

# Diamond trees and the forest expansion

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Cumulants in Stochastic Analysis  
Zoom, February 25, 2021

# Outline of this talk

- The diamond product
- The  $\mathbb{G}$ -expansion
  - Trees and forests
- The  $\mathbb{K}$ -expansion
  - Third cumulant
- The  $\mathbb{F}$ -expansion and stochastic volatility
  - Triple joint MGF
  - The leverage swap
- Explicit computations in affine forward variance models

## My 2017 TU Berlin talk

- In January 2018, I presented the diamond product and the exponentiation theorem for the first time at a TU Berlin seminar.
- Manipulations were formal and the convergence properties of the resulting forest expansion unclear.
  - Eventually published as Elisa Alòs, Jim Gatheral, and Radoš Radoičić, Exponentiation of conditional expectations under stochastic volatility, *Quantitative Finance* 20(1):13–27, 2020.
- This time we explain the remarkably simple origin of the forest expansion, we give its convergence properties and attempt to give a sense of its wide applicability.

# The diamond product

## Definition

Given two continuous semimartingales  $A, B$  with integrable covariation process  $\langle A, B \rangle$ , the diamond product<sup>a</sup> of  $A$  and  $B$  is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}_t[\langle A, B \rangle_{t,T}] = \mathbb{E}_t[\langle A, B \rangle_T] - \langle A, B \rangle_t,$$

where  $\langle A, B \rangle_{t,T} = \langle A, B \rangle_T - \langle A, B \rangle_t$ .

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<sup>a</sup>Warning. Our diamond product is (very) different from the Wick product.

# Properties of the diamond product

- Commutative:  $A \diamond B = B \diamond A$ .
- Non-associative:  $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$ .
- $A \diamond B$  depends only on the respective martingale parts of  $A$  and  $B$ .
- $A \diamond B$  is in general not a martingale.

# The $\mathbb{G}$ -forest expansion

## Theorem 1 (Theorem 1.1 of [FGR20])

Let  $Y_T$  be a real-valued,  $\mathcal{F}_T$ -measurable random variable with associated martingale  $Y_t = \mathbb{E}_t[Y_T]$ . Under natural integrability conditions, with  $a, b$  small enough, there is a.s. convergence of

$$\log \mathbb{E}_t \left[ e^{aY_T + b\langle Y \rangle_T} \right] = aY_t + b\langle Y \rangle_t + \sum_{k \geq 2} \mathbb{G}_t^k(T), \quad (1)$$

where

$$\begin{aligned} \mathbb{G}^2 &= \left( \frac{1}{2}a^2 + b \right) (Y \diamond Y)_t(T), \\ \mathbb{G}^k &= \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (aY \diamond \mathbb{G}^{k-1}) \text{ for } k > 2. \end{aligned} \quad (2)$$

## Idea of the proof

For a generic (continuous) semimartingale  $Z$ , sufficiently integrable, let

$$\Lambda_t^T = \log \mathbb{E}_t \left[ e^{Z_{t,T}} \right].$$

Then, noting that  $\Lambda_T^T = 0$ ,

$$\mathbb{E}_t \left[ e^{Z_T} \right] = \mathbb{E}_t \left[ e^{Z_T + \Lambda_T^T} \right] = e^{Z_t + \Lambda_t^T}.$$

The stochastic logarithm  $\mathcal{L}(\mathbb{E}_\bullet(Z_T)) = Z + \Lambda^T + \frac{1}{2}\langle Z + \Lambda^T \rangle$  is a martingale. Thus,

$$\begin{aligned} \Lambda_t^T &= \mathbb{E}_t \left[ Z_{t,T} + \frac{1}{2}\langle Z + \Lambda^T \rangle_{t,T} \right] \\ &= \mathbb{E}_t [Z_{t,T}] + \frac{1}{2}((Z + \Lambda^T) \diamond (Z + \Lambda^T))_t(T). \end{aligned}$$

Now with<sup>1</sup>  $Z = \epsilon a Y + \epsilon^2 b \langle Y \rangle$  we get

$$\Lambda_t^T(\epsilon) = \epsilon a \mathbb{E}_t[Y_{t,T}] + \epsilon^2 b (Y \diamond Y)_t(T) + \frac{1}{2} \left( \epsilon a Y + \Lambda_t^T(\epsilon) \right)_t^{\diamond 2}(T).$$

