Signature Cumulants and Ordered Partitions

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Overview

Classical cumulants as tensors

Classical cumulants as tensors

Given a random variable X on $\mathbb{R}$, its n:th moment and cumulant can be defined as

$$\mathbb{E}[X^n] = \left(\frac{\partial}{\partial u}\right)^n \mathbb{E}[e^{uX}]|_{u=0}$$
$$\kappa^n(X) = \left(\frac{\partial}{\partial u}\right)^n \log \mathbb{E}[e^{uX}]|_{u=0}$$

For multivariate X:

$$\kappa^{i_1,\ldots,i_n}(X) = \frac{\partial^n}{\partial i_1 \cdots \partial i_n} \log \mathbb{E}[e^{\langle u,X\rangle}]|_{u=0}$$

Classical cumulants as tensors

Moments of an $\mathbb{R}^d$-valued random variable X as a series of tensors:

$$\mu_X = (\mathbb{E}X^{\otimes m})_{m\geq 0},$$

so that $\mathbb{E}[X^{i_1}\cdots X^{i_m}] = \langle \mu_X, e_{i_1}\otimes\cdots\otimes e_{i_m}\rangle$.

The *tensor algebra* of $\mathbb{R}^d$:

$$\mathrm{T}(\!(\mathbb{R}^d)\!) = \prod_{n\geq 0}(\mathbb{R}^d)^{\otimes n}.$$

Duality between $\mathrm{T}(\!(\mathbb{R}^d)\!)$ and words on $1,\ldots,d$ by

$$e_{i_1\cdots i_m} = e_{i_1}\otimes\cdots\otimes e_{i_m}.$$

Classical cumulants as tensors

The tensor logarithm and exponential

$$\exp(x) = \sum_{m\geq 1} \frac{1}{m!} x^{\otimes m}, \quad \log(1+x) = \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} x^{\otimes m}.$$

The cumulants can be defined in a similar fashion

$$\kappa_X = \mathrm{Usym} \circ \log \mathbb{E} \exp(X),$$

$$\mathrm{Usym}(v_1 \otimes \cdots \otimes v_m) = \sum_{\sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$$

With this notation we have

$$\kappa(X^{i_1}, \ldots, X^{i_m}) = \langle \kappa_X, e_{i_1 \cdots i_m} \rangle := \langle \kappa_X, e_{i_1} \otimes \cdots \otimes e_{i_m} \rangle$$

Classical cumulants as tensors

- Given a set S, a partition π of S is a collection of disjoint subsets such that their union equals S.
- The partitions of the set $\{1, \ldots, d\}$ is denoted $P(d)$.
- The relation $\pi \leq \tau$ if every block of π is contained in a block of τ makes $P(S)$ into a partially ordered set (poset)

Classical cumulants as tensors

Notation: for $S = \{S_1, \ldots, S_k\} \subseteq \{1, \ldots, d\}$

$$X^S := X^{S_1} \cdots X^{S_k}, \quad X_S := (X^{S_1}, \ldots, X^{S_k})$$

For $\pi = \{\pi_1, \ldots, \pi_k\} \in P(d)$:

$$\mu_X(\pi) := \prod_{i=1}^{k} \mathbb{E}[X^{\pi_i}], \quad \kappa_X(\pi) := \prod_{i=1}^{k} \kappa(X_{\pi_i})$$

Example

$$\mu_X(\{\{1,2\},\{3\}\}) = \mathbb{E}[X^1 X^2]\mathbb{E}[X^3]$$
$$\kappa_X(\{\{1,2\},\{3\}\}) = \kappa(X^1, X^2)\kappa(X^3)$$

Classical cumulants as tensors

The Shiryaev relations

$$\langle \mu_X, e_{1\cdots d}\rangle = \mathbb{E}[X^1 \cdots X^d] = \sum_{\pi \in P(d)} \kappa_X(\pi)$$

$$\langle \kappa_X, e_{1\cdots d}\rangle = \kappa(X^1, \ldots, X^d) = \sum_{\pi \in P(d)} (-1)^{|\pi|-1}(|\pi|-1)!\mu_X(\pi)$$

Generalised, for any $\pi \in P(S)$ it holds that

$$\mu_X(\pi) = \sum_{\sigma \leq \pi} \kappa_X(\sigma),$$

$$\kappa_X(\pi) = \sum_{\sigma \leq \pi} m(\sigma, \pi)\mu_X(\sigma).$$

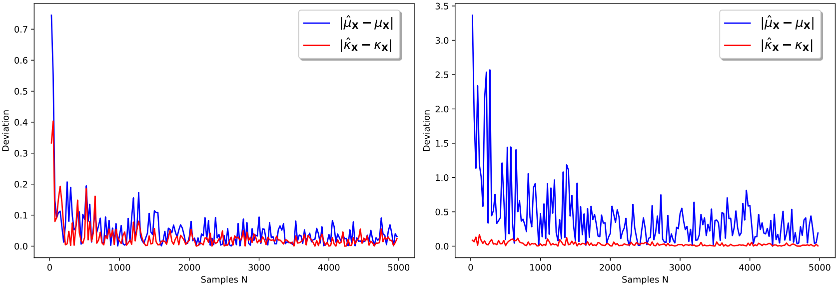
where m is the Möbius function associated the partition poset $P(S)$.

