Cumulants for Lévy-type processes

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Lévy-type processes of non-commutative random variables.

$(\mathcal{A}, \mathbb{E})$ a finite von Neumann algebra with a tracial state (e.g. bounded random variables $(L^\infty(\Lambda, \Sigma), \mathbb{E})$).

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$\{X^{(j)}(t) : t \geq 0, 1 \leq j \leq d\}$ a d-dimensional Lévy-type process:

- $X^{(j)}(0) = 0$.
- Stationary increments.
- Increments independent in some sense.
- Continuous: either each $t \mapsto \mathbb{E}\left[X^{(u(1))}(t) \ldots X^{(u(n))}(t)\right]$ continuous (differentiable), or $d = 1$ and for each $\varepsilon > 0$, $\mathbb{E}\left[\mathbf{1}_{[-\varepsilon,\varepsilon]^c}(X(t))\right] \to 0$ as $t \to 0$.

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Throughout denote $X^{(j)}(1) = X^{(j)}$.

Cumulants as generators.

- Michael Schürmann, Quantum stochastic processes with independent additive increments, J. Multivariate Anal. 38 (1991), no. 1, 15–35.
- Peter Glockner, Michael Schürmann, and Roland Speicher, Realization of free white noises, Arch. Math. (Basel) 58 (1992), no. 4, 407–416.
- A, Partition-dependent dtochastic measures and q-deformed cumulants, Doc. Math. 6 (2001), 343-384.
- Takahiro Hasebe, Hayato Saigo, Joint cumulants for natural independence, Electr. Comm. in Probab. 16 (2011), 491-506 (and other works).

Random variables with finite moments.

On the algebra of (non-commutative) polynomials $\mathbb{C}\langle x_1, \ldots, x_d \rangle$, define a positive linear functional φ_t by

$$\varphi_t \left[x_{u(1)} \ldots x_{u(n)} \right] = \mathbb{E} \left[X^{(u(1))}(t) \ldots X^{(u(n))}(t) \right].$$

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Then the joint cumulant functional is

$$\rho \left[x_{u(1)} \ldots x_{u(n)} \right] = \left. \frac{d}{dt} \right|_{t=0} \varphi_t \left[x_{u(1)} \ldots x_{u(n)} \right].$$

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The joint cumulants are

$$K \left[X^{(u(1))}(t_1), \ldots, X^{(u(n))}(t_n) \right] = t_1 \ldots t_n \rho \left[x_{u(1)} \ldots x_{u(n)} \right].$$

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Schürmann (1991):

$$\rho \rightsquigarrow \left(\mathcal{H}, \{\lambda_j\}_{j=1}^d \subset \mathbb{R}, \{\xi_j\}_{j=1}^d \subset \mathcal{H}, \{T_j\}_{j=1}^d \subset \mathcal{L}(\mathcal{H})\right).$$

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Plus a Fock space construction for each type of independence:

- Usual (Parthasarathy?)
- Free (Glockner, Schürmann, Speicher 1992)
- q-deformed: $q = 1$ usual, $q = 0$ free. (A 2001)
- Boolean, monotone.

Extensivity.

Takahiro Hasebe with Saigo, Lehner (2011, 2017+): similar approach without requiring infinite divisibility.

Rough idea:

$$\rho\left[x_{u(1)} \ldots x_{u(n)}\right] = \left.\frac{d}{dt}\right|_{t=0} \varphi_t\left[x_{u(1)} \ldots x_{u(n)}\right]$$

because $\varphi_t\left[x_{u(1)} \ldots x_{u(n)}\right] =$ polynomial in t, without constant term, with linear term $\rho\left[x_{u(1)} \ldots x_{u(n)}\right]$.

Enough to consider $t \in \mathbb{N}$.

Random variables without moments: transforms.

Independent:

$$\mathcal{F}_t(\theta_1, \ldots, \theta_d) = \varphi_t\left[\exp(i\theta \cdot \mathbf{x})\right] = \int \exp(i\theta \cdot \mathbf{x})\, d\mu_t(\mathbf{x}).$$

$C(\theta_1, \ldots, \theta_d) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_t(\theta_1, \ldots, \theta_d) =$ Lévy-Hinchin representation.

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Free, $d = 1$:

$$G_t(z) = \varphi\left[\frac{1}{z - x}\right] = \int \frac{1}{z - x}\, d\mu_t(x),$$

$$R(z) = \left.\frac{d}{dt}\right|_{t=0} G_t(z).$$

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For $d > 1$, can (in principle) use matrix-valued versions of G and R.