Put  $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{G}_t^2 + \epsilon^3 \mathbb{G}_t^3 + \dots$ , and match coefficients of  $\epsilon^n$ .

$$[\epsilon^2]: \mathbb{G}_t^2 = b (Y \diamond Y)_t(T) + \frac{1}{2} a^2 (Y \diamond Y)_t(T).$$

$$[\epsilon^3]: \mathbb{G}_t^3 = (a Y \diamond \mathbb{G}_t^2)_t(T).$$

$$[\epsilon^4]: \mathbb{G}_t^4 = (a Y \diamond \mathbb{G}_t^3)_t(T) + \frac{1}{2} (\mathbb{G}_t^2 \diamond \mathbb{G}_t^2)_t(T).$$

- We see the recursion (2) emerge!

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<sup>1</sup>Recall that terms of bounded variation such as  $\langle Y \rangle$  do not contribute to diamond products.



## Special cases

Interesting special cases include

- The exponential martingale:  $b = -\frac{1}{2}a^2$ . All corrector terms  $\mathbb{G}^k$  vanish.
  - The  $\mathbb{G}$ -expansion can thus be seen as a “broken exponential martingale” expansion.
- The  $\mathbb{F}$ -forest expansion of [AGR2020] (working paper 2017):  $\frac{1}{2}a + b = 0$ .
  - The  $\mathbb{F}$ -forest expansion gives a general expression for the characteristic function of the log-stock price in a stochastic volatility model written in forward variance form.
- The cumulant ( $\mathbb{K}$ -forest) expansion of Lacoïn-Rhodes-Vargas [LRV19]:  $b = 0$ .
  - Their expansion was derived in the context of renormalization of the sine-Gordon model in quantum physics.

## Further applications

- In [FGR20], we give a number of applications.
- Other possible applications include
  - computation of likelihood functions in statistics,
  - computation of correlation functions in statistical physics,
  - computation of amplitudes in quantum field theory.
- It's very satisfying that problems in quantitative finance and quantum physics lead to the same nice mathematics!

# Trees and forests

- The general term  $\mathbb{G}_t^n(T)$  in (2) is naturally written as a linear combination of binary diamond trees<sup>2</sup>.
- Hence the terminology  $\mathbb{G}$ -forest expansion for (1).
- Specifically, writing  $\bullet$  as a short-hand for  $Y$ , interpreted as single leaf, we have

$$\begin{aligned}
 \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b\right) \bullet \text{---} \bullet \\
 \mathbb{G}^3 &= a\left(\frac{1}{2}a^2 + b\right) \bullet \text{---} \bullet \text{---} \bullet \\
 \mathbb{G}^4 &= \frac{1}{2}\left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + a^2\left(\frac{1}{2}a^2 + b\right) \bullet \text{---} \bullet \text{---} \bullet \\
 \mathbb{G}^5 &= a\left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet + \frac{1}{2}a\left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\
 &\quad + a^3\left(\frac{1}{2}a^2 + b\right) \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad (3)
 \end{aligned}$$

<sup>2</sup>Trees stolen from [Hai13]!

# The $\mathbb{K}$ -forest expansion

As mentioned earlier, the  $\mathbb{K}$ -forest expansion ( $\mathbb{K}$  for “Kumulant”) is obtained by setting  $b = 0$  in (1). This gives

$$\mathbb{K}^2 = \frac{1}{2} a^2 \text{ (diagram: two nodes connected by a line with a red arrow pointing from the left node to the right node)}$$

$$\mathbb{K}^3 = \frac{1}{2} a^3 \text{ (diagram: three nodes in a chain, with red arrows pointing from the left node to the middle node, and from the middle node to the right node)}$$

$$\mathbb{K}^4 = \frac{1}{8} a^4 \text{ (diagram: four nodes in a chain, with red arrows pointing from the left node to the second, third, and fourth nodes)} + \frac{1}{2} a^4 \text{ (diagram: four nodes in a chain, with red arrows pointing from the second, third, and fourth nodes to the left node)}$$

$$\mathbb{K}^5 = \frac{1}{4} a^5 \text{ (diagram: five nodes in a chain, with red arrows pointing from the left node to the second, third, and fourth nodes)} + \frac{1}{8} a^5 \text{ (diagram: five nodes in a chain, with red arrows pointing from the second, third, and fourth nodes to the left node)} + \frac{1}{2} a^5 \text{ (diagram: five nodes in a chain, with red arrows pointing from the second, third, and fourth nodes to the left node, and from the fifth node to the left node)}$$

With  $\mathbb{K}^1 = \bullet$ , the  $\mathbb{K}$ -recursion follows naturally.