Classical cumulants as tensors

The Shiryaev relations are useful for computing cumulants in practise!
Cumulants have the following property

$$X, Y \text{ are independent } \iff \text{ all their cross-cumulants vanish.}$$

which is used in practise for independence testing in e.g Independent Component Analysis.



(a) $\mu = 1$ (b) $\mu = 10$

Figure: $X \sim N(\mu, 1)$

Classical cumulants as tensors

When X is a stochastic process:

- The signature $S(X)$ is a natural replacement for the exponential
- The expected signature takes the place of the moment generating function

Definition

The *signature cumulant* of a stochastic process X is defined as:

$$\kappa(X) = \log \mathbb{E}S(X)$$

Can we derive similar relations for $\kappa(X)$?

Signatures

Signatures

The signature of a path $x : [0, T] \to \mathbb{R}^d$ is a natural extension of the exponential function to paths.

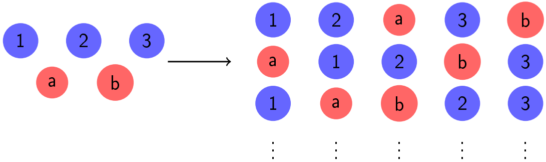
$$
\begin{aligned}
S(x) = \ (S(x)_m)_{m\geq 0} &= (\int \mathrm{d}x^{\otimes m})_{m\geq 0} \\
&= \Big(\int\limits_{0<t<T} \mathrm{d}x_t, \int\limits_{0<t_1<t_2<T} \mathrm{d}x_{t_1} \otimes \mathrm{d}x_{t_2}, \\
&\qquad \int\limits_{0<t_1<t_2<t_3<T} \mathrm{d}x_{t_1} \otimes \mathrm{d}x_{t_2} \otimes \mathrm{d}x_{t_3}, \dots
\end{aligned}
$$

If $x_t = tv$, then

$$
S(x) = \exp(v).
$$

Definition

A *shuffle* of $w_1 = i_1 \cdots i_m$ and $w_2 = j_1 \cdots j_n$ is any possible way of interlacing w_1 and w_2 while preserving their internal order.



Definition

The shuffle product ш is defined by linear extension of the map

$$e_{w_1} ш e_{w_2} = \sum_{w \in Sh(w_1, w_2)} e_w$$

where the sum is taken over all shuffles of w_1 and w_2.

Signatures

If x is a bounded variation path on $\mathbb{R}^d$ and $f, g \in T((\mathbb{R}^d)^\star)$ one can check that integration by parts yields

$$\langle S(x), f\rangle \langle S(x), g\rangle = \langle S(x), f \sqcup\!\sqcup g\rangle. \tag{1}$$

e.g

$$\langle S(x), e_1\rangle \langle S(x), e_2\rangle = x^1 x^2 = \int x^1 \mathrm{d}x^2 + \int x^2 \mathrm{d}x^1 = \langle S(x), e_{12} + e_{21}\rangle$$

More generally, if x is a rough path, we say that it's *geometric* if 1 holds.

Signatures

The expected signature

$$\mu_X := \mathbb{E}[S(X)].$$

characterises the law of a stochastic process X like the moments of a random variable.
We expect the signature cumulant

$$\kappa_X := \log \mu_X.$$

To have similar properties to classical cumulants.
The natural commutative product on $\mathrm{T}((\mathbb{R}^d)^\star)$ is the shuffle, so we consider expressions of the form

$$\langle \kappa_X, f_1 ш \cdots ш f_n \rangle$$

with $f_1, \ldots, f_n \in \mathrm{T}((\mathbb{R}^d)^\star)$.

Classical Shiryaev relations:

$$\langle \kappa_X, e_{1\dots d}\rangle = \sum_{\pi \in P(d)} (-1)^{|\pi|-1} \frac{|\pi|!}{|\pi|} \mu_X(\pi)$$

Signature Shiryaev relations:

Theorem

Given a geometric rough path X, it holds that

$$\langle \kappa_X, e_{w_1} ш \cdots ш e_{w_k}\rangle = \sum_{\pi \in \mathrm{Orp}(w_1,\dots,w_k)} (-1)^{|\pi|-1} \frac{\pi!}{|\pi|} \mu_X(\pi)$$

Shuffles and Ordered Partitions

Shuffles and Ordered Partitions

- A poset is a tuple $(P, \leq)$ where P is a set and $\leq$ is a partial order on P.
- A function $f : P_1 \to P_2$ between two posets is said to be *order-preserving* if $f(x) \geq f(y)$ in P_2 whenever $x \geq y$ in P_1, the set of such functions is denoted $\mathrm{Hom}(P_1, P_2)$.

