!

Definition/Notation (augmented algebra)

- unital algebra with character (non-zero homomorphism to $\mathbb{C}$)
- augmented algebra ≡ unitization of its *augmentation ideal* (kernel of the character)
- denote the augmentation ideal simply by B
- denote the augmented algebra $\widetilde{B} = \mathbb{C}1 \oplus B$
- For this talk: think of $\widetilde{B}$ as polynomial algebra, then $B = \widetilde{B}_0 = \{p \in B : p(0) = 0\}$

Non-commutative independence

Fix product operation for states on unital (augmented) $*$-algebras

$$\bigtimes_i \widetilde{B}_i' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\widetilde{\bigsqcup_i B_i}\Big)'$$

Definition ($\odot$-**independence** of random variables $j_i \colon B_i \to \mathcal{A}$)

$$\Phi \circ \widetilde{\bigsqcup_i j_i} = \bigodot_i (\Phi \circ \widetilde{j}_i)$$

joint distribution = product of marginals

Examples

tensor, free, monotone, anti-monotone, Boolean

Non-commutative independence

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Classification theorem (Muraki)

certain axioms $\Longrightarrow$ tensor, free, (anti-)monotone or Boolean

Non-commutative probability: $*$-algebraic setting

Definition (non-commutative probability space)

pair $(\mathcal{A}, \Phi)$ with

- unital $*$-algebra $\mathcal{A}$
- state Φ on $\mathcal{A}$

Definition (*m*-faced random variable)

$*$-homomorphism $j\colon B \to \mathcal{A}$
($B = B^{(1)} \sqcup \cdots \sqcup B^{(m)}$ is *m*-faced $*$-algebra)

- m-tuple of selfadjoint elements $a = (a^{(1)}, \ldots, a^{(m)}) \in \mathcal{A}^m \rightsquigarrow$ $j_a\colon \mathbb{C}\langle x^{(1)}, \ldots, x^{(m)}\rangle_0 \to \mathcal{A}, x^{(k)} \mapsto a^{(k)}$
- $a^\delta := a^{(\delta_1)} \cdots a^{(\delta_m)}$, $\delta \in [m]^*$
- distribution of $j_a \leftrightsquigarrow$ collection of *moments* $(\Phi(a^\delta))_{\delta \in [m]^*}$
- m-tuple of $*$-subalgebras $(B_1, \ldots, B_m) \rightsquigarrow \iota_1 \sqcup \cdots \sqcup \iota_m$

m-Independence

Fix product operation for states on unital (augmented) m-faced $*$-algebras

$$\bigtimes_i \widetilde{B}_i' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \left(\widetilde{\bigsqcup_i B_i}\right)'$$

Definition (⊙-independence of m-faced rv's $j_i\colon B_i \to \mathcal{A}$)

$$\Phi \circ \widetilde{\bigsqcup_i j_i} = \bigodot_i (\Phi \circ \widetilde{j}_i)$$

joint distribution = product of marginals

Examples

bi-freeness, bi-monotone independence, list growing

m-Independence

Fix product operation for states on unital (augmented) m-faced $*$-algebras

$$\bigtimes_i \widetilde{B}_i' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\widetilde{\bigsqcup_i B_i}\Big)'$$

Definition (⊙-independence of m-faced rv's $j_i \colon B_i \to \mathcal{A}$)

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joint distribution = product of marginals

Classification results?

first results coming soon (Varšo)

Overview

uau-products

Definition (m-1-**uau-product** / m-**faced universal product**, cf. Manzel & Schürmann 2017)

$$\widetilde{B}_1' \times \widetilde{B}_2' \ni (\varphi_1, \varphi_2) \mapsto \varphi_1 \odot \varphi_2 \in (\widetilde{B_1 \sqcup B_2})'$$

product operation (for arbitrary m-faced algebras B_1, B_2) which is

- **u**nital in the sense that $1 \odot \varphi = \varphi = \varphi \odot 1$
- **a**ssociative
- **u**niversal in the sense that (for $*$-hom's $j_i \colon B_i \to A_i$)

$$(\varphi_1 \odot \varphi_2) \circ (\widetilde{j_1 \sqcup j_2}) = (\varphi_1 \circ \widetilde{j}_1) \odot (\varphi_2 \circ \widetilde{j}_2)$$