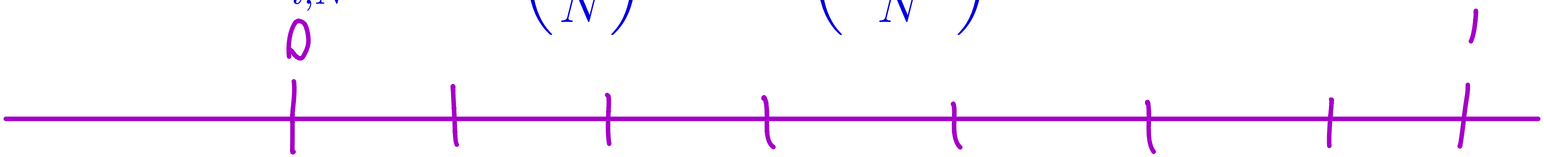
Cumulants from higher variations.

- R. Speicher, A new example of “independence” and “white noise”, Probab. Theory Related Fields 84 (1990), 141–159.
- Gian-Carlo Rota and Timothy C. Wallstrom, Stochastic integrals: a combinatorial approach, Ann. Probab. 25 (1997), no. 3, 1257–1283. 15–35.
- A, Free stochastic measures via noncrossing partitions, Adv. Math. 155 (2000), no. 1, 154–179.

Higher variations (diagonal measures).

For $N \in \mathbb{N}$, denote

$$X_{i,N}^{(j)} = X^{(j)}\left(\frac{i}{N}\right) - X^{(j)}\left(\frac{i-1}{N}\right), \quad 1 \leq j \leq N.$$



Denote

$$\Delta_n^{(N)}(\mathbf{u}) = \sum_{i=1}^{N} X_{i,N}^{u(1)} \dots X_{i,N}^{u(n)}.$$

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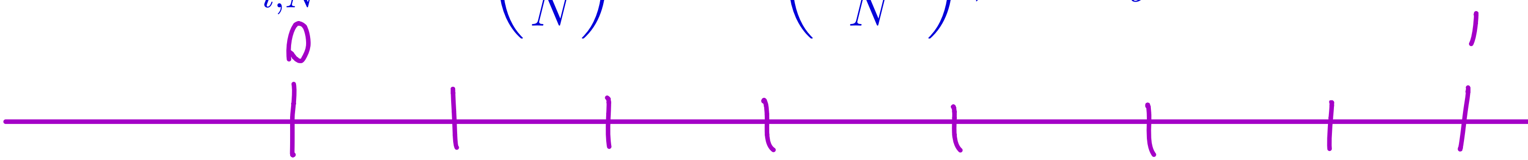

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Or can use triangular arrays converging in distribution to $(X^{(1)}, \ldots, X^{(d)})$, but then limits below may only exist in distribution.

Random variables with finite moments.

Define the cumulants by

$$K_n(X) = \lim_{N \to \infty} \mathbb{E}\left[\Delta_n^{(N)}\right] = \lim_{N \to \infty} N \mathbb{E}\left[X(1/N)\right]$$

and more generally

$$\begin{aligned} K\left[X^{u(1)}, \ldots, X^{u(n)}\right] &= \lim_{N \to \infty} \mathbb{E}\left[\Delta_n^{(N)}(\mathbf{u})\right] \\ &= \lim_{N \to \infty} N \mathbb{E}\left[X^{u(1)}(1/N), \ldots, X^{u(n)}(1/N)\right]. \end{aligned}$$

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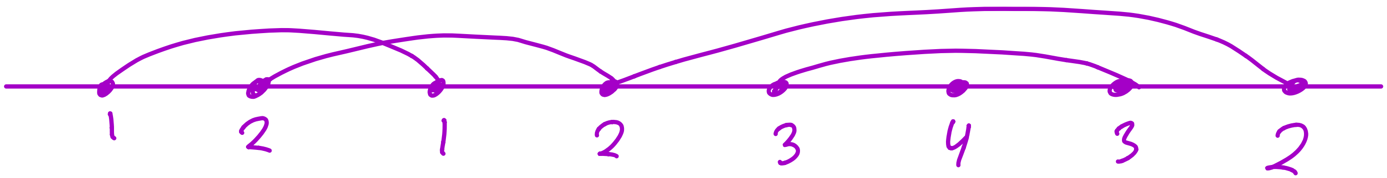
Note this is again the time derivative at $t = 0$.