$(\mathbb{N}, \leq)$ is a poset with its usual total order.

- P is called a *chain* if it is totally ordered
- P is called an *antichain* if no distinct elements are comparable

Shuffles and Ordered Partitions

For a finite set S, the lattice of partitions of S can be defined as

$$\mathcal{P}(S) := \{\ker(f) \mid f : S \to \mathbb{N}\}$$

A natural definition when P is a poset is

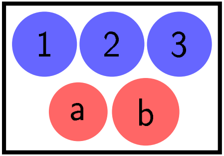
Definition

Let $(P, \leq)$ be a finite poset. We call

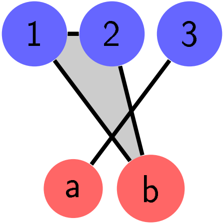
$$\mathrm{Orp}(P) := \{\ker(f) \mid f \in \mathrm{Hom}(P, \mathbb{N})\}$$

the set of ordered partions of P.
Equivalently: "The partitions that can be enumerated without breaking the partial order"

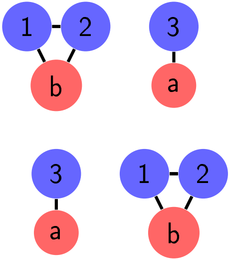
Shuffles and Ordered Partitions



(a) A set with a partial order

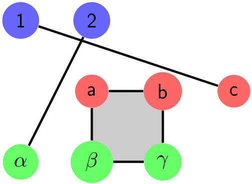


(b) Not an ordered partition

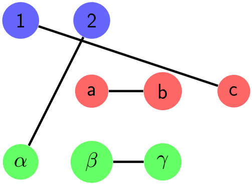


(c) Any enumeration breaks the internal order

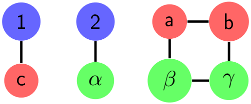
Shuffles and Ordered Partitions



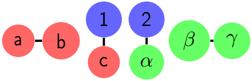
(a) Not an ordered partition



(b) An ordered partition



(c) can't be enumerated



(d) can be enumerated

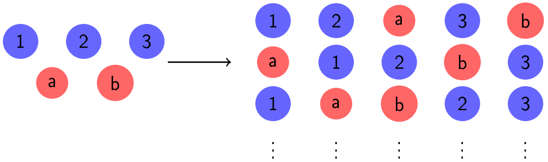
Shuffles and Ordered Partitions

Definition

We denote by $\mathsf{P}_{(n_1,\ldots,n_k)}$ the poset $P = C_1 \cup \cdots \cup C_k$ where each C_i is a chain of length n_i and $x \in C_i, y \in C_j$ are comparable if and only if $i = j$.

If $\tau_1 = i_1 \cdots i_m, \tau_2 = i_{m+1} \cdots i_{m+n}$ then their shuffles can equivalently be defined as

$$\mathrm{Sh}(\tau_1, \tau_2) := \{f(\tau_1, \tau_2) = (i_{f^{-1}(1)}, \ldots, i_{f^{-1}(m+n)}) \mid \text{bijections } f \in \mathrm{Hom}(\mathsf{P}_{(m,n)}, \mathsf{P}_{(m+n)})\}.$$



Shuffles and Ordered Partitions

Definition

For $\pi \in \mathrm{Orp}(P)$, we define its *factorial* $\pi!$ as

$$\pi! := \#\{f \in \mathrm{Hom}(P, [|\pi|]) \big| \ker(f) = \pi\}.$$

- If P is an antichain, then $\pi! = |\pi|!$
- If P is a chain, then $\pi! = 1$
- If $P = C_1 \cup \cdots \cup C_k = \mathsf{P}_{(|C_1|, \dots, |C_k|)}$, and $\pi = (\pi \cap C_1) \cup \cdots \cup (\pi \cap C_k)$ is a disjoint union of its intersection with each chain, then

$$\pi! = \frac{|\pi|!}{\prod_{i=1}^{k} |\pi \cap C_i|!}.$$

Signature Cumulants

Signature Cumulants

Recall that the signature cumulant of a geometric rough path X is defined as

$$\kappa_X = \log \mu_X = \log \mathbb{E}S(X)$$

Using the notation $\mu_X(\pi) = \langle \mu_X, e_{\pi_1} \rangle \cdots \langle \mu_X, e_{\pi_n} \rangle$ we can write down the signature Shiryaev relations.