$\odot$ is called **positive** if φ_1, φ_2 states $\implies$ $\varphi_1 \odot \varphi_2$ state

Note: Unital linear functionals on augmented m-faced algebras can be identified with linear functionals on m-faced algebras.

Convolution exponentials

Consider $A = \mathbb{C}\langle x^{(1)}, \dots, x^{(m)} \rangle$

Definition

- **convolution** for $\varphi_1, \varphi_2 \in A', \varphi_1(1) = \varphi_2(1) = 1$

$$\varphi_1 * \varphi_2(x^\delta) := \varphi_1 \odot \varphi_2((x_1 + x_2)^\delta)$$

- **linearized convolution** for $\psi_1, \dots, \psi_n \in A', \psi_k(1) = 0$

$$\psi_1 \boxplus \cdots \boxplus \psi_n := \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (t_1 \psi_1) * \cdots * (t_n \psi_n) \Big|_{t=0}$$

- **convolution exponential** of $\psi \in A'$ with $\psi(1) = 0$

$$\exp_\odot \psi := \sum_{k=0}^{\infty} \frac{1}{k!} \psi^{\boxplus k}$$

Hopf algebra realization

Central Lemma (Manzel & Schürmann 2017, Lachs 2015, Ben Ghorbal & Schürmann 2005)

For an (m-faced) universal product $\odot$:
$\exists$ homomorphisms $\sigma_{B_1,B_2}\colon \mathcal{S}(B_1 \sqcup B_2) \to \mathcal{S}(B_1) \otimes \mathcal{S}(B_2)$ s.t.

- $\mathcal{S}(\varphi_1 \odot \varphi_2) = \big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ \sigma_{B_1,B_2}$
- $\mathcal{S}(A)$ becomes a Hopf algebra with comultiplication

$$\Delta(x^\delta) = \sigma_{A,A}\big((x_1 + x_2)^\delta\big)$$

- $\varphi_1 * \varphi_2 = \big(\mathcal{S}(\varphi_1) \otimes \mathcal{S}(\varphi_2)\big) \circ \Delta \restriction A$
- $\psi_1 \boxplus \cdots \boxplus \psi_n = \big(D(\psi_1) \otimes \cdots \otimes D(\psi_n)\big) \circ \Delta^n \restriction A$

In particular:

$$\mathcal{S}\big(\exp_\odot(t\psi)\big) = \exp_\Delta\big(tD(\psi)\big)$$

Cumulants I

Corollary (cf. Manzel & Schürmann 2017, Lachs 2015)

If $\odot$ is an (m-faced) universal product, then

$$\exp_\odot \colon \{\psi \in A' \mid \psi(1) = 0\} \to \{\varphi \in A' \mid \varphi(1) = 1\}$$

is a well-defined bijection.

Idea of proof.

From $\varphi = \exp_\odot(\psi)$ we get the recursive formula

$$\psi(x^\delta) = \varphi(x^\delta) - \sum_{k=2}^{n} \frac{1}{k!} \psi^{\boxplus k}(x^\delta)$$

with $\psi^{\boxplus k}(x^\delta)$ determined by $\psi \restriction \{p \in A \mid \deg(p) < |\delta|\}$. □

Cumulants II

Definition

For $(\mathcal{A}, \Phi)$ a ncps and $a \in \mathcal{A}^m$ put

- $\varphi_a := \Phi \circ \widetilde{j}_a \in \mathbb{C}\langle x^{(1)}, \dots, x^{(m)} \rangle'$
- $\psi_a := \log_\odot \varphi_a$, i.e. ψ_a is determined by $\exp_\odot \psi_a = \varphi_a$

Then $\psi_a(x^\delta)$, $\delta \in [m]^*$ are called $\odot$-**cumulants** of a.