# The $\mathbb{K}$ -forest expansion

## Theorem 2 (Theorem 1.2 of [FGR20])

Let  $A_T$  be  $\mathcal{F}_T$ -measurable with  $N \in \mathbb{N}$  finite moments. Then the recursion

$$\mathbb{K}_t^{n+1}(T) = \frac{1}{2} \sum_{k=1}^n (\mathbb{K}^k \diamond \mathbb{K}^{n+1-k})_t(T), \quad \forall n > 0$$

with  $\mathbb{K}_t^1(T) := \mathbb{E}_t[A_T]$  is well-defined up to  $\mathbb{K}^N$  and, for  $a \in \mathbb{R}$ ,

$$\log \mathbb{E}_t \left[ e^{iaA_T} \right] = \sum_{n=1}^N (ia)^n \mathbb{K}_t^n(T) + o(|a|^N)$$

which identifies  $n! \times \mathbb{K}_t^n(T)$  as the (time  $t$ -conditional)  $n$ .th cumulant of  $A_T$ .

## Example: $\mathbb{K}^3$ and the third central moment

- For higher  $n$ , the forest expansion encodes relations that are increasingly complex to derive by hand.
- For example, from the forest expansion we have

$$\mathbb{K}_t^3(T) = \frac{1}{2} (Y \diamond (Y \diamond Y))_t(T)$$

and also, since the third cumulant is the third central moment,

$$\mathbb{K}_t^3(T) = \frac{1}{3!} \mathbb{E}_t [Y_{t,T}^3].$$

- On the other hand, the relation

$$\frac{1}{2} (Y \diamond (Y \diamond Y))_t(T) = \frac{1}{3!} \mathbb{E}_t [Y_{t,T}^3]$$

is not so obvious.

## Another application: MGF of the Lévy area

### Theorem (P. Lévy)

Let  $\{X, Y\}$  be 2-dimensional standard Brownian motion, and stochastic (“Lévy”) area be given by

$$\mathcal{A}_t = \int_0^t (X_s dY_s - Y_s dX_s) .$$

Then, for  $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,

$$\mathbb{E}_0 [e^{\mathcal{A}_T}] = \frac{1}{\cos T} .$$

- In particular, we will see how to compute trees in practice.

# First term

First,

$$\begin{aligned}\mathbb{K}^2 &= \frac{1}{2} \text{◇} = \frac{1}{2} (\mathcal{A} \diamond \mathcal{A})_t(T) \\ &= \frac{1}{2} \int_t^T (\mathbb{E}_t [X_s^2] + \mathbb{E}_t [Y_s^2]) ds \\ &= \frac{1}{2} (T - t)^2 + \frac{1}{2} (X_t^2 + Y_t^2) (T - t).\end{aligned}$$

In particular,

$$d\mathbb{K}_s^2 = (X_s dX_s + Y_s dY_s)(T - s) + \text{BV},$$


where BV denotes a bounded variation term.

- Note that BV terms do not contribute to diamond trees.



## Second term

Similarly, recalling that  $d\mathbb{K}_s^1 = X_s dY_s - Y_s dX_s$ ,

$$\begin{aligned}
 \mathbb{K}^3 &= \mathbb{K}^1 \diamond \mathbb{K}^2 = \text{

- It is easy to check that all odd forests vanish.$$

$\mathbb{K}^4$ 

$$\begin{aligned}
 \mathbb{K}^4 &= \frac{1}{2} \mathbb{K}^2 \diamond \mathbb{K}^2 = \frac{1}{2} \begin{array}{c} \bullet \bullet \bullet \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\
 &= \frac{1}{2} \mathbb{E}_t \left[ \int_t^T [X_s^2 d\langle X \rangle_s + Y_s^2 d\langle Y \rangle_s] (T-s)^2 \right] \\
 &= \frac{1}{2} \int_t^T (\mathbb{E}_t [X_s^2] + \mathbb{E}_t [Y_s^2]) (T-s)^2 ds \\
 &= \int_t^T (s-t)(T-s)^2 ds + \frac{1}{2} (X_t^2 + Y_t^2) \int_t^T (T-s)^2 ds \\
 &= \frac{1}{12} (T-t)^4 + \frac{1}{2} (X_t^2 + Y_t^2) \frac{1}{3} (T-t)^3.
 \end{aligned}$$

- It is now clear how to extend this computation to all orders.

