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http://page.math.tu-berlin.de/~sullivan/L/11W/DG2/ WS 11

Differential Geometry II: Analysis and Geometry on Manifolds

Exercise Sheet 7

(preparation for the test)

Exercise 1

Let real projective space $\mathbb{RP}^n$ be the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation

$$(y_1, \dots, y_{n+1}) \sim (\lambda y_1, \dots, \lambda y_{n+1}),$$

for any non–zero real number λ. Let $[y_1 : \dots : y_{n+1}]$ denote the equivalence class of $(y_1, \dots, y_{n+1})$. Consider the maps

$$\phi_i \colon V_i := \{[y_1 : \dots : y_i : \dots : y_{n+1}] \mid y_i \neq 0\} \to \mathbb{R}^n,$$
$$[y_1 : \dots : y_i : \dots : y_{n+1}] \mapsto \frac{1}{y_i}(y_1, \dots, \hat{y}_i, \dots, y_{n+1})$$

for all $i = 1, \dots, n+1$. Show that the collection $(\phi_i, V_i)_{1 \leq i \leq n+1}$ defines a smooth structure on $\mathbb{RP}^n$. Is $\mathbb{RP}^n$ compact? What happens if one replaces the real numbers by the complex numbers? What is the dimension of $\mathbb{CP}^n$?

Exercise 2

Show that $\mathbb{RP}^1$ and $\mathbb{S}^1$ are diffeomorphic.

Exercise 3

Let A be a real symmetric $n \times n$ matrix and $b \neq 0$. Show that

$$M := \{x \in \mathbb{R}^n \mid x^T A x = b\}$$

is an $n-1$ dimensional submanifold of $\mathbb{R}^n$. If all eigenvalues of A are greater zero then M is diffeomorphic to $\mathbb{S}^n$.

Exercise 4

Let $f\colon M \to N$ be an embedding with $f(p) = q$. Show

i) $f^*\colon C^\infty(q) \to C^\infty(q)$ is onto,

ii) $f_*\colon T_pM \to T_qM$ is injective.

Exercise 5

On $\mathbb{R}^3$ consider the vector fields

$$X := x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad Y := x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad Z := [X, Y].$$

Compute the coordinate expression of Z and describe (in words, geometrically) the flows of all three vector fields.

Exercise 6

Let X be a vector field on $\mathbb{S}^2$, which is never tangent to the equator $\mathbb{S}^1 := \mathbb{S}^2 \bigcap (\mathbb{R}^2 \times \{0\})$. Show that the integral curves of X intersect the equator at most once.

Exercise 7

We consider functions P, Q defined on $\mathbb{R}^4$ and coordinates x, y, u, v. Let

$$\omega_1 := dx - Pdu + Qdv, \omega_2 := dy - Qdu - Pdv.$$

Show that $\omega_1 = \omega_2 = 0$ defines a two-plane distribution. Determine the conditions on P and Q under which this distribution is completely integrable.

Exercise 7

Let $i\colon \mathbb{S}^3 \to \mathbb{R}^4$ denote the inclusion map

i) Show that for every $p \in \mathbb{S}^3$ the kernel of the map $i^*\colon T_p^*\mathbb{R}^4 \to T_p^*\mathbb{S}^3$ equals $\lambda(x_1dx_1 + x_2dx_2 + x_3dx_3 + x_4dx_4)$, $\lambda \in \mathbb{R}$.

ii) Prove that the restriction $i^*\sigma$ of

$$\sigma = x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3$$

to $\mathbb{S}^3$ is nowhere zero.