Remarks

- formula for $\exp_\odot$ is also called **moment cumulant formula**
- even if convolution is not commutative, we have

$$\log_\odot(\varphi_1 * \varphi_2) = CBH(\log_\odot \varphi_1, \log_\odot \varphi_2),$$

where $[\psi_1, \psi_2] := \psi_1 \boxplus \psi_2 - \psi_2 \boxplus \psi_1$

Schoenberg correspondence

Theorem

Let $\odot$ be a positive m-faced universal product, $\psi \in \mathbb{C}\langle x^{(1)}, \ldots, x^{(m)}\rangle'$. Then the following are equivalent:

1. $\exp_\odot(t\psi)$ is a state for all $t \geq 0$
2. ψ is hermitian, $\psi(1) = 0$ and ψ is **conditionally positive**, i.e.,

$$\psi(p^* p) = 0 \text{ for all } p \text{ with } p(0) = 0$$

Idea of proof (overlooked even for $m{=}1$).

use "transformation" (cf. Schürmann & Voß 2014, Skeide, Schürmann & Volkwardt 2010) to prove

$$\exp_\odot(t\psi) = \lim_{n\to\infty} \exp_\otimes(\tfrac{t}{n}\psi) *_\odot \ldots *_\odot \exp_\otimes(\tfrac{t}{n}\psi)$$

□

Comparison with other approaches

Shuffle algebra approach (Ebrahimi-Fard & Patras, Gilliers)

- pros: interrelations, operator valued generalization
- cons: restricted to monotone, free & boolean case

Exchangeability system approach (Lehner & Hasebe)

- pros: very general
- cons: in general no convlolution, maybe harder to prove results for a large number of independences simultaneously?

Cumulants as generators of Lévy type processes (Anshelevich)

- pros: applicable to certain non-universal examples
- cons: weaker properties in general

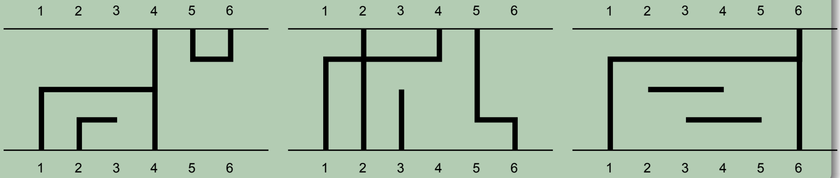
Overview

Partitions

Definition (m-faced partition, ordered m-faced partition)

- m-**faced partition** of a set X:
  set partition of X together with a map (*face-map*)
  $$\delta: X \to \{1, \dots, m\}$$
- **ordered** m-**faced partition** of X:
  m-faced partition with total order between blocks

Visualization examples



Definition (**easy universal product**)

m-faced universal product & $\exists$ lattice of m-faced partitions $\mathcal{P}$ s.t.

$$\pi_a \in \mathcal{P} \implies \varphi_1 \odot \cdots \odot \varphi_k(a) = \varphi_1\Big(\prod_{a_i \in A_1} a_i\Big) \cdots \varphi_k\Big(\prod_{a_i \in A_k} a_i\Big)$$

$$\pi_a \notin \mathcal{P} \implies \varphi_1 \boxdot \cdots \boxdot \varphi_k(a) = 0$$

where

- $a = a_1 \dots a_n \in A_1 \sqcup \cdots \sqcup A_k,\ a_i \in A_{\varepsilon_i}^{(\delta_i)}$
- π_a the m-faced partition associated with (ε, δ)
- $\boxdot$ is the "linear part" of $\odot$, i.e.,

$$\varphi_1 \boxdot \cdots \boxdot \varphi_k := \frac{\partial^n}{\partial t_1 \cdots \partial t_n} (t_1\varphi_1) \odot \cdots \odot (t_k\varphi_k)\Big|_{t=0}$$