## The general pattern

We see that for each even  $n$ ,  $\mathbb{K}_t^n(T) = a_n I_t^{(n)}(T)$  for some  $a_n \in \mathbb{Q}$  where

$$\begin{aligned} I_t^{(n)}(T) &= \frac{1}{2} \int_t^T (\mathbb{E}_t[X_s^2] + \mathbb{E}_t[Y_s^2]) (T-s)^{n-2} ds \\ &= \frac{(T-t)^n}{n(n-1)} + \frac{1}{2} (X_t^2 + Y_t^2) \frac{1}{n-1} (T-t)^{n-1}. \end{aligned}$$

To compute the forests  $\mathbb{K}^n$ , we need the following lemma.

### Lemma

$$\left( I^{(m)} \diamond I^{(n)} \right)_t(T) = \frac{2}{(m-1)(n-1)} I_t^{(n+m)}(T).$$

## More terms

- Note from above that  $\mathbb{K}^2 = I^{(2)}$  and  $\mathbb{K}^4 = I^{(4)}$ .
- Applying the lemma

$$\begin{aligned}\mathbb{K}^6 &= I^{(4)} \diamond I^{(2)} = \frac{2}{3 \cdot 1} I^{(6)} \\ &= \frac{(T-t)^6}{45} + \frac{2}{3} \frac{1}{2} (X_t^2 + Y_t^2) \frac{1}{5} (T-t)^5.\end{aligned}$$

- In principle, we could go on for ever, computing forests (or cumulants) in this way.
  - As we show in [FGR20], without much extra effort, we can sum all these cumulants and so recover Lévy's theorem.

### Remark

As a comparison, Levin and Wildon[LW08] obtain Lévy's theorem from (a much harder) moment expansion.

# A bivariate $\mathbb{K}$ -expansion

Let  $\mathbb{K}_t^1 = \mathbb{E}_t [a Y_T + b \langle Y \rangle_{t,T}] \equiv a \bullet + b \curvearrowright$ . Then

$$\mathbb{K}^1 = a \bullet + b \curvearrowright$$

$$\mathbb{K}^2 = \frac{1}{2} (a \bullet + b \curvearrowright)^{\diamond 2} = \frac{1}{2} a^2 \curvearrowright + ab \curvearrowright + \frac{1}{2} b^2 \curvearrowright$$

$$\mathbb{K}^3 = \frac{1}{2} a^3 \curvearrowright + \frac{1}{2} a^2 b \curvearrowright + a^2 b \curvearrowright + ab^2 \curvearrowright + \frac{1}{2} ab^2 \curvearrowright + \dots$$

$$\mathbb{K}^4 = \frac{1}{2} a^4 \curvearrowright + \frac{1}{2^3} a^4 \curvearrowright + \frac{1}{2} a^3 b \curvearrowright + \frac{1}{2} a^3 b \curvearrowright + a^3 b \curvearrowright + \frac{1}{2} a^3 b \curvearrowright + \dots$$

$$\mathbb{K}^5 = \frac{1}{2} a^5 \curvearrowright + \frac{1}{2^3} a^5 \curvearrowright + \frac{1}{2^2} a^5 \curvearrowright + \dots \quad (4)$$

# Forest reordering

- We see that the  $\mathbb{G}$ -recursion is equivalent to the bivariate  $\mathbb{K}$ -recursion applied to  $A_T = aY_T + b\langle Y \rangle_T$ , after forest reordering.
  - Reorder by collecting all trees with the same number of leaves.
  - $\mathbb{G}$ -forests consist of trees which are homogenous in the number of leaves • but not in  $a, b$ .
- Note also that forest reordering resolves the infinite cancellations present in the bivariate  $\mathbb{K}$ -expansion.
  - To see this put  $b = -\frac{1}{2}a^2$  in (4) – we see a very complicated expression which must sum to zero.
  - On the other hand putting  $b = -\frac{1}{2}a^2$  in (3) trivially results in zero.

