Realized cumulants for martingales

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Introduction: cumulants in financial stochastics

- cumulants of asset returns
  - guide for statistical modeling of price processes
  - the 3rd and 4th cumulants characterize high-frequency asymptotic error distribution of discrete hedging; cf.
    Fukasawa, 2011, "Discretization error of stochastic integrals"
- cumulants of price under pricing measure
  .... characterize deviation from the Black-Scholes model
  - 3rd cumulant as the at-the-money implied volatility skew (slope)
  - 4th cumulant as the at-the-money implied volatility curvature
  - Edgeworth expansion of option price
  - model selection
  - control variate for the Monte-Carlo pricing
- cumulant "risk premium" characterizing the aggregate utility
  - expected return interpreted as a risk premium
  - realized and implied variance deviation as the variance risk premium
  - higher-order ?

Introduction: realized variance

Let $\Pi_{a,b}$ be a partition of an interval $(a, b] \subset \mathbb{R}$, that is, a collection of disjoint subintervals of $(a, b]$ with

$$\bigcup_{(s,t]\in\Pi_{a,b}} (s, t] = (a, b].$$

Denote $|\Pi_{a,b}| = \max_{(s,t]\in\Pi_{a,b}} |t - s|$.
For a process Z, denote $Z_{s,t} = Z_t - Z_s$.
Let M be a (log) price process. The realized variance (a.k.a realized quadratic variation) on a period $(a, b]$ (associated with $\Pi_{a,b}$) is defined as

$$\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2 .$$

For a sequence of partitions $\Pi^n_{a,b}$ with $|\Pi^n_{a,b}| \to 0$, we have

$$\sum_{(s,t]\in\Pi^n_{a,b}} |M_{s,t}|^2 \to [M]_{a,b} = \langle M^c\rangle_{a,b} + \sum_{t\in(a,b]} |\Delta M_t|^2 \text{ in prob.}$$

Introduction: high-frequency data for estimating low-frequency distribution

If M is an $L^2(Q)$ martingale, then

$$E^Q[|M_{s,t}|^2] = E^Q[|M_{0,t}|^2 - |M_{0,s}|^2]$$

and so

$$E^Q\left[\sum_{(s,t]\in\Pi_{a,b}} |M_{s,t}|^2\right] = E^Q[|M_{a,b}|^2]$$

that connects high and low frequency distributions.
Neuberger (2012) introduced the notion of the aggregation property:

$$E[g(\mathbb{X}_{s,u})|\mathcal{F}_s] = E[g(\mathbb{X}_{s,t})|\mathcal{F}_s] + E[g(\mathbb{X}_{t,u})|\mathcal{F}_s]$$

for $s \leq t \leq u$. This property is met by $g(x) = x^2$ and $\mathbb{X} = M$ if M is a martingale.

Introduction: realized skewness

The aggregation property implies

$$E\left[\sum_{(s,t]\in\Pi_{a,b}} g(\mathbb{X}_{s,t})\right] = E\left[g(\mathbb{X}_{a,b})\right]$$

for any partition $\Pi_{a,b}$. Neuberger also found that the aggregation property is met by

$$g_2(x,y) = x^3 + 3xy, \quad \mathbb{X}^{(2)} = (M, M^{(2)})$$

if M is an L^3 martingale, where $M_t^{(n)} = E[(M_T - M_t)^n|\mathcal{F}_t]$ for $t \leq T$. Noticing

$$E[g_2(\mathbb{X}^{(2)}_{0,T})] = E[(M_T - M_0)^3],$$

Neuberger named $\sum_{(s,t]\in\Pi_{0,T}} g_2(\mathbb{X}^{(2)}_{s,t})$ the realized skewness.