Easy moment cumulant formula

Theorem

For $\odot$ easy with partitions $\mathcal{P}$:

$$\exp_{\odot}(\psi)(x^{\delta}) = \sum_{\pi \in \mathcal{P}(\delta)} \hat{\psi}(x^{\pi})$$

Proof.

$$\begin{aligned}\exp_{\odot}(\psi)(x^{\delta}) &= \sum_{k} \frac{1}{k!} \psi^{\boxplus k}(x^{\delta}) \\ &= \sum_{k,\varepsilon} \frac{1}{k!} \psi^{\boxdot k}(x^{(\varepsilon,\delta)}) \\ &= \sum_{\pi \in \mathcal{P}(\delta)} \hat{\psi}(x^{\pi})\end{aligned}$$

□

Easy mixed moments formula

Lemma

For symmetric universal products: mixed cumulants vanish!

Theorem

For A_i $\odot$-independent, $a = a_1 \dots a_n$ s.t. π describes membership and faces:

$$\Phi(a) = \sum_{\emptyset \neq R \subseteq S} (-1)^{\#R-1} \hat{\Phi}(a_{\wedge R})$$

where

- S is the set of maximal (coarsest) refinements of π which belong to $\mathcal{P}$
- $\wedge R$ is the maximal common refinement of partitions in R

Proof.

- lift Φ to a polynomial algebra with a_i as indeterminates
- plug in the moment cumulant formula:

$$\sum_{R \subseteq S} (-1)^{\#R} \hat{\Phi}(a_{\wedge R})$$

$$= \sum_{R \subseteq S} \sum_{\substack{\sigma \leq \wedge R \\ \sigma \in \mathcal{P}}} (-1)^{\#R} \hat{\psi}(a_\sigma) = \sum_{\substack{\sigma \leq \pi \\ \sigma \in \mathcal{P}}} \underbrace{\sum_{k=0}^{n(\sigma)} (-1)^k \binom{n(\sigma)}{k}}_{=0} \hat{\psi}(a_\sigma)$$

where $n(\sigma) = \#\{\rho \in S : \sigma \leq \rho\}$

□

Remarks

- A mixed moment formula determines the universal product.
- All 1-faced positive symmetric universal products are easy (tensor, free, Boolean).
- Most known 2-faced examples of positive symmetric universal products are easy.
- In many examples (not free!), for every π there is a unique maximal refinement ρ in $\mathcal{P}$. In this case, the formula simplifies significantly:
$$\Phi(a) = \hat{\Phi}(a_\rho)$$
- In examples, the formulas hold (up to small adjustments due to block orders) also for non-symmetric universal products determined by partitions (e.g., monotone and bi-monotone).
- Philipp Varšo systematically studies the possible classes $\mathcal{P}$ and found several new examples for his PhD thesis.

Overview

1. Multifaced random variables
2. Moments and cumulants
3. Easy universal products
4. Examples

Positive examples

- type II bi-monotone (2-faced, G 2017) and tbma (4-faced)

- bi-free (2-faced, Voiculescu 2014) and free-free-Boolean (3-faced, Liu 2018)

Non-positive examples

- r-s-independence (deformed Boolean independence, Lachs 2015, G & Lachs 2015)
- c-bi-free independence; defined via a (2,2) uau product (Gu & Skoufranis 2017)
- bi-Boolean independence (Gu & Skoufranis 2019)
- type I bi-monotone independence (Gu, Skoufranis & Hasebe 2020)

Positivity not known

From Varšo's classification of *universal classes of partitions*:

- "pure noncrossing partitions"-independence
- "pure crossing partitions"-independence
- "crossing noncrossing partitions"-independence
- 2-faced deformations of tensor and free independence

Thank you!

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