# Forward variance models

- Let  $S$  be a strictly positive continuous martingale.
- Then  $X := \log S$  is a semimartingale with quadratic variation process  $\langle X \rangle$ .
- Following [BG12], it is natural to specify a model in forward variance form.

$$v_t dt := d\langle X \rangle_t$$
$$\xi_t(T) = \mathbb{E}_t[v_T].$$

- Forward variances are tradable assets (unlike spot variance).
- We get a family of martingales indexed by their individual time horizons  $T$ .

# VIX squared

- Consider the payoff of a forward-starting variance swap

$$\begin{aligned}\zeta_T(T) &= \frac{1}{\Delta} \int_T^{T+\Delta} \xi_T(u) du \\ &= \frac{1}{\Delta} \int_T^{T+\Delta} \mathbb{E}_T \int_T^{T+\Delta} v_u du \\ &= \frac{1}{\Delta} \int_T^{T+\Delta} \mathbb{E}_T \langle X \rangle_{T, T+\Delta},\end{aligned}$$

which, when  $\Delta$  is 30 days, is just *VIX* squared.

- The  $\mathbb{G}$ -expansion gives us the joint MGF of  $VIX^2$ ,  $X$  and  $\langle X \rangle$  as follows.



# Triple joint MGF

## Theorem 3 (Theorem 4.4 of [FGR20])

For  $a, b, c \in \mathbb{R}$  sufficiently small,

$$\mathbb{E}_t \left[ e^{aX_T + b \langle X \rangle_{t,T} + c \zeta_T(T)} \right] = \exp \left\{ aX_t + c \zeta_t(T) + \sum_{k=2}^{\infty} \mathbb{G}_t^k \right\},$$

where

$$\mathbb{G}^2 = \left( \frac{1}{2} a(a-1) + b \right) (X \diamond X)_t(T) + ac X \diamond \zeta + \frac{1}{2} c^2 \zeta \diamond \zeta,$$

$$\mathbb{G}^k = \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (aX \diamond \mathbb{G}^{k-1}) \text{ for } k > 2.$$

## Proof.

This is a direct consequence of Theorem 1: The time- $T$  quantity of interest is

$$A_T := a X_T + b \langle X \rangle_{t,T} + c \zeta_T(T)$$

and it suffices to compute (using that  $X + \frac{1}{2} \langle X \rangle$  is martingale),

$$\mathbb{E}_t[A_T] = a X_t + (b - \frac{1}{2} a) (X \diamond X)_t(T) + c \zeta_t(T).$$



- Theorem 3 is completely *model-independent*!
  - It is useful in particular when the diamond trees are easy to compute or approximate.
- We can get the joint MGF of any set of random variables of interest in the same way.
  - For example, VIX futures are martingales. So the joint MGF of SPX and VIX is in principle computable!

# Trees with colored leaves

Denote  $X \equiv \circ$  and  $\zeta \equiv \bullet$ .

- In Theorem 3 we wrote

$$\mathbb{G}^2 = \left(\frac{1}{2}a(a-1) + b\right) \circ \swarrow \circ + ac \circ \swarrow \bullet + \frac{1}{2}c^2 \bullet \swarrow \bullet.$$

- We could define  $(X \diamond X) = M$ , or  $\circ \swarrow \circ = \bullet$ , resulting in trees with leaves of three different colors.
  - In a forward variance model,  $X_t$  represents the log-stock price and  $M_t(T)$ , the expected total variance  $\int_t^T \xi_t(u) du$ .
- Then

$$\mathbb{G}^2 = \left(\frac{1}{2}a(a-1) + b\right) \bullet + ac \circ \swarrow \bullet + \frac{1}{2}c^2 \bullet \swarrow \bullet.$$

- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).

# F-recursion

Putting  $b = -\frac{1}{2}a$  in the G-recursion gives the F-recursion.

## Theorem 4

With  $\mathbb{F}^2 = \frac{1}{2}a(a-1) \heartsuit = \frac{1}{2}a(a-1) \bullet$  and  $\forall k > 2$ ,

$$\mathbb{F}^k = \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{F}^{k-j} \diamond \mathbb{F}^j + (a Y \diamond \mathbb{F}^{k-1}), \quad (5)$$

and we have, for sufficiently small  $a$ ,

$$\log \mathbb{E}_t \left[ e^{aX_T} \right] = a X_t + \sum_{k \geq 2} \mathbb{F}^k. \quad (6)$$

On the other hand, Corollary 3.1 of [AGR2020] reads:

### Corollary

The cumulant generating function (CGF) is given by

$$\psi_t(T; a) = \log \mathbb{E}_t \left[ e^{ia X_T} \right] = ia X_t - \frac{1}{2} a(a+i) M_t(T) + \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}_{\ell}(a). \quad (7)$$

where the  $\tilde{\mathbb{F}}_{\ell}$  satisfy the recursion

$$\tilde{\mathbb{F}}_0 = -\frac{1}{2} a(a+i) M_t = -\frac{1}{2} a(a+i) \bullet \text{ and for } k > 0,$$

$$\tilde{\mathbb{F}}_{\ell} = \frac{1}{2} \sum_{j=0}^{\ell-2} \left( \tilde{\mathbb{F}}_{\ell-2-j} \diamond \tilde{\mathbb{F}}_j \right) + ia \left( X \diamond \tilde{\mathbb{F}}_{\ell-1} \right). \quad (8)$$

- With the identification  $\tilde{\mathbb{F}}_{\ell} = \mathbb{F}^{\ell+2}$ , formulae (6) and (7), and the recursions (5) and (8) are equivalent.

Applying the recursion (8), the first few  $\tilde{\mathbb{F}}$  forests are given by

$$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+i) \bullet$$

$$\tilde{\mathbb{F}}_1 = -\frac{i}{2}a^2(a+i) \bullet \text{---} \bullet$$

$$\tilde{\mathbb{F}}_2 = \frac{1}{2^3}a^2(a+i)^2 \bullet \text{---} \bullet + \frac{1}{2}a^3(a+i) \bullet \text{---} \bullet \text{---} \bullet$$

$$\tilde{\mathbb{F}}_3 = (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + ia \bullet \diamond \tilde{\mathbb{F}}_2$$

$$= \frac{i}{2^2}a^3(a+i)^2 \bullet \text{---} \bullet + \frac{i}{2^3}a^3(a+i)^2 \bullet \text{---} \bullet \text{---} \bullet + \frac{i}{2}a^4(a+i) \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$$

- Note that the total probability and martingale constraints are satisfied for each tree.
  - That is  $\psi_t^T(0) = \psi_t^T(-i) = 0$ .

## Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}_t[X_T] = (-i) \psi_t^{T'}(0) = X_t - \frac{1}{2} M_t(T)$$

and the gamma swap (wlog set  $X_t = 0$ ) by

$$\mathbb{E}_t \left[ X_T e^{X_T} \right] = -i \psi_t^{T'}(-i).$$

### Remark

We can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

# The gamma swap

It is easy to see that only trees containing a single ● leaf will survive in the sum after differentiation when  $a = -i$  so that

$$\begin{aligned} \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}'_{\ell}(-i) &= \frac{i}{2} \sum_{\ell=1}^{\infty} X^{\diamond \ell} M \\ &= \frac{i}{2} \left\{ \text{●} + \text{●} + \text{●} + \dots \right\} \end{aligned}$$

Then the fair value of a gamma swap is given by

$$G_t(T) = 2 \mathbb{E}_t \left[ X_T e^{X_T} \right] = \text{●} + \text{●} + \text{●} + \dots \quad (9)$$

## Remark

Equation (9) allows for explicit computation of the gamma swap for any model written in forward variance form.



# The leverage swap

We deduce that the fair value of a leverage swap is given by

$$\begin{aligned} \mathcal{L}_t(T) &= \mathcal{G}_t(T) - M_t(T) = \sum_{\ell=1}^{\infty} X^{\diamond \ell} M \\ &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \end{aligned} \quad (10)$$

- The leverage swap is expressed explicitly in terms of covariance products of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.

An explicit model-free expression for the leverage swap!

## $\mathcal{L}_t(T)$ directly from the smile

- Let

$$d_{\pm}(k) = \frac{-k}{\sigma_{\text{BS}}(k, T)\sqrt{T}} \pm \frac{\sigma_{\text{BS}}(k, T)\sqrt{T}}{2}$$

and following Fukasawa [Fuk12], denote the inverse functions by  $g_{\pm}(z) = d_{\pm}^{-1}(z)$ . Further define

$$\sigma_{\pm}(z) = \sigma_{\text{BS}}(g_{\pm}(z), T)\sqrt{T}.$$

- It is a well-known corollary of Matytsin's characteristic function representation in [Mat00], that

$$M_t(T) = \int_{\mathbb{R}} dz N'(z) \sigma_-^2(z).$$

- The gamma swap is given by

$$\mathcal{G}_t(T) = \int_{\mathbb{R}} dz N'(z) \sigma_+^2(z).$$

## Fast calibration

- For each  $T$ ,  $\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T)$  may be estimated from the observed smile.
  - In the case of SPX, there are currently between 30 and 40 listed expirations.
- Also,  $\mathcal{L}_t(T) = \sum_{l=1}^{\infty} X^{\diamond l} M$ .
- For models (such as affine forward variance models) where diamond trees are easily computable, fast calibration is then possible.