Theorem

Given a geometric rough path X*, it holds that*

$$\langle \kappa_{\mathrm{X}}, e_{w_1} ш \cdots ш e_{w_k} \rangle = \sum_{\pi \in \mathrm{Orp}(w_1, \dots, w_k)} (-1)^{|\pi|-1} \frac{\pi!}{|\pi|} \mu_{\mathrm{X}}(\pi)$$

where $(w_1, \dots, w_k)$ has the poset structure of $\mathrm{P}_{(n_1, \dots, n_k)}$.

Signature Cumulants

Example:

$$
\begin{aligned}
\langle \kappa_X, e_{12} ш e_3 \rangle =& \mathbb{E}\Big[\langle S(X), e_{12}\rangle \langle S(X), e_3\rangle\Big] \\
&- \mathbb{E}\Big[\langle S(X), e_{12}\rangle\Big]\mathbb{E}\Big[\langle S(X), e_3\rangle\Big] \\
&- \frac{1}{2}\mathbb{E}\Big[\langle S(X), e_1\rangle \langle S(X), e_3\rangle\Big]\mathbb{E}\Big[\langle S(X), e_2\rangle\Big] \\
&- \frac{1}{2}\mathbb{E}\Big[\langle S(X), e_1\rangle\Big]\mathbb{E}\Big[\langle S(X), e_2\rangle \langle S(X), e_3\rangle\Big] \\
&+ \mathbb{E}\Big[\langle S(X), e_1\rangle\Big]\mathbb{E}\Big[\langle S(X), e_2\rangle\Big]\mathbb{E}\Big[\langle S(X), e_3\rangle\Big]
\end{aligned}
$$

Compare with the classic cumulant

$$
\kappa(X_1, X_2, X_3) = \mathbb{E}[X_1 X_2 X_3] - \mathbb{E}[X_1 X_2]\mathbb{E}[X_3]
$$
$$
-\mathbb{E}[X_1 X_3]\mathbb{E}[X_2] - \mathbb{E}[X_2 X_3]\mathbb{E}[X_1] + 2\mathbb{E}[X_1]\mathbb{E}[X_2]\mathbb{E}[X_3]
$$

Signature Cumulants

Corollary

$$\langle \kappa_X, e_{i_1} ш \cdots ш e_{i_m} \rangle = \kappa(X^{i_1}_{0,T}, \ldots, X^{i_m}_{0,T}).$$

Proposition

Two rough paths X, Y are independent if and only if their signature cross-cumulants (of shuffles) vanish (+ extra conditions)

Used in the recent paper [1]

[1] Nonlinear Independent Component Analysis For Continuous-Time Signals, A. Schell, H. Oberhauser, 2021

Estimation

Estimation

Easy to show that

$$\hat{\kappa}_n(w_1, \ldots, w_k) := \sum_{\pi \in \mathrm{Orp}(w_1, \ldots, w_k)} (-1)^{|\pi|-1} \frac{\pi!}{|\pi|} \hat{\mu}_n(\pi)$$

is the minimum variance unbiased estimator for

$$\langle \kappa_X, e_{w_1} ш \cdots ш e_{w_k} \rangle.$$

- Asymptotically normal with explicit asymptotic covariance.
- Typically lower variance than moment estimators.

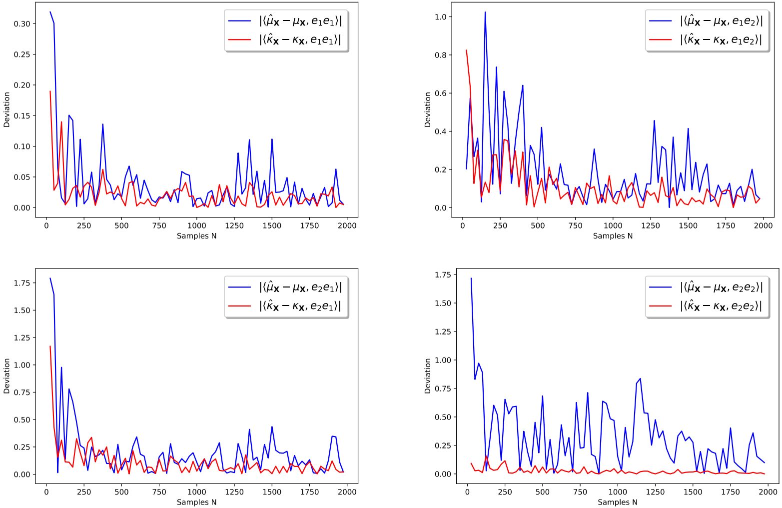


Figure: $\vec{X} = \vec{b}t + \vec{B}_t$, $\vec{b} = \binom{1}{10}$.

Thank you!

P. Bonnier and H. Oberhauser,
"Signature Cumulants, Ordered Partitions,
and Independence of Stochastic Processes",
arXiv:1908.06496, 2019