Moments in terms of cumulants.

Let $\pi \in \mathcal{P}(n)$ be a set partition of n elements.

For $\mathbf{i} \in [N]^n$, write $\mathbf{i} \sim \pi$ if

$$i(j_1) = i(j_2) \quad \Leftrightarrow \quad j_1 \overset{\pi}{\sim} j_2.$$



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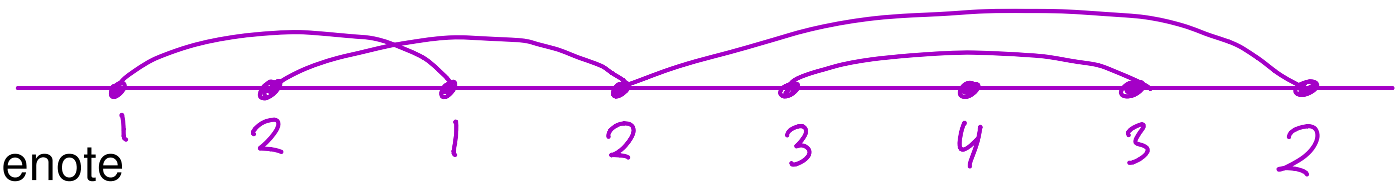

Denote

$$\mathrm{St}_\pi^{(N)}(\mathbf{u}) = \sum_{\mathbf{i} \sim \pi} X_{i(1),N}^{u(1)} \ldots X_{i(n),N}^{u(n)}.$$

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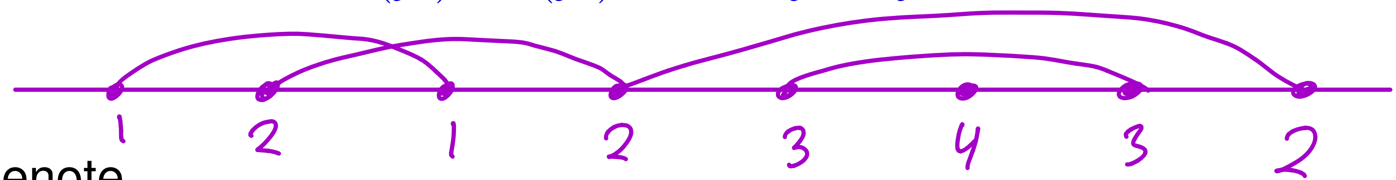

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Clearly

$$\sum_{\pi \in \mathcal{P}(n)} \mathrm{St}_\pi^{(N)}(\mathbf{u}) = X^{u(1)} \dots X^{u(n)}.$$

Partitioned cumulants.

Thus

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In particular, denote

$$K_\pi\left[X^{u(1)}, \dots, X^{u(n)}\right] = \lim_{N \to \infty} \mathbb{E}\left[\mathrm{St}_\pi^{(N)}(\mathbf{u})\right]$$

if the limit exists. Then

$$\sum_{\pi \in \mathcal{P}(n)} K_\pi\left[X^{u(1)}, \dots, X^{u(n)}\right] = \mathbb{E}\left[X^{u(1)} \dots X^{u(n)}\right].$$

Theorem: Partitioned cumulant factorization.

- **Classical.**

$$K_\pi\left[X^{u(1)}, \ldots, X^{u(n)}\right] = \prod_{V \in \pi} K\left[X^{u(i)} : i \in V\right].$$

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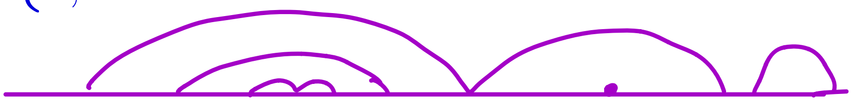
$$K_\pi\left[X^{u(1)}, \dots, X^{u(n)}\right] = \begin{cases} \prod_{V \in \pi} K\left[X^{u(i)} : i \in V\right], & \pi \text{ non-crossing}, \\ 0, & \text{otherwise}. \end{cases}$$



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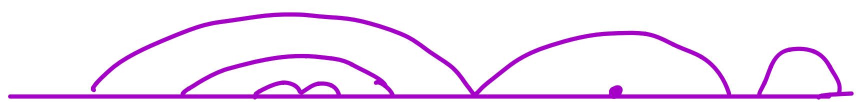

- **q-deformed.** (A 2001)

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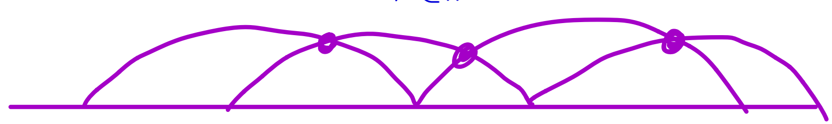

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- **Boolean/monotone?**

Beyond cumulants.