# Affine forward variance models

Following [GKR19] consider *affine forward variance models* of the form

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t} dZ_t \\ d\xi_t(u) &= \kappa(u-t) \sqrt{v_t} dW_t,\end{aligned}$$

with  $d\langle W, Z \rangle_t = \rho dt$ .

- This class of models includes classical and rough Heston.
- As we will see, diamond trees are particularly easy to compute in AFV models.

# Affine trees

Lemma 5 (Lemma 4.5 of [FGR20] )

*In an affine forward variance model, all diamond trees take the form*

$$\int_t^T \xi_t(u) h(T - u) du$$

*for some function  $h$ .*

# Classical Heston

## Example (Classical Heston)

*In this case,*

$$d\xi_t(u) = \nu e^{-\lambda(u-t)} \sqrt{v_t} dW_t.$$

*Then, for example,*

$$\mathbb{Q} = (X \diamond M)_t(T) = \frac{\rho \nu}{\lambda} \int_t^T \xi_t(u) \left[ 1 - e^{-\lambda(T-u)} \right] du.$$

# Rough Heston

## Example (Rough Heston)

In this case, with  $\alpha = H + 1/2 \in (1/2, 1)$  (and with  $\lambda = 0$ ),

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)} (u - t)^{\alpha-1} \sqrt{v_t} dW_t.$$

Then, for example,

$$\bullet = M_t(T) = (X \diamond X)_t(T) = \int_t^T \xi_t(u) du,$$

$$\begin{aligned} \bullet \bullet &= \frac{\nu^2}{\Gamma(\alpha)^2} \int_t^T \xi_t(u) du \left( \int_u^T (s - u)^{\alpha-1} ds \right)^2 \\ &= \frac{\nu^2}{\Gamma(1 + \alpha)^2} \int_t^T \xi_t(u) (T - u)^{2\alpha} du. \end{aligned}$$

- For a bounded forward variance curve  $\xi$  one then sees that diamond trees with  $k$  leaves are of order  $(T - t)^{1+(k-2)\alpha}$ .
- In this case, the  $\mathbb{F}$ -expansion (forest reordering according to number of leaves) has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter  $\alpha = H + 1/2 \in (1/2, 1)$ , cf. [CGP21, GR19].



# The triple joint MGF in affine forward variance models

- Lemma 5 combined with Theorem 3 characterize the triple-joint MGF of  $X_T$ ,  $\langle X \rangle_T$  and  $\zeta_T(T)$  for an affine forward variance model.
  - Compare with Theorem 4.3 of [AJLP2019] and Proposition 4.6 of [GKR19].

- We obtain the convolutional form

$$\mathbb{E}_t \left[ e^{aX_T + b\langle X \rangle_{t,T} + c\zeta_T(T)} \right] = \exp \{ aX_t + (\xi \star g)(\tau; a, b, c)_t(T) \} .$$

- This is consistent with (and generalizes) Theorem 2.6 of [GKR19] where the same convolution Riccati equation appears, but with  $g = g(\tau; a)$  instead of  $(\tau; a, b, c)$  and different boundary conditions.

# Computation of trees under rough Heston

Abbreviating bounded variation terms as 'BV', we have

$$\begin{aligned}dX_t &= \sqrt{v_t} dZ_t + \text{BV} \\dM_t &= \int_t^T d\xi_t(u) du + \text{BV} \\&= \frac{\nu}{\Gamma(\alpha)} \sqrt{v_t} \left( \int_t^T \frac{du}{(u-t)^\gamma} \right) dW_t + \text{BV} \\&= \frac{\nu(T-t)^\alpha}{\Gamma(1+\alpha)} \sqrt{v_t} dW_t + \text{BV}.\end{aligned}$$

# The first order forest

There is only one tree in the forest  $\tilde{\mathbb{F}}_1$ .