Introduction: realized kurtosis

Recently, Bae and Lee (2020) further extended the idea to find that the aggregation is met by

$$g_3(x,y,z) = x^4 + 6x^2y + 3y^2 + 4xz, \quad \mathbb{X} = (M, M^{(2)}, M^{(3)}).$$

Further,

$$E[g_3(\mathbb{X}^{(3)}_{0,T})] = E[(M_T - M_0)^4] - 3E[(M_T - M_0)^2]^2.$$

Comments:

- the way to find those polynomials was brute-force; actually they showed that there is no other analytic function of $\mathbb{X}^{(3)}$ with the aggregation property (up to a linear combination).
- moments (and cumulants) of an asset under the pricing measure can be computed from its option market data. A bias of the realized cumulant is interpreted as a risk premium.

Introduction: our finding

Our finding is that

$$g_n(x_1, \dots, x_n) = B_{n+1}(x_1, \dots, x_n, 0)$$

satisfies the aggregation property with

$$\mathbb{X}^{(n)} = (X^{(1)}, \dots, X^{(n)})$$

for any $n \in \mathbb{N}$, where B_{n+1} is the $(n+1)$-th complete Bell polynomial and $X^{(i)}$ is the ith conditional cumulant process of an L^{n+1} integrable r.v. X. In fact, we show

$$E[g_n(\mathbb{X}^n_{s,t})|\mathcal{F}_s] = -E[X^{(n+1)}_{s,t}|\mathcal{F}_s].$$

When $X = M_T$, then $\mathbb{X}^{(3)} = (M, M^{(2)}, M^{(3)})$, $X^{(n)}_T = 0$ for $n \geq 2$, and

$$X^{(n+1)}_s = E[g_n(\mathbb{X}^{(n)}_{s,T})|\mathcal{F}_s] = E\left[\sum_{(t,u]\in\Pi_{s,T}} g_n(\mathbb{X}^{(n)}_{t,u})\middle|\mathcal{F}_s\right]$$

for any partition $\Pi_{s,T}$.

The complete Bell polynomials

Definition.

For $(x_1, \dots, x_n) \in \mathbb{R}^n$, the n th complete Bell polynomial $B_n(x_1, \dots, x_n)$ is defined by

$$B_n(x_1, \dots, x_n) = \frac{\partial^n}{\partial z^n} \exp\left(\sum_{i=1}^n x_i \frac{z^i}{i!} \right) \Bigg|_{z=0}$$

with $B_0 = 1$

Examples are

$$\begin{aligned}
&B_1(x_1) = x_1, \\
&B_2(x_1, x_2) = x_1^2 + x_2, \\
&B_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3, \\
&B_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4, \\
&\dots
\end{aligned}$$

$$B_n(x_1, 0, \ldots, 0) = \frac{\partial^n}{\partial z^n} \sum_{k=0}^{\infty} \frac{1}{k!} (x_1 z)^k \bigg|_{z=0} = \frac{\partial^n}{\partial z^n} \frac{1}{n!} (x_1 z)^n \bigg|_{z=0} = (x_1)^n,$$

$$B_n(0, \ldots, 0, x_n) = \frac{\partial^n}{\partial z^n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(x_n \frac{z^n}{n!} \right)^k \bigg|_{z=0} = \frac{\partial^n}{\partial z^n} \left(x_n \frac{z^n}{n!} \right) \bigg|_{z=0} = x_n.$$

Proposition. (binomial property)

Let $n \in \mathbb{N}$ and $(x_1, \ldots x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then,

$$B_n(x_1 + y_1, \ldots, x_n + y_n) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j}(x_1, \ldots, x_{n-j}) B_j(y_1, \ldots, y_j).$$

In particular, we have a push-out property:

$$\begin{aligned} B_n(x_1, \ldots, x_{n-1}, x_n) &= B_n(x_1, \ldots, x_{n-1}, 0) + B_n(0, \ldots, 0, x_n) \\ &= B_n(x_1, \ldots, x_{n-1}, 0) + x_n. \end{aligned}$$

Cumulants

Let $p \geq 1$ and $X \in L^p$. For $n \leq p$, the n th cumulant κ_n of X is defined by

$$\kappa_n = (-\sqrt{-1})^n \frac{\partial^n}{\partial z^n} \log E[e^{\sqrt{-1}zX}]\big|_{z=0}.$$

We have

$$E[X^n] = B_n(\kappa_1, \ldots, \kappa_n) = B_n(\kappa_1, \ldots, \kappa_{n-1}, 0) + \kappa_n$$

because

$$\exp\left(\sum_{n=1}^{\infty} x_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{z^n}{n!}$$

as long as convergent, and it is convergent for $x_n = E[X^n]$ when $X \in L^\infty$ (Note L^∞ is dense in L^p).

On a filtered probability space, we define the n th conditional cumulant process $X^{(n)} = \{X_t^{(n)}\}$ by $X_t^{(1)} = E[X|\mathcal{F}_t]$ and

$$X_t^{(n)} = E[X^n|\mathcal{F}_t] - B_n(X_t^{(1)}, \ldots, X_t^{(n-1)}, 0).$$

Key lemma

Denote $\mathbb{X}^{(n)} = (X^{(1)}, \dots, X^{(n)})$.

Lemma.

For any stopping times $\tau \leq \upsilon$,

$$E[B_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\tau] = 0$$

Consequences:

- Let $g_n(x_1, \dots, x_n) = B_{n+1}(x_1, \dots, x_n, 0)$. Then,

  $$E[g_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\tau] + E[X^{(n+1)}_{\tau,\upsilon}|\mathcal{F}_\tau] = E[B_{n+1}(\mathbb{X}^{(n+1)}_{\tau,\upsilon})|\mathcal{F}_\tau] = 0$$

  hence the aggregation property:

  $$E[g_n(\mathbb{X}^{(n)}_{\sigma,\tau})|\mathcal{F}_\sigma] + E[g_n(\mathbb{X}^{(n)}_{\tau,\upsilon})|\mathcal{F}_\sigma] = E[g_n(\mathbb{X}^{(n)}_{\sigma,\upsilon})|\mathcal{F}_\sigma].$$

- Not only $B_n(\mathbb{X}^{(n)}_t) = E[X^n|\mathcal{F}_t]$ but also $B_n(\mathbb{X}^{(n)}_{0,t})$ is a martingale.

Proof of Lemma. When $X \in L^\infty$,

$$\frac{1}{E[e^{zX}|\mathcal{F}_\tau]} = \left(\sum_{n=0}^{\infty} E[X^n|\mathcal{F}_\tau]\frac{z^n}{n!}\right)^{-1} = \exp\left(-\sum_{n=1}^{\infty} X_\tau^{(n)}\frac{z^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty} B_n(-\mathbb{X}_\tau^{(n)})\frac{z^n}{n!}$$

on a neighborhood of $z = 0$. This implies

$$\left(\sum_{n=0}^{\infty} E[X^n|\mathcal{F}_\tau]\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(-\mathbb{X}_\tau^{(n)})\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E[X^{n-j}|\mathcal{F}_\tau]B_j(-\mathbb{X}_\tau^{(j)})\frac{z^n}{n!}$$

is equal to 1, and so for $n \geq 1$,

$$0 = \sum_{j=0}^{n}\binom{n}{j}E[X^{n-j}|\mathcal{F}_\tau]B_j(-\mathbb{X}_\tau^{(j)}) = E\left[\sum_{j=0}^{n}\binom{n}{j}E[X^{n-j}|\mathcal{F}_\upsilon]B_j(-\mathbb{X}_\tau^{(j)})\middle|\mathcal{F}_\tau\right].$$

The right hand side coincides with $E[B_n(\mathbb{X}_\upsilon^{(n)} - \mathbb{X}_\tau^{(n)})|\mathcal{F}_\tau]$ by Proposition.

Example: Lévy process

When $X = L_T$ for a Lévy process L with triplet (μ, σ^2, ν),

$$X_t^{(1)} = L_t + (T - t)\mu,$$
$$X_t^{(2)} = (T - t)\left(\sigma^2 + \int x^2 \nu(\mathrm{d}x)\right),$$
$$X_t^{(n)} = (T - t)\int x^n \nu(\mathrm{d}x) \quad (n \geq 3)$$

and so,

$$X_{0,t}^{(1)} = L_t - L_0 - \mu t,$$
$$X_{0,t}^{(2)} = -t\left(\sigma^2 + \int x^2 \nu(\mathrm{d}x)\right),$$
$$X_{0,t}^{(n)} = -t\int x^n \nu(\mathrm{d}x) \quad (n \geq 3).$$

We have that $B_n(\mathbb{X}_{0,t}^{(n)})$ is a martingale. In particular for $L = W$ (a Brownian motion), $B_n(\mathbb{X}_{0,t}^{(n)}) = t^{n/2} H_n(t^{-1/2} W_t)$, where H_n is the n th Hermite polynomial.

Main result

Theorem.

Let $p > 2$, $T > 0$ and $X \in L^p$ be $\mathcal{F}_T$ measurable. Then, for any (possibly stochastic) partition $\Pi_{\sigma,T}$ and for any $n \leq p - 1$,

$$X_\sigma^{(n+1)} = E\left[\sum_{(\tau,\upsilon]\in\Pi_{\sigma,T}} g_n(\mathbb{X}_{\tau,\upsilon}^{(n)})\middle|\mathcal{F}_\sigma\right]$$

Proof: Use the aggregation property and the fact that

$$E[g(\mathbb{X}_{\sigma,T}^{(n)})|\mathcal{F}_\sigma] = -E[X_{\sigma,T}^{(n+1)}|\mathcal{F}_\sigma] = X_\sigma^{(n+1)}.$$

Realized cumulants

If M is a martingale and $X = M_T$, then $X^{(1)} = M$. We name

$$\sum_{(\tau,\upsilon]\in\Pi_{\sigma,T}} g_n(\mathbb{X}^{(n)}_{\tau,\upsilon})$$

the $n+1$ th realized cumulant for M associated with the partition $\Pi_{\sigma,T}$ of the period $(\sigma, T]$.

A cumulant recursion formula

By taking the high-frequency limit $|\Pi_{\sigma,T}| \to 0$, we have the following.

Theorem.

If $X \in \bigcap_{p>1} L^p$ and $\mathcal{F}_T$ measurable, then

$$X_\sigma^{(n+1)} = E\left[\sum_{s\in(\sigma,T]} g_n(\Delta\mathbb{X}_s^{(n)}) + \frac{1}{2}\sum_{j=1}^{n}\binom{n+1}{j}\langle X^{(n+1-j),c}, X^{(j),c}\rangle_{\sigma,T}\middle|\mathcal{F}_\sigma\right]$$

for any $n \in \mathbb{N}$, where $X^{(j),c}$ is the continuous local martingale part of the semimartingale $X^{(j)}$.

- An independent derivation by Friz, Hager and Tapia (2021).
- Continuous case ($\Delta\mathbb{X}^{(n)} = 0$) by Lacoin, Rhodes and Vargas (2019), and Friz, Gatheral and Radoičic (2020).

The second term of the RHS comes from the fact that the quadratic terms contained in $B_{n+1}(x_1, \ldots, x_{n+1})$ are

$$\frac{1}{2}\sum_{j=1}^{n}\binom{n+1}{j}x_{n+1-j}x_j$$

since

$$\exp\left(\sum_{i=1}^{\infty}x_i\frac{z^i}{i!}\right) = 1 + \sum_{i=1}^{\infty}x_i\frac{z^i}{i!} + \frac{1}{2}\left(\sum_{i=1}^{\infty}x_i\frac{z^i}{i!}\right)^2 + \ldots.$$