- F. Avram and M. S. Taqqu, Symmetric polynomials of random variables attracted to an infinitely divisible law, Probab. Theory Related Fields 71 (1986), 491–500.
- A, Free stochastic measures via noncrossing partitions II, Pacific J. Math. 207 (2002), 13-30.
- O. E. Barndorff-Nielsen and S. Thorbjørnsen, The Lévy–Itô decomposition in free probability, Probab. Theory Related Fields 131 (2005), 197–228.
- A and Zhichao Wang, Higher variations for free Lévy processes, Studia Math. 252 (2020) 49-81.

Existence of Δ_n.

Theorem. For processes with free increments:

- If random variables are bounded (distributions compactly supported),

$$\Delta_n(\mathbf{u}) = \lim_{N\to\infty} \Delta_n^{(N)}(\mathbf{u})$$

  exist as limits in the operator norm (A 2002).
- For a free compound Poisson process, exist as almost uniform limits (by Egorov, n.c. version of “almost surely”) (A, Wang 2020).
- For (appropriate) triangular arrays, exists as limits in distribution (A, Wang 2020).
- In general, do they exist as limits in probability (n.c. version)?
- In general, do they exist as joint limits in distribution?

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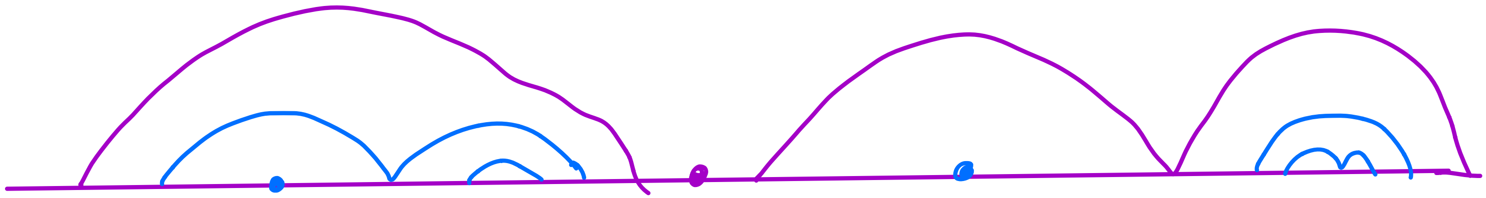
- Moreover, denoting

$$\Psi_n(\mathbf{u}) = \lim_{N \to \infty} \sum_{\substack{i(1), \ldots, i(n) \\ \text{distinct}}} X_{i(1),N}^{u(1)} \ldots X_{i(n),N}^{u(n)}$$

$$\text{``} = \text{''} \int_{[0,1]^d} dX^{(u(1))}(t_1) \ldots dX^{(u(n))}(t_n),$$

Existence of St_π.

$$\mathrm{St}_\pi(\mathbf{u}) = \begin{cases} \prod_{\substack{V \in \pi \\ \text{inner}}} K\left[X^{u(i)} : i \in V\right] \Psi\left[\Delta_U : U \in \pi \text{ outer}\right], & \pi \text{ non-crossing}, \\ 0, & \text{otherwise}. \end{cases}$$



Additivity and vanishing properties.

Theorem. Assume $\{X^{(1)}(t), \ldots, X^{(d)}(t)\}$ are free.

- If all the random variables are bounded,

$$\Delta_n(X^{u(1)}, \ldots, X^{u(n)}) = 0$$

  unless all $u(1) = u(2) = \ldots = u(n)$.

- In particular, if $X(t)$ and $Y(t)$ are free,

$$\Delta_n(X(t) + Y(t)) = \Delta_n(X(t)) + \Delta_n(Y(t)).$$

Same for (possibly unbounded) free compound Poisson process.

General unbounded free Lévy processes?

Theorem. (A, Wang 2020) If the distributions of all $X(t)$ are symmetric,

then $\Delta_2(X(t)$ exists as a limit in probability,

and for two free, symmetric processes,

$$\Delta_2(X(t) + Y(t)) = \Delta_2(X(t)) + \Delta_2(Y(t)).$$

For $n > 2$??

Thank you!