$$\begin{aligned}\tilde{\mathbb{F}}_1 = \text{tree} &= (X \diamond M)_t(T) = \mathbb{E}_t \left[ \int_t^T d\langle X, M \rangle_s \right] \\ &= \frac{\rho\nu}{\Gamma(1+\alpha)} \mathbb{E}_t \left[ \int_t^T v_s (T-s)^\alpha ds \right] \\ &= \frac{\rho\nu}{\Gamma(1+\alpha)} \int_t^T \xi_t(s) (T-s)^\alpha ds.\end{aligned}$$

## Higher order forests

Define for  $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds \xi_t(s) (T - s)^{j\alpha}.$$

Then

$$\begin{aligned} dl_s^{(j)}(T) &= \int_s^T du d\xi_s(u) (T - u)^{j\alpha} + \text{BV} \\ &= \frac{\nu \sqrt{v_s}}{\Gamma(\alpha)} dW_s \int_s^T \frac{(T - u)^{j\alpha}}{(u - s)^\alpha} du + \text{BV} \\ &= \frac{\Gamma(1 + j\alpha)}{\Gamma(1 + (j + 1)\alpha)} \nu \sqrt{v_s} (T - s)^{(j+1)\alpha} dW_s + \text{BV}. \end{aligned}$$

With this notation,

$$\text{orange dot} = \frac{\rho \nu}{\Gamma(1 + \alpha)} I_t^{(1)}(T).$$

## The second order forest

There are two trees in  $\tilde{\mathbb{F}}_2$ :

$$\begin{aligned} \text{🌳} &= \mathbb{E}_t \left[ \int_t^T d\langle M, M \rangle_s \right] \\ &= \frac{\nu^2}{\Gamma(1 + \alpha)^2} \int_t^T \xi_t(s) (T - s)^{2\alpha} ds \\ &= \frac{\nu^2}{\Gamma(1 + \alpha)^2} I_t^{(2)}(T) \end{aligned}$$

and

$$\begin{aligned} \text{🌳} &= \frac{\rho \nu}{\Gamma(1 + \alpha)} \mathbb{E}_t \left[ \int_t^T d\langle X, I^{(1)} \rangle_s \right] \\ &= \frac{\rho^2 \nu^2}{\Gamma(1 + 2\alpha)} I_t^{(2)}(T). \end{aligned}$$

# The third order forest

Continuing to the forest  $\tilde{\mathbb{F}}_3$ , we have the following.

$$\begin{aligned}
 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} &= \frac{\rho \nu^3}{\Gamma(1 + \alpha) \Gamma(1 + 2\alpha)} I_t^{(3)}(T) \\
 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} &= \frac{\rho^3 \nu^3}{\Gamma(1 + 3\alpha)} I_t^{(3)}(T) \\
 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} &= \frac{\rho \nu^3 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2 \Gamma(1 + 3\alpha)} I_t^{(3)}(T).
 \end{aligned}$$

In particular, we readily identify the pattern

$$\left( X^{\diamond \ell} M \right)_t(T) = \frac{(\rho \nu)^\ell}{\Gamma(1 + \ell \alpha)} I_t^{(\ell)}(T).$$

# The leverage swap under rough Heston

Using (10), we have

$$\begin{aligned}\mathcal{L}_t(T) &= \sum_{\ell=1}^{\infty} \left( X^{\diamond \ell} M \right)_t (T) \\ &= \sum_{\ell=1}^{\infty} \frac{(\rho \nu)^\ell}{\Gamma(1 + \ell \alpha)} \int_t^T du \xi_t(u) (T - u)^{\ell \alpha} \\ &= \int_t^T du \xi_t(u) \{ E_\alpha(\rho \nu (T - u)^\alpha) - 1 \}\end{aligned}$$

where  $E_\alpha(\cdot)$  denotes the Mittag-Leffler function.

An explicit expression for the leverage swap!

- Since we can impute the leverage swap  $\mathcal{L}_t(t)$  from the smile for each expiration  $T$ , fast calibration of the rough Heston model is possible.

# Summary

- We introduced the diamond product.
- We defined the  $\mathbb{G}$ -expansion and gave an idea of its proof.
  - The cumulant expansion of [LRV19] and the Exponentiation Theorem of [AGR2020] are special cases.
- We showed how easy computations can be in affine forward variance models.
  - Quick calibration of such models is one application.



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