Cumulant operators for stochastic integrals

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Cumulant operators and Malliavin Calculus

We consider the Malliavin Derivative D and its adjoint δ:

$$\mathbb{E}(F\delta(v)) = \mathbb{E}(\langle DF, v\rangle_H), \quad F \in \mathrm{Dom}(D),\ v \in \mathrm{Dom}(\delta)$$

where $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$. For the next definiton, one takes $F \in D_{2,1}$ and $u \in D_{k,2}(U)$. The map

$$\Gamma_k^u : D_{2,1} \to L^2(\Omega)$$

is given in terms of u and with k D-derivatives.

Moments formula

The main formula is

$$\mathbb{E}(F\delta(u)^n) = \sum_{k=1}^{n} \frac{n!}{k!} \sum_{d_1+\ldots+d_k=n} \frac{1}{d_1 \cdots d_k} \mathbb{E}(\Gamma_{d_1}^u \cdots \Gamma_{d_k}^u F)$$

In the specific case, $u \in D_{n+1,2}(H)$ with $\|u\|_H$ deterministic and $\textbf{Trace}(Du)^{k+1} = 0$, $1 \leq k \leq n$:

$$\mathbb{E}(\delta(u)^{n+1}) = n\|u\|_H^2 \mathbb{E}(\delta(u)^{n-1})$$

the same moments as for $\mathcal{N}(0, \|u\|_H^2)$.

Application to the Stein approximation

Edgeworth type expansions: formula for $\mathbb{E}(F\delta(u)\Phi(\delta(u)))$. One can have precise bounds on the quantity

$$d_w(\delta(u), \mathcal{N}) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}[h(\delta(u))] - \mathbb{E}[h(\mathcal{N})]|,$$

for $u \in D_{k,2}(H)$ with $k \in \{1, 2, 3\}$.

Poisson Point process

Let consider $\Omega^{\mathbb{X}} = \{\omega = \sum_{k\geq 1} \delta_{x_k},\ x_k \in \mathbb{X}\}$ equipped with the Poisson probability measure with intensity measure σ on $\mathbb{X}$. We set

$$\varepsilon_x^+ F(\omega) = F(\omega \cup \{x\}), \quad x \in \mathbb{X}^n$$

For $n \geq 1$ and $u : \Omega^{\mathbb{X}} \times \mathbb{X} \to \mathbb{R}$

$$\mathbb{E}\left[\left(\sum_{x\in\omega} u(x,\omega)\right)^n\right]$$

$$= \sum_{k=1}^{n} \sum_{P_1\cup\cdots\cup P_k=\{1,\ldots,n\}} \mathbb{E}\left(\int_{\mathbb{X}^k} \varepsilon^+_{x_1,\ldots,x_k}(u^{|P_1|}(x_1)\cdots u^{|P_k|}(x_k)) \prod_{i=1}^{k} \sigma(dx_i)\right)$$

Some applications: Random stopping sets, Computation of the moments of shot noise.

Questions

- Other examples of point processes?
- Cumulant operators connections with free probability?
- Malliavin calculus exists for Rough Paths and Regularity Structures. Any potential benefit from